APPROXIMATE GREATEST COMMON DIVISOR OF POLYNOMIALS AND THE STRUCTURED SINGULAR VALUE

G. Halikias*, **S. Fatouros*** **and N. Karcanias***

* Control Engineering Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, U.K.
e-mail addresses: g.halikias@city.ac.uk, s.fatouros@city.ac.uk, n.karcanias@city.ac.uk

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Abstract

In this note the following problem is considered: Given two monic coprime polynomials $a(s)$ and $b(s)$ with real coefficients, find the smallest (in magnitude) perturbation in their coefficients so that the perturbed polynomials have a common root. It is shown that the problem is equivalent to the calculation of the structured singular value of a matrix, which can be performed using efficient existing techniques of robust control. A simple numerical example illustrates the effectiveness of the method. The generalisation of the method to calculate the approximate greatest common divisor (GCD) of polynomials is finally discussed.

1 Introduction

The computation of the greatest common divisor (GCD) of two polynomials $a(s)$ and $b(s)$ is a non-generic problem, in the sense that a generic pair of polynomials has greatest common divisor one. Thus, the notion of *approximate* common factors and GCD's has to be introduced for the purpose of effective numerical calculations.

In the context of Systems and Control applications, the main motivation for developing non-generic techniques for calculating the GCD arises from the study of almost zeros [3]. The numerical computation of the GCD of polynomials in a robust way has been considered using a variety of methodologies, such as ERES [5], extended ERES [5] and matrix pencil methods [4]. The main characteristic of all these techniques is that they transform exact procedures to their numerical versions.

In this work, we formalise the notion of "approximate coprimeness" and "approximate GCD" of two polynomials, by considering the minimum-magnitude perturbation in the polynomial's coefficients, such that the perturbed polynomials have a common root. The calculation of the minimal perturbation is shown to correspond to the distance of a structured matrix from singularity, or, equivalently to the calculation of the structured singular value of a matrix [6, 7]. The later problem has been studied extensively in the context of robust-stability and robust-performance design of control systems, and efficient methods have been developed for its numerical solution. Generalising the definition of the structured singular value by imposing additional rank constraints, leads naturally to the formulation of a problem involving the calculation of the approximate GCD of some minimal degree k.

2 Main results

The problem considered in this paper is the following:

Problem 1: Let $a(s)$ and $b(s)$ be two monic and coprime polynomials with real coefficients, such that $\partial a(s) = m$ and $\partial b(s) = n$ where $m \geq n$. What is the smallest real perturbation (in magnitude) in the coefficients of $a(s)$ and $b(s)$ so that the perturbed polynomials have a common root?

Formally write:

$$a_0(s) = s^m + \alpha_{m-1}s^{m-1} + \alpha_{m-2}s^{m-2} + \ldots + \alpha_0 \quad (1)$$

and

$$b_0(s) = s^m + \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_0 \quad (2)$$

and assume that the coefficients $\{\alpha_i, i = 0, 1, \ldots, m-1\}$ and $\{\beta_i, i = 0, 1, \ldots, n-1\}$ are subjected to real perturbations $\delta_0, \delta_1, \ldots, \delta_{m-1}$ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$, respectively, i.e. the perturbed polynomials are:

$$\begin{aligned} a(s;\delta_0,\ldots,\delta_{m-1}) = s^m &+ (\alpha_{m-1}+\delta_{m-1})s^{m-1} \\ &+ \ldots + (\alpha_0+\delta_0) \end{aligned}$$

and

$$\begin{aligned} b(s;\epsilon_0,\ldots,\epsilon_{n-1}) = s^n &+ (\beta_{n-1}+\epsilon_{n-1})s^{n-1} \\ &+ \ldots + (\beta_0+\epsilon_0) \end{aligned}$$

respectively. Let also:

$$\gamma = \max\{|\delta_0|,\ |\delta_1|,\ldots,|\delta_{m-1}|,|\epsilon_0|,|\epsilon_1|,\ldots,|\epsilon_{n-1}|\} \quad (3)$$

Then Problem 1 is equivalent to: $\min\gamma$ such that $a(s;\delta_0,\ldots,\delta_{m-1})$ and $b(s;\epsilon_0,\ldots,\epsilon_{n-1})$ have a common root.

In the sequel, it is shown that Problem 1 is equivalent to the calculation of the (real) structured singular value of an appropriate matrix which can be performed efficiently using existing algorithms.

Problem 2: (real structured singular value). Let $M \in \mathcal{R}^{n\times n}$ and define the "structured" set:

$$\mathcal{D} = \{\mathrm{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathcal{R}, i = 1, 2, \ldots, s\}$$

where the r_i are positive integers such that $\sum_{i=1}^{s} r_i = n$. The structured singular value of M (relative to "structure" $\mathcal{D}$) is defined as:

$$\mu_{\mathcal{D}}(M) = \frac{1}{\min\{\|\Delta\| \ : \ \Delta \in \mathcal{D}, \ \det(I_n - M\Delta) = 0\}} \quad (4)$$

unless no $\Delta \in \mathcal{D}$ makes $I_n - M\Delta$ singular, in which case $\mu_{\mathcal{D}}(M) = 0$. The problem here is to calculate $\mu_{\mathcal{D}}(M)$ and, provided that $\mu_{\mathcal{D}}(M) \neq 0$, to find a $\Delta \in \mathcal{D}$ of minimal norm such that $\det(I_n - M\Delta) = 0$.

Before stating the equivalence between Problem 1 and Problem 2, we need the following result which relates the existence of common factors of two polynomials to the rank of their corresponding Sylvester resultant matrix:

Theorem 1: Consider the monic polynomials $a(s)$ and $b(s)$ with $\partial a(s) = m$ and $\partial b(s) = n$ where $m \geq n$, and let A be their Sylvester resultant matrix,

$$A = \begin{pmatrix} 1 & \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 \\ 1 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 & 0 & \cdots & 0 \\ 0 & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix}$$

Then, if $\phi(s)$ denotes the GCD of $a(s)$ and $b(s)$ the following properties hold true:

1. $\mathrm{Rank}(A) = n + m - \partial\phi(s)$.
2. The polynomials $a(s)$ and $b(s)$ are coprime if and only if $\mathrm{Rank}(A) = n + m$.
3. The GCD $\phi(s)$ is invariant under elementary row operations on A. Furthermore, if we reduce A to its row echelon form, the last non-vanishing row defines the coefficients of $\phi(s)$.

Proof: See [1, 5, 2]. □

Using Theorem 1, the equivalence of Problem 1 and Problem 2 can now be established:

Theorem 2: Problem 1 is equivalent to Problem 2 by defining:

1. The structured set $\mathcal{D}$ as:

$$\mathcal{D} = \{\mathrm{diag}(\delta_{m-1} I_n, \ \delta_{m-2} I_n, \ldots, \delta_0 I_n, \epsilon_{n-1} I_m, \\ \epsilon_{n-2} I_m, \ldots, \epsilon_0 I_m) : \delta_i \in \mathcal{R}, \epsilon_i \in \mathcal{R}\}$$

   i.e. $s = m + n$, $r_i = n$ for $1 \leq i \leq m$ and $r_i = m$ for $m + 1 \leq i \leq m + n$.

2. $M = -ZA^{-1}\Theta$ where:

$$\Theta = \left(\begin{array}{ccc|ccc} I_n & \cdots & I_n & O_{n,m} & \cdots & O_{n,m} \\ \hline O_{m,n} & \cdots & O_{m,n} & I_m & \cdots & I_m \end{array}\right)$$
$$Z' = \left((Z_{nm}^0)' \ \cdots \ (Z_{nm}^{m-1})' \ (Z_{mn}^0)' \ \cdots \ (Z_{mn}^{n-1})'\right)$$

   with $\Theta \in \mathcal{R}^{n+m,2nm}$ and $Z' \in \mathcal{R}^{n+m,2nm}$, where:

$$Z_{nm}^k = \begin{pmatrix} O_{n,k+1} & I_n & O_{n,m-k-1} \end{pmatrix}$$

   for $k = 0, 1, \ldots, m-1$ and A denotes the (non-singular) Sylvester's resultant matrix of polynomials $a(s)$ and $b(s)$ defined above.

3. With these definitions:

$$\gamma^\star = \frac{1}{\mu_{\mathcal{D}}(M)} \quad (5)$$

   where $\gamma^\star$ denotes the minimum-magnitude real perturbation in the coefficients of $a(s)$ and $b(s)$ such that the perturbed polynomials have a common root.

Proof: Since $a(s)$ and $b(s)$ are assumed coprime, their Sylvester resultant matrix A is nonsingular. The Sylvester resultant matrix $\hat{A}$ of the perturbed polynomials $a(s; \delta_0, \ldots, \delta_{m-1})$ and $b(s; \epsilon_0, \ldots, \epsilon_{n-1})$ can be decomposed as $\hat{A} = A + E$ where E denotes the "perturbation matrix":

$$E = \begin{pmatrix} 0 & \delta_{m-1} & \delta_{m-2} & \cdots & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 \\ 0 & \epsilon_{n-1} & \epsilon_{n-2} & \cdots & \epsilon_0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 \end{pmatrix}$$

Matrix E can now be factored as $E = \Theta\Delta Z$ where Θ and Z are defined in Theorem 2 part 2 above, and

$$\Delta = \mathrm{diag}(\delta_{m-1} I_n, \ \delta_{m-2} I_n, \ldots, \delta_0 I_n, \\ \epsilon_{n-1} I_m, \ \epsilon_{m-2} I_m, \ldots, \epsilon_0 I_m)$$

Clearly $\Delta \in \mathcal{D}$, i.e. it has a block-diagonal structure $s = m + n$, $r_i = n$ for $1 \leq i \leq m$ and $r_i = m$ for $m + 1 \leq i \leq m + n$. Note also that:

$$\gamma = \max\{ \ |\delta_0|, \ |\delta_1|, \ldots, |\delta_{m-1}|, \ |\epsilon_0|, \ |\epsilon_1|, \ldots, |\epsilon_{n-1}| \ \} \\ = \|\Delta\|$$

Since the resultant Sylvester matrix $\hat{A}$ loses rank if and only if there is a common factor between $a(s; \delta_0, \ldots, \delta_{m-1})$ and $b(s; \epsilon_0, \ldots, \epsilon_{m-1})$, Problem 1 is equivalent to:

$$\min \|\Delta\| \ \text{ such that } \ \det(A + \Theta\Delta Z) = 0 \ \text{ and } \ \Delta \in \mathcal{D} \quad (6)$$

Using the matrix identity

$$\det(I + BC) = \det(I + CB) \tag{7}$$

which holds for any two matrices B and C of compatible dimensions, and the fact that A is non-singular, we have that:

$$\begin{aligned} \det(A + \Theta\Delta Z) = 0 \;\; &\Leftrightarrow \;\; \det(I + ZA^{-1}\Theta\Delta) = 0 \\ &\Leftrightarrow \;\; \det(I - M\Delta) = 0 \end{aligned}$$

Hence Problem 1 becomes:

$$\min\{\ \|\Delta\| \ : \ \det(I - M\Delta) = 0,\ \Delta \in \mathcal{D}\ \} \tag{8}$$

which is equivalent to Problem 2, the minimum being $\mu_{\mathcal{D}}^{-1}(M)$.

□

Example 1: Let $a(s) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ ($m = 3$) and $b(s) = s^2 + \beta_1 s + \beta_0$ ($n = 2$). The Sylvester resultant matrix of the perturbed polynomials is:

$$\hat{A} = \begin{pmatrix} 1 & \alpha_2 + \delta_2 & \alpha_1 + \delta_1 & \alpha_0 + \delta_0 & 0 \\ 0 & 1 & \alpha_2 + \delta_2 & \alpha_1 + \delta_1 & \alpha_0 + \delta_0 \\ 1 & \beta_1 + \epsilon_1 & \beta_0 + \epsilon_0 & 0 & 0 \\ 0 & 1 & \beta_1 + \epsilon_1 & \beta_0 + \epsilon_0 & 0 \\ 0 & 0 & 1 & \beta_1 + \epsilon_1 & \beta_0 + \epsilon_0 \end{pmatrix} \tag{9}$$

which can be written as:

$$\hat{A} = \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \end{pmatrix} + \begin{pmatrix} 0 & \delta_2 & \delta_1 & \delta_0 & 0 \\ 0 & 0 & \delta_2 & \delta_1 & \delta_0 \\ 0 & \epsilon_1 & \epsilon_0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & \epsilon_0 & 0 \\ 0 & 0 & 0 & \epsilon_1 & \epsilon_0 \end{pmatrix}$$

Let A and E denote the first and second matrices, respectively, in the RHS of this equation. The "perturbation" matrix E can be factored as:

$$E = \left(\begin{array}{cc|cc|cc|ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$$\begin{pmatrix} \delta_2 I_2 & 0 & 0 & 0 & 0 \\ 0 & \delta_1 I_2 & 0 & 0 & 0 \\ 0 & 0 & \delta_0 I_2 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 I_3 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_0 I_3 \end{pmatrix}$$
$$\left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

which is of the required form $E = \Theta\Delta Z$ with $\Delta \in \mathcal{D}$. The minimum coefficient perturbation is the inverse of the structured singular value of the matrix:

$$M = -\left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \end{pmatrix}^{-1}$$
$$\left(\begin{array}{cc|cc|cc|ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

and can be computed numerically using efficient existing techniques [6, 7].

Example 2: Here the effectiveness of the method is tested with a numerical example. Consider the polynomials

$$a_0(s) = s^3 - 6s^2 + 11s - 6 \quad \text{and} \quad b_0(s) = s^2 - 6s + 8$$

with roots $\{3,\ 2,\ 1\}$ and $\{4,\ 2\}$ respectively. Since there is a common root ($s = 2$) the resultant Sylvester matrix is singular. The polynomials were perturbed to:

$$a(s) = s^3 - 6.05s^2 + 11.1s - 5.95$$

and

$$b(s) = s^2 - 6.06s + 8.1$$

which have roots

$$\{3.0512,\ 2.0454,\ 0.9534\}$$

and

$$\{4.0301,\ 2.0099\}$$

respectively. The singular values of the corresponding Sylvester resultant matrix,

$$A = \begin{pmatrix} 1 & -6.05 & 11.1 & -5.95 & 0 \\ 0 & 1 & -6.05 & 11.1 & -5.95 \\ 1 & -6.04 & 8.1 & 0 & 0 \\ 0 & 1 & -6.04 & 8.1 & 0 \\ 0 & 0 & 1 & -6.04 & 8.1 \end{pmatrix}$$

were obtained as:

$$\{22.7997,\ 12.3247,\ 5.4710,\ 0.2264,\ 0.0007\}$$

indicating a numerical rank of 4 and, hence, an approximate GCD of degree one (Theorem 1), as expected.

Next the results of Theorem 2 were applied to $a(s)$ and $b(s)$. Note that since the maximum perturbation of the coefficients of

$a(s)$ and $b(s)$ relative to those of $a_0(s)$ and $b_0(s)$ (which have a common root) is 0.1, we expect that $\gamma^* \leq 0.1$.

Two functions from Matlab's μ-optimisation toolbox (*mu.m* and *unwrap.m*) were used to calculate the (real) structured singular value of matrix M and the corresponding minimum-norm singularising perturbation Δ_0. The lower and upper bounds of the structured singular value were obtained as:

$$222.991497161 \leq \mu_{\mathcal{D}}(M) = \frac{1}{\gamma^*} \leq 222.991497162$$

corresponding to $\gamma^* = 0.0044844759227$. It was also checked that the singularising perturbation Δ_0 corresponding to $\mu_{\mathcal{D}}(M)$ had the right structure (i.e. $\Delta_0 \in \mathcal{D}$) and in fact $\delta_0 = \delta_1 = \delta_2 = -\gamma^*$ and $\epsilon_0 = \epsilon_1 = \gamma^*$. Polynomials with a common factor "nearest" to $a(s)$ and $b(s)$ were obtained with the help of Δ_0 as:

$$\hat{a}(s) = s^3 - 6.05448447592s^2 + 11.0955155241s - 5.95448447592$$

and

$$\hat{b}(s) = s^2 - 6.03551552408s + 8.10448447592$$

The roots of $\hat{a}(s)$ and $\hat{b}(s)$ were calculated as:

$$\{3.07886773569, \;\; 2.01656975051, \;\; 0.959046989722\}$$

and

$$\{4.01894577356, \;\; 2.01656975051\}$$

respectively corresponding to an "optimal" approximate GCD, $\phi(s) = s - 2.01656975051$.

3 Approximate GCD of polynomials

The technique developed in section 2 may be used to define a conceptual algorithm to calculate the numerical GCD of any two polynomials $a(s)$ and $b(s)$. This sequentially extracts approximate common factors $\phi_i(s)$, by calculating the corresponding structured singular value $\mu_{\mathcal{D}}$ and the minimum-norm singularising matrix perturbation Δ_0. After the extraction of each factor, the quotients $a_{i+1} = a_i(s)/\phi_i(s)$ and $b_{i+1}(s) = b_i(s)/\phi_i(s)$ are calculated, ignoring possible (small) remainders of the divisions. The procedure is initialised by setting $a_0(s) = a(s)$, $b_0(s) = b(s)$, and iterates by constructing at each step of the algorithm the reduced-dimension Sylvester matrix corresponding to the polynomial pair $(a_{i+1}(s), b_{i+1}(s))$, followed by calculating the new $\mu_{\mathcal{D}}$ and Δ_0, which in turn results to the extraction of the new factor $\phi_{i+1}(s)$. The whole process is repeated until a tolerance condition is met, at which stage the approximate GCD $\phi(s)$ can be constructed by accumulating the extracted common factors $\phi_i(s)$. Special care is needed in the real case, to ensure that any complex roots in $\phi(s)$ appear in conjugate pairs.

Compared to this procedure, a more elegant approach would be to extract the approximate common factor via a non-iterative algorithm. This requires the refinement of the definition of the structured singular value by introducing rank constraints. Specifically, we define:

Definition: (generalised real structured singular value). Let $M \in \mathcal{R}^{n\times n}$ and define the "structured" set:

$$\mathcal{D} = \{\mathrm{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathcal{R}, i = 1, 2, \ldots, s\}$$

where the r_i are positive integers such that $\sum_{i=1}^{s} r_i = n$. The generalised structured singular value of M relative to "structure" $\mathcal{D}$ and for a non-negative integer k is defined as:

$$\hat{\mu}_{\mathcal{D},k}(M) = \frac{1}{\min\{\|\Delta\| : \Delta \in \mathcal{D}, \mathrm{null}(I_n - M\Delta) > k\}}$$

unless there does not exist a $\Delta \in \mathcal{D}$ such that $\mathrm{null}(I_n - M\Delta) > k$, in which case $\hat{\mu}_{\mathcal{D},k}(M) = 0$.

It follows immediately from the definition that $\hat{\mu}_{\mathcal{D},0}(M) = \mu_{\mathcal{D}}(M)$ and that $\hat{\mu}_{\mathcal{D},k}(M) \geq \hat{\mu}_{\mathcal{D},k+1}(M)$ for each integer $k \geq 0$. Further if for some integer k, $\hat{\mu}_{\mathcal{D},k}(M) > 0$ and $\hat{\mu}_{\mathcal{D},k+1}(M) = 0$, then for any minimiser Δ of $\hat{\mu}_{\mathcal{D},k}(M)$, $\mathrm{null}(I_n - M\Delta) = k + 1$.

Theorem 3: Let $a_0(s)$ and $b_0(s)$ be two monic coprime polynomials of degrees $\partial a_0(s) = m$ and $\partial b_0(s) = n$ with $m \geq n$ defined in equations (1) and (2). Let $a(s)$ and $b(s)$ be two perturbed polynomials, and set

$$\gamma = \max\{|\delta_0|, |\delta_1|, \ldots, |\delta_{m-1}|, |\epsilon_0|, |\epsilon_1|, \ldots, |\epsilon_{n-1}|\} \quad (10)$$

where $\{\delta_i\}$ and $\{\epsilon_i\}$ denote the perturbed coefficients of $a_0(s)$ and $b_0(s)$ respectively. Further, let $\gamma^*(k)$ denote the the minimum value of γ such that $a(s)$ and $b(s)$ have a common factor $\phi(s)$ of degree $\partial\phi(s) > k$ ($k = 0, 1, \ldots, n-1$). Then,

$$\gamma^*(k) = \frac{1}{\hat{\mu}_{\mathcal{D},k}(M)} \quad (11)$$

where $\hat{\mu}_{\mathcal{D},k}(M)$ denotes the *generalised structured singular value* of $M = -ZA^{-1}\Theta$ with respect to the structure $\mathcal{D}$ defined in equation (8), and A, Θ and Z are the matrices defined in equations (7), (9) and (10) respectively. Further, $\phi(s)$ may be constructed from any $\Delta_0 \in \mathcal{D}$ which minimises $\|\Delta\|$ subject to the constraint $\mathrm{null}(I_n - M\Delta) > k$.

Proof: This is almost identical to the proof of Theorem 2, on noting that the transformations in equation (18) do not change the nullity of the corresponding matrices. $\square$

Theorem 3 suggests that the GCD of two polynomials $a(s)$ and $b(s)$ can be obtained by calculating successively $\hat{\mu}_{\mathcal{D},k}(M)$ for $k = 0, 1, \ldots n-1$. The procedure terminates when either $k = n-1$ is reached, or when the generalised structured singular value falls below a pre-specified tolerance level. The effective numerical calculation of $\hat{\mu}_{\mathcal{D},k}$ (or at least of tight upper and lower bounds) will be the subject of future research. A further generalisation of our method involves the calculation of the approximate GCD for an arbitrary number of polynomials (rather than just two). This problem can also be translated to our framework using some recent results on generalised resultants [2] and will also be a topic of future work.

4 Conclusions

In this paper we have proposed a new method for calculating numerically the approximate GCD of a set of polynomials. It was shown that, for two coprime polynomials, the problem of calculating the smallest l_∞-norm perturbation in the polynomials' coefficient vector so that the perturbed polynomials have a common root, is equivalent to the calculation of the structured singular value of an appropriate matrix. This is a fundamental problem in the area of robust control and various techniques have been successfully developed for its solution. The effectiveness of one such method for calculating the GCD of low-degree polynomials has been demonstrated via a numerical example. We have further shown that calculating the minimum l_∞-norm perturbation in the coefficient vector of two (or more) coprime polynomials so that the perturbed polynomials have a GCD of degree at least k, reduces to a structured singular value-type calculation with additional rank constraints. This is a non-standard problem and developing effective algorithms for its solution will be the topic of future research work.

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