CONTROLLABILITY AND REACHABILITY CRITERIA FOR LINEAR PIECEWISE CONSTANT IMPULSIVE SYSTEMS

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Abstract

Impulsive differential systems are an important class of mathematical models for many practical systems in physics, chemistry, biology, engineering, and information science which exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the dynamical processes. In this paper, controllability and reachability of linear piecewise constant impulsive systems are studied. Necessary and sufficient criteria for reachability and controllability are established, respectively. It is proved that reachability is equivalent to controllability for such systems under some mild conditions. Our criteria are in geometric forms, they can be transformed into algebraic type conveniently.

1 Introduction

In recent years, there has been increasing interest in the analysis and synthesis of impulsive systems, or impulsive control systems, due to their significance both in theory and applications([1-3],[6-9],[11-12] and [14-18]).

Different from another type of systems associated with the impulses, i.e., the singular systems or the descriptor systems, impulsive control systems are described by impulsive ordinary differential equations. Many real systems in physics, chemistry, biology, engineering, and information science exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the continuous dynamical processes. This kind of impulsive behaviors can be modelled by impulsive systems.

Controllability and observability of impulsive control systems have been studied by a number of papers. [9] investigated the controllability of a class of time-invariant impulsive systems with the assumption that the impulses of impulsive control are regulated at discontinuous points. [12] improved [9]'s results. Then [7] extended the results to the linear impulsive systems with time-varying coefficients and nonlinear perturbations. [16] studied the null controllability of the linear impulsive systems with the control impulses only acting at the discontinuous points. [18] investigated the controllability and observability of linear time-varying impulsive systems. Sufficient and necessary conditions for controllability and observability are established and their applications to time-invariant impulsive control systems are also discussed.

Controllability and observability are the two most fundamental concepts in modern control theory [10][13]. They have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design, etc. In this paper, we aim to derive necessary and sufficient criteria for controllability and observability of linear piecewise constant impulsive control systems. First, reachability of such systems is investigated, a geometric type necessary and sufficient condition is established. Next, controllability is investigated and an equivalent condition is established as well. Moreover, it is shown that controllability is not equivalent to reachability for such systems in general case, but is equivalent under some extra conditions.

This paper is organized as follows. Section 2 formulates the problem and presents the preliminary results. Section 3 and Section 4 investigate reachability and controllability, respectively. Two numerical examples are given in Section 5. Finally, the conclusion of the paper is provided in Section 6.

2 Preliminaries

Consider the piecewise linear impulsive system given by

$$\left\{\begin{array}{rcll} \dot{x}(t) & = & A_k x(t) + B_k u(t), & t \in [t_{k-1}, t_k) \\ x(t_k^+) & = & E_k x(t_k^-) + F_k u(t_k^-), & \\ x(t_0^+) & = & x_0, & t_0 \geq 0, \end{array}\right. \tag{1}$$

where $k = 1, 2, \cdots$, A_k, B_k, E_k, F_k are known $n \times n$, $n \times p$, $n \times n$ and $n \times p$ constant matrices; $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the input vector; $x(t^+) := \lim_{h \to 0^+} x(t+h)$, $x(t^-) := \lim_{h \to 0^+} x(t-h)$ and the discontinuity points

$$t_1 < t_2 < \cdots < t_k < \cdots, \quad \lim_{k \to \infty} t_k = \infty$$

where $t_0 < t_1$ and $x(t_k^-) = x(t_k)$, which implies that the solution of (1) is left-continuous at t_k. As usual, the admissible control input are limited to piecewise continuous(p.c.) functions.

First, consider the solution of the system (1).

Lemma 1. *For any $t \in (t_{k-1}, t_k]$, $k = 1, 2, \cdots$, the general solution of the system (1) is given by* *a) if* $k = 1$,

$$x(t) = e^{A_1(t-t_0)}x(t_0^+) + \int_{t_0}^{t} e^{A_1(t-s)}B_1u(s)ds \qquad (2)$$

b) if $k = 2, 3, \cdots$,

$$\begin{aligned} x(t) &= e^{A_k(t-t_{k-1})}\Big\{ \prod_{i=k-1}^{1} E_i e^{A_i h_i} x(t_0^+) \\ &+ \sum_{i=1}^{k-2} \Big[\prod_{j=k-1}^{i+1} E_j e^{A_j h_j} \\ &(E_i \int_{t_{i-1}}^{t_i} e^{A_i(t_i-s)}B_iu(s)ds + F_iu(t_i))\Big] \\ &+ E_{k-1}\int_{t_{k-2}}^{t_{k-1}} e^{A_{k-1}(t_{k-1}-s)}B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\Big\} \\ &+ \int_{t_{k-1}}^{t} \exp[A_k(t-s)]B_ku(s)ds \end{aligned} \qquad (3)$$

where $h_j = t_j - t_{j-1}$, $j = 1, 2, \cdots$.

Proof. See Appendix A. □

If $t_f \in (t_0, t_1]$, then a linear time-invariant system is just concerned with. Controllability and observability criteria can be found in standard text books[17,18]. Thus, in the remaining of the paper, the case $t_f \in (t_{k-1}, t_k]$, $k = 2, 3, \cdots$ is only considered.

Now, some mathematical preliminaries are given as the basic tools in the following discussion.

Given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, denote $\mathcal{I}m(B)$ as the *range* of B, i.e., $\mathcal{I}m(B) = \{y|y = Bx, x \in \mathbb{R}^p\}$; and denote $\langle A|B\rangle$ as the *minimal invariant subspace* of A on $\mathcal{I}m(B)$, i.e., $\langle A|B\rangle = \sum_{i=0}^{n-1} \mathcal{I}m(A^iB)$.

The following lemma is a generalization of Theorem 7.8.1 in [5], which is the starting point for deriving the criteria for reachability and controllability.

Lemma 2. *Given matrices* $A, E \in \mathbb{R}^{n\times n}$, $B, F \in \mathbb{R}^{n\times p}$, *for any* $0 \le t_0 < t_f < +\infty$, *we have*

$$\begin{aligned} &\{x|x = E\int_{t_0}^{t_f} \exp[A(t_f-s)]Bu(s)ds + Fu(t_f), \\ &\textit{for some p.c. } u\} = E\,\langle A|B\rangle + \mathcal{I}m(F) \end{aligned} \qquad (4)$$

Proof. See Appendix A. □

3 Reachability

In this Section, reachability of system (1) is investigated.

Definition 1 (Reachability). *The system (1) is said to be (completely) reachable on* $[t_0, t_f]$ ($t_0 < t_f$)*, if for any terminal state* $x_f \in \mathbb{R}^n$*, there exists a piecewise continuous input* $u(t) : [t_0, t_f] \to \mathbb{R}^p$ *such that the system is driven from* $x(t_0) = 0$ *to* $x(t_f) = x_f$.

Definition 2 (Reachable Set). *The set of all the reachable states on* $[t_0, t_f]$ *is said to be the* reachable set *on* $[t_0, t_f]$*, denoted as* $\mathcal{R}_{[t_0,t_f]}$.

Theorem 1. *For the system (1), the reachable set on* $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$ *is given by*

$$\begin{aligned} &\mathcal{R}_{[t_0,t_f]} = \\ &e^{A_k(t_f-t_{k-1})}\Big\{ \sum_{i=1}^{k-2} \Big[\prod_{j=k-1}^{i+1} E_j e^{A_j h_j} (E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i))\Big] \\ &+ E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1})\Big\} + \langle A_k|B_k\rangle \end{aligned} \qquad (5)$$

Proof. By Lemma 1, let $x(t_0) = 0$, we have

$$\begin{aligned} x(t) &= e^{A_k(t-t_{k-1})}\Big\{ \sum_{i=1}^{k-2} \Big[\prod_{j=k-1}^{i+1} E_j \exp(A_jh_j) \\ &\Big(E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i-s)]B_iu(s)ds + F_iu(t_i)\Big) \Big] \\ &+ E_{k-1}\int_{t_{k-2}}^{t_{k-1}} e^{A_{k-1}(t_{k-1}-s)}B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\Big\} \\ &+ \int_{t_{k-1}}^{t} \exp[A_k(t-s)]B_ku(s)ds \end{aligned} \qquad (6)$$

It follows that

$$\begin{aligned} &\mathcal{R}[t_0, t_f] = \{x|x = \exp[A_k(t_f - t_{k-1})]\Big(\\ &\sum_{i=1}^{k-2} \Big(\prod_{j=k-1}^{i+1} E_j e^{A_jh_j} (E_i \int_{t_{i-1}}^{t_i} e^{A_i(t_i-s)}B_iu(s)ds + F_iu(t_i))\Big) \\ &+ E_{k-1}\int_{t_{k-2}}^{t_{k-1}} e^{A_{k-1}(t_{k-1}-s)}B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\Big) \\ &+ \int_{t_{k-1}}^{t_f} e^{A_k(t_f-s)}B_ku(s)ds, \text{for some p.c. } u\} \\ &= \exp[A_k(t_f - t_{k-1})]\Big(\sum_{i=1}^{k-2} \Big(\prod_{j=k-1}^{i+1} E_j \exp(A_jh_j) \times \\ &\{x|x = E_i \int_{t_{i-1}}^{t_i} e^{A_i(t_i-s)}B_iu(s)ds + F_iu(t_i), \text{for some p.c. } u\}\Big) \\ &+ \{x|x = E_{k-1} \int_{t_{k-2}}^{t_{k-1}} e^{A_{k-1}(t_{k-1}-s)}B_{k-1}u(s)ds + F_{k-1}u(t_{k-1}) \\ &\text{for some p.c. } u\}\Big) \\ &+ \{x|x = \int_{t_{k-1}}^{t_f} e^{A_k(t_f-s)}B_ku(s)ds, \text{for some p.c. } u\} \end{aligned}$$

By Lemma 2, we get (5). □

According to the geometric form of the reachable set, a geometric type criterion is established as follows.

Theorem 2. *The system (1) is reachable on* $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$*, if and only if*

$$\begin{aligned} &\sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_jh_j)\,(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) \\ &+ E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k|B_k\rangle = \mathbb{R}^n \end{aligned} \qquad (7)$$

Proof. Since $\exp[A_k(t_f - t_{k-1})]\langle A_k|B_k\rangle = \langle A_k|B_k\rangle$, we get

$$\mathcal{R}_{[t_0,t_f]} = e^{A_k(t_f-t_{k-1})}\Big(\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j e^{A_j h_j}(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) + E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1})\Big) + \langle A_k|B_k\rangle$$
$$= e^{A_k(t_f-t_{k-1})}\Big(\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j e^{A_j h_j}(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) + E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k|B_k\rangle\Big)$$

Since the matrix $\exp[A_k(t_f - t_{k-1})]$ is nonsingular, we know that $\mathcal{R}_{[t_0,t_f]} = \mathbb{R}^n$ if and only if (10) holds. □

Remark 1. *Theorem 2 is a geometric type condition, by simple transformation, an algebraic type condition is derived. In fact, for* $i = 1, 2, \cdots$, *denote*

$$Q_i = [B_i, A_iB_i, \cdots, A_i^{n-1}B_i], \tag{8}$$

for $i = 1, 2, \cdots, k-2$, *denote*

$$H_i = \left[\prod_{j=k-1}^{i+1} E_j e^{A_j h_j} E_i Q_i, \quad \prod_{j=k-1}^{i+1} E_j e^{A_j h_j} F_i\right], \tag{9}$$

$$H_{k-1} = [E_{k-1}Q_{k-1}, \quad F_{k-1}], \tag{10}$$

finally, denote

$$Q_{[t_0,t_f]} = [H_1, H_2, \cdots, H_{k-1}, Q_k] \tag{11}$$

Then it is easy to verify that

$$\exp[A_k(t_f - t_{k-1})]\mathcal{I}m(Q_{[t_0,t_f]}) = \mathcal{R}_{[t_0,t_f]}.$$

Thus, the algebraic type criterion is as follows.

Corollary 1. *The system (1) is reachable on* $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$, *if and only if* $rank(Q_{[t_0,t_f]}) = n$.

4 Controllability

In this Section, controllability of the system (1) is investigated.

Definition 3 (Controllability). *The system (1) is said to be (completely) controllable on* $[t_0, t_f]$ ($t_0 < t_f$), *if for any initial state* $x_0 \in \mathbb{R}^n$, *there exists a piecewise continuous input* $u(t) : [t_0, t_f] \to \mathbb{R}^p$ *such that the system is driven from* $x(t_0) = x_0$ *to* $x(t_f) = 0$.

Definition 4 (Controllable Set). *The set of all the controllable states on* $[t_0, t_f]$ *is said to be the* controllable set *on* $[t_0, t_f]$, *denoted as* $\mathcal{C}_{[t_0,t_f]}$.

First, the relationship between the controllable set and the reachable set is shown in the following Theorem.

Theorem 3. *For the system (1), we have*

$$\Big(e^{A_k(t-t_{k-1})}\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\Big)\mathcal{C}_{[t_0,t_f]} \subseteq \mathcal{R}_{[t_0,t_f]} \tag{12}$$

Proof. By Lemma 1, it is easy to see that if the state x_0 is controllable then the corresponding state $\exp[A_k(t - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)x_0$ is reachable. Based on this fact, we know that (12) holds. □

Based on Theorem 3, a criterion for controllability can be established as follows.

Theorem 4. *The system (1) is controllable on* $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$, *if and only if*

$$\begin{aligned}&\mathcal{I}m(\prod_{i=k-1}^{1} E_i \exp(A_i h_i)) \subseteq \\ &\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)\,(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) \\ &+E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k|B_k\rangle\end{aligned} \tag{13}$$

Proof. Since $e^{A_k(t-t_{k-1})}$ is nonsingular, we know that (12) is equivalent to

$$\begin{aligned}&\Big(\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\Big)\mathcal{C}_{[t_0,t_f]} \subseteq \\ &\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)\,(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) \\ &+E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k|B_k\rangle\end{aligned} \tag{14}$$

Moreover, $\mathcal{C}_{[t_0,t_f]} = \mathbb{R}^n$ if and only if $\Big(\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\Big)\mathcal{C}_{[t_0,t_f]} = \mathcal{I}m(\prod_{i=k-1}^{1} E_i \exp(A_i h_i))$. Thus, we know that $\mathcal{C}_{[t_0,t_f]} = \mathbb{R}^n$ if and only if (13) holds. □

In the general case, for system (1), controllability is not equivalent to reachability. But under some mild conditions, they are equivalent each other.

Corollary 2. *For the system (1), if* E_i *is nonsingular,* $i = 1, 2, \cdots, k-1$, *then the following statements are equivalent:*

a) the system is reachable on $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$;

b) the system is controllable on $[t_0, t_f]$, $t_f \in (t_{k-1}, t_k]$;

c) the following equation holds,

$$\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)\,(E_i\langle A_i|B_i\rangle + \mathcal{I}m(F_i)) + E_{k-1}\langle A_{k-1}|B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k|B_k\rangle = \mathbb{R}^n$$

Proof. Since E_i is nonsingular, $i = 1, 2, \cdots, k-1$, we have $\exp[A_k(t - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)$ is nonsingular. It follows that

$$\exp[A_k(t - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\mathcal{C}_{[t_0,t_f]} = \mathcal{R}_{[t_0,t_f]}$$

It is easy to see that $\mathcal{C}_{[t_0,t_f]} = \mathbb{R}^n \iff \mathcal{R}_{[t_0,t_f]} = \mathbb{R}^n$. □

Remark 2. *In system (1), if $(A_i, B_i) = (A, B)$, $i = 1, \cdots, k$, the system is reduced to the impulsive time-invariant systems studied in [18]. It is easy to see that Theorem 4 concludes the results of Theorem 3.4 in [18] as a special case.*

Remark 3. *In system (1), if $E_i = I$, $F_i = 0$, $i = 1, \cdots, k$, the system is reduced to the piecewise linear systems studied in [4]. It is easy to see that Theorem 5 in [4] is a special case of Corollary 2.*

5 Illustrating Examples

In this section, two numerical examples are presented to illustrate how to utilize our criteria.

Example 1. *Consider a 3-dimensional linear piecewise constant impulsive system with*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $t_i = i$, $i = 0, 1, 2, 3$.

Now, we try to use our criteria to investigate the reachability and controllability on $[0, t_f]$, where $t_f \in (2, 3]$ of the system in Example 1.

First, consider the reachability. By a simple calculation, we have

$$\begin{aligned} &E_2 \exp(A_2)(E_1\langle A_1|B_1\rangle + \mathcal{I}m(F_1)) + E_2\langle A_2|B_2\rangle \\ &+\mathcal{I}m(F_2) + \langle A_3|B_3\rangle = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\} \end{aligned}$$

By Theorem 2, the system should not be reachable. In fact, for any piecewise continuous input $u(t)$, $t \in [0, t_f]$ and any nonzero initial state $x_0 = [x_1^0 \ \ x_2^0 \ \ x_3^0]^T$, we have

$$x(t_f) = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} \tag{15}$$

This fact shows that the system is indeed not reachable.

Next, consider the controllability. By a simple calculation, we have

$$\mathcal{I}m(E_2 \exp(A_2)E_1 \exp(A_1)) = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$$

It is easy to see that

$$\begin{aligned} &\mathcal{I}m(E_2 \exp(A_2)E_1 \exp(A_1)) \subseteq E_2 \exp(A_2)(E_1\langle A_1|B_1\rangle \\ &+\mathcal{I}m(F_1)) + E_2\langle A_2|B_2\rangle + \mathcal{I}m(F_2) + \langle A_3|B_3\rangle \end{aligned}$$

By Theorem 4, the system should be controllable. In fact, we can take the piecewise constant input

$$u(t) = \begin{cases} c_1, & t \in (0, 1]; \\ 0, & t \in (1, 2]; \\ c_3, & t \in (2, 3]. \end{cases}$$

Then, for any nonzero initial state $x_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}$, we have

$$x(t_f) = \begin{bmatrix} x_1^0 + 0.5c_1 \\ x_2^0 + 1.5c_1 + (2 - 2t_f + 0.5t_f^2)c_3 \\ 0 \end{bmatrix} \tag{16}$$

Obviously, if $c_1 = -2x_1^0$, $c_3 = (-x_2^0 - 1.5c_1)/(2 - 2t_f + 0.5t_f^2)$, then $x(t_f) = 0$. This fact shows that the system is indeed controllable.

Example 2. *Consider a 4-dimensional switched linear impulsive system with $N = 3$ and*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $t_i = i$, $i = 0, 1, 2, 3$.

First, it is easy to see that E_1, E_2 are nonsingular. By simple calculation, we have

$$\begin{aligned}&E_2\exp(A_2)(E_1\langle A_1|B_1\rangle+\mathcal{I}m(F_1))+E_2\langle A_2|B_2\rangle\\&+\mathcal{I}m(F_2)+\langle A_3|B_3\rangle\\&=\text{span}\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\}\end{aligned}$$

By Corollary 2, the system is controllable and reachable on $[0,3]$. In fact, let x_0 be the initial state, x_f be the terminal state, we can take the piecewise constant input

$$u(t)=\begin{cases}c_1, & t\in(0,1);\\ c_2, & t=1;\\ c_3, & t\in(1,2];\\ c_4, & t\in(2,3].\end{cases}$$

Then we have

$$x_f=\exp(A_3)E_2\exp(A_2)E_1\exp(A_1)x_0+H\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}$$

where

$$H=\begin{bmatrix}2e^2(e-1) & 0 & e(e-1) & 0\\ 0 & e^2 & 0 & 0\\ e^2(e-1) & 0 & 0 & 0\\ 0 & 0 & 0 & 1-e\end{bmatrix}$$

It is easy to verify that the matrix H is nonsingular. Thus, we can select appropriate $c_1,\cdots,c_4$ such that the system can be driven from any initial state x_0 to any terminal state x_f. This fact shows that the system is controllable and reachable indeed.

6 Conclusion

This paper has studied the controllability and observability of linear piecewise constant impulsive systems. Necessary and sufficient criteria for reachability and controllability have been established, respectively. Moreover, It has been proved that the reachability is equivalent to the controllability under some mild conditions. Our criteria are of the geometric type, they can be transformed into algebraic type conveniently.

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Appendix A

Proof of Lemma 1. For $t\in(t_0,t_1]$, we have

$$x(t)=e^{A_1(t-t_0)}x(t_0^+)+\int_{t_0}^t e^{A_1(t-s)}B_1u(s)ds \tag{17}$$

For $t=t_1^+$, we have

$$\begin{aligned}&x(t_1^+)=E_1(e^{A_1h_1}x(t_0^+)\\&+\int_{t_0}^{t_1}e^{A_1(t_1-s)}B_1u(s)ds)+F_1u(t_1^-)\end{aligned} \tag{18}$$

Similarly, for $t\in(t_{i-1},t_i]$, $i=2,3,\cdots,k$, we have

$$\begin{aligned}&x(t)=e^{A_i(t-t_{i-1})}x(t_{i-1}^+)\\&+\int_{t_{i-1}}^t e^{A_i(t-s)}B_iu(s)ds\end{aligned} \tag{19}$$

and for $t=t_i^+$, $i=2,3,\cdots,k$, we have

$$\begin{aligned}&x(t_i^+)=E_i(e^{A_ih_i}x(t_{i-1}^+)\\&+\int_{t_{i-1}}^{t_i}e^{A_i(t_i-s)}B_iu(s)ds)+F_iu(t_i^-)\end{aligned} \tag{20}$$

Thus, by (17), (18), (19) and (20), it is easy to verify (3). □

Appendix B

Proof of Lemma 2. By Theorem 7.8.1 in [5], we have

$$\{x|x=\int_{t_0}^{t_f}e^{A(t_f-s)}Bu(s)ds, \textit{for some p.c. } u\}=\langle A|B\rangle \tag{21}$$

Thus, it is easy to see that

$$\begin{aligned}&\{x|x=E\int_{t_0}^{t_f}e^{A(t_f-s)}Bu(s)ds+Fu(t_f),\\&\textit{for some p.c. } u\}\subseteq E\langle A|B\rangle+\mathcal{I}m(F)\end{aligned} \tag{22}$$

Moreover, we have

$$\{x|x=\int_{t_0}^{t_E}e^{A(t_E-s)}Bu(s)ds, \text{for some p.c. } u\}=\langle A|B\rangle$$

where $t_E=(t_0+t_f)/2$. Then, for any $x\in E\langle A|B\rangle+\mathcal{I}m(F)$, there exist a piecewise continuous function $u(t)$, $t\in[t_0,t_E]$, and vector $z\in\mathbb{R}^n$ such that

$$x=E\int_{t_0}^{t_E}\exp[A(t_E-s)]Bu(s)ds+Fz$$

Then we can take

$$v(t)=\begin{cases}u(t), & t\in[t_0,t_E]\\ 0, & t\in(t_E,t_f)\\ z, & t=t_f\end{cases}$$

such that

$$x=E\int_{t_0}^{t_f}\exp[A(t_f-s)]Bv(s)ds+Fv(t_f)$$

This implies that

$$\begin{aligned}&x\in\{x|x=E\int_{t_0}^{t_f}\exp[A(t_f-s)]Bu(s)ds+Fu(t_f),\\&\text{for some p.c. } u\}\end{aligned}$$

It follows that

$$\begin{aligned}&\{x|x=E\int_{t_0}^{t_f}\exp[A(t_f-s)]Bu(s)ds+Fu(t_f),\\&\text{for some p.c. } u\}\supseteq E\langle A|B\rangle+\mathcal{I}m(F)\end{aligned} \tag{23}$$

By (22) and (23), we know that (4) holds. □

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