RELATIVE STRUCTURE AT INFINITY AND NONLINEAR SEMI-IMPLICIT DAE'S

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Abstract

A notion of *relative structure at infinity* and the concept of *relative output rank* with respect to a subsystem are introduced. We introduce the *problem of relative decoupling*, showing that this problem is solvable if and only if the relative output rank $\widetilde{\rho}(z)$ coincides with $\dim z$. The *Relative Dynamic Extension Algorithm (RDEA)* is presented and a geometric interpretation is also given, showing that it computes the relative structure at infinity. We develop necessary and sufficient conditions for testing if a given output z of system is relatively flat with respect to Y. As a byproduct, we obtain conditions for the decoupling problem and for flatness of a class of Differential Algebraic Equations (DAE's). It is shown the the RDEA may be used for constructing dynamic linearizing feedback laws and/or the solution of the dynamic input-output decoupling problem for implicit systems.

1 Introduction

The notion of subsystem was introduced in [13] using the infinite dimensional differential geometric setting of [6], which is also the approach of the present paper. Given a system S with output y, the existence and uniqueness of the *output subsystem* Y is studied there. The concept of relative flatness with respect to a subsystem was also introduced in [13] and it is shown that relative flatness with respect to Y implies flatness of a class of Differential-Algebraic Equations (DAEs) obtained by making $y \equiv 0$. Sufficient conditions of relative flatness based on *Relative Derived Flags* were also presented there. However, the following question remains open. Given a system S with subsystem Y and output z, how to test is z is a relatively flat output with respect to Y? Closely related to this question is the problem of relative decoupling of the output z with respect to subsystem Y, *i. e.*, to control each component of z independently of each other and independently of the output $y(t)$ (that determines the trajectory of the output subsystem). As in the case of flatness, one may ask if relative decoupling implies the decoupling of the corresponding DAE.

In this paper it is shown that a system S for which the relative decoupling of the output z is solvable with respect to susbsystem Y possesses the structure of figure 1, where subsystem Z is flat with flat output z and $S/(Y \cup Z)$ is a generalization of the zero dynamics for this problem[1]. Note that, according to the definitions of [13], z is a relatively flat output with respect to y if and only if Z is flat with flat output z and the "*zero dynamics*" $S/(Y \cup Z)$ is absent.

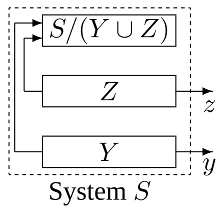

Figure 1: *Structure of of a system S for which the output z is relatively decoupled with respect to subsystem Y. The subsystem Z is flat with flat output z.*

Decoupling of nonlinear DAE's has been considered for instance in [9][2]. Feedback linearization and flatness of implicit systems has been studied for instance by [8, 7, 13]. In this paper we consider a semi-implicit DAE, *i. e.*, a system Γ of the form

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t) \quad (1a)$$
$$y(t) = a(t, x(t)) + b(t, x(t))u(t) = 0 \quad (1b)$$
$$z(t) = \phi(x(t)) + \psi(x(t))u(t) \quad (1c)$$

where $x(t) \in I\!R^n$ is the pseudo-state of the system, $u(t) \in I\!R^m$ is the pseudo-input[3], $z(t) \in I\!R^p$ is the output and $y_i(t), i = 1, \ldots, r$ are the constraints.

One can associate to Γ an explicit system S with outputs y and z given by

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t) \quad (2a)$$
$$y(t) = a(t, x(t)) + b(t, x(t))u(t) \quad (2b)$$
$$z(t) = \phi(x(t)) + \psi(x(t))u(t) \quad (2c)$$

[1]Note $S/(Y \cup Z)$ is only a notation suggesting a quotient, but it does not have any precise meaning.

[2]In [9] the static decoupling for DAEs is considered. To solve this problem, [9] introduces an algorithm that is closely related to the RDEA.

[3]Note that u is not a differentially independent input for Γ, since the constraints $y \equiv 0$ induce differential relations linking the components of u. By the same reasons, x is not a state of Γ.

In this entire paper we consider the system S with output[4] y the system defined by (2), in the framework of [6]. Then $y^{(k)}$ stands for the function $\frac{d^k}{dt^k}y$ defined on S, which may depend $t, x, u^{(0)}, u^{(1)}, \ldots$.

Definition 1 *In the sequel we shall consider the following sequences of codistributions defined on* S

$$\mathcal{Y}_{-1} = \text{span}\,\{dt, dx\} \tag{3a}$$
$$\mathcal{Y}_k = \text{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\}, k \in I\!N \tag{3b}$$
$$Y_{-1} = \text{span}\,\{dt\} \tag{3c}$$
$$Y_k = \text{span}\left\{dt, dy, \ldots, dy^{(k)}\right\},\ k \in I\!N \tag{3d}$$
$$\mathbb{Y}_{-1} = \{0\} \tag{3e}$$
$$\mathbb{Y}_k = \text{span}\left\{dy, \ldots, dy^{(k)}\right\},\ k \in I\!N \tag{3f}$$
$$\mathbb{Z}_{-1} = \{0\} \tag{3g}$$
$$\mathbb{Z}_k = \text{span}\left\{dz, \ldots, dz^{(k)}\right\},\ k \in I\!N \tag{3h}$$

♠

Let $\xi \in S$ be a regular point of the codistributions Y_k and $\mathcal{Y}_k$ for $k = 0, \ldots, n$, where $n = \dim x(t)$. According [4] (see also [1, 12, 16]), the sequence of integers $\{\sigma_0, \ldots, \sigma_n\}$, where $\sigma_k = \dim \mathcal{Y}_k|_\xi - \mathcal{Y}_{k-1}|_\xi$ is called the *algebraic structure at infinity* at ξ. It can be shown that the sequence σ_k is nondecreasing and converges for $k^* \leq n$ to the integer $\rho(y) = \sigma_{k^*} = \max\{\sigma_0, \ldots, \sigma_n\}$. One calls $\rho(y)$ by *output rank* at ξ [5] and k^* by the *convergence index*[5]. The dynamic input-output decoupling problem is solvable if and only if $\rho(y)$ coincides with the dimension of the output y. By the results of [10] (see also [12]), an output $y = h(t, x)$ of S is a flat output if and only if span $\{dt, dx, du\} \subset$ span $\left\{dt, dy, \ldots, dy^{(n)}\right\}$ and $\rho(y) = \dim u$. Furthermore, [11] shows that the last condition is equivalent to $n + \sum_{i=1}^{k^*-1} \sigma_i = mk^*$ and $\sigma_k^* = m$, where $m = \dim u$ and $n = \dim x$.

Instrumental for nonlinear control synthesis is the *dynamic extension algorithm*. It computes the algebraic structure at infinity and it constructs solutions of the dynamic input-output decoupling problem when such problem is solvable [4, 2]. Furthermore, when y is a flat output, this algorithm constructs an endogenous dynamic feedback that is a solution of the feedback linearization problem [3]. In this paper, these results are generalized for a class of DAEs called regular semi-implicit systems.

By the results of [13, 16], one may identify the semi-implicit system Γ given by (1) with the subset of S defined by

$$\Gamma = \{\xi \in S \mid y^{(k)} = 0, k \in I\!N\} \tag{4}$$

Definition 2 *Let* S *be the explicit system defined by* (2) *and consider the codistributions defined by* (3). *A point* $\xi \in S$ *is said to be* r*-regular if*

(i) *The codistributions* $\mathbb{Y}_k$, Y_k *and* $\mathcal{Y}_k$, *defined on* S *by* (3) *are nonsingular around* ξ *for* $k = 0, \ldots, n$.

(ii) *Let* k^* *be the convergence index of output* y *of system* S. *The codistributions* $L_k = Y_{k^*+k} + \mathbb{Z}_k$ *and* $\mathcal{L}_k = \mathcal{Y}_{k^*+k} + \mathbb{Z}_k$ *are nonsingular around* ξ *for* $k = 0, \ldots, n$ *for every point* $\xi \in \Gamma$.

The semi-implicit system given by (1) *is said to be regular if every point* ξ *of* $\Gamma \subset S$, *is* r*-regular, where* $\Gamma \subset S$ *is defined by* (4).

Definition 3 *Let* S *be an explicit system with outputs* y *and* z *given by* (2). *Assume that the codistributions defined by* (3) *are nonsingular around* $\xi \in S$. *Let* $\rho(y)$ *be the output rank of* S. *The sequence of integers* $\{\widetilde{\sigma}_0, \ldots, \widetilde{\sigma}_n\}$, *where* $\widetilde{\sigma}_k = \dim \mathcal{L}_k - \dim \mathcal{L}_{k-1} - \rho(y)$, *computed around* $\xi \in S$ *is called local* relative structure at infinity *(of output z) at* ξ *with respect to the output subsystem* Y. *The integer* $\widetilde{\rho}(z) = \widetilde{\sigma}_n$ *is called* relative *output rank*[6].

In this paper we show that the relative decoupling problem is solvable if and only if $\widetilde{\rho}(z) = \dim z$. It is also shown that the decoupling problem for the DAE given by (1) is solvable under the same conditions (see Thm. 1).

Let $m = \dim u$. We show that z is a relatively flat output of S if and only if card $z = \widetilde{\rho}(z) = m - \rho(y)$ and furthermore span $\{dx, du\} \subset L_n$. These conditions can be deduced directly from the dimensions of the codistributions L_k and $\mathcal{L}_k$. In particular, z is a flat output of the implicit system Γ given by (1) under the same conditions (see Thm. 2).

2 Preliminaries and notation

The field of real numbers will be denoted by $I\!R$. The set of real matrices of n rows and m columns is denoted by $I\!R^{n\times m}$. The matrix (or vector) M^T stands for the transpose of M. The set of natural numbers $0, 1, 2, \ldots$ is denoted by $I\!N$ and the subset $\{1, \ldots, k\} \subset I\!N$ will be denoted by $\lfloor k \rceil$. Our approach will follows the infinite dimensional geometric setting of [17, 6]. We will use the standard notations of differential geometry in the finite and infinite dimensional case. The cardinal of a set Z is denoted by card Z.

For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. In the same vein, if $x^i = (x_1^i, \ldots, x_{p_i}^i)$ for $i = 1, 2, \ldots$, are sets of functions, then $\{dx^1, dx^2, \ldots\}$ stands for the set $\{dx_1^1, \ldots, dx_{p_1}^1, dx_1^2, \ldots, dx_{p_2}^2, \ldots\}$.

[4] We regard y as an output instead of being a constraint.

[5] In [16] it is shown that k^* is the *differential index* of the DAE (1).

[6] Note that a first version of the RDEA has been introduced in [14], and the connection of the RDEA with flatness of and decoupling of DAEs has also been pointed out there. This paper improves these results and introduces the relationship between the RDEA and relative flatness, and relative decoupling.

3 Relative Dynamic Extension Algorithm (RDEA)

The following algorithm is instrumental for studying relative flatness and relative decoupling:

Algorithm 1 (Relative Dynamic Extension Algorithm)
Preparation Process. *Execute k^* steps of the dynamic extension algorithm [4, 12] for the explicit system S with output y,* [7] *obtaining the state representation $(\widetilde{x}_{-1}, \widetilde{u}_{-1})$, where $x_{-1} = x_{k^*-1}$, $\widetilde{u}_{-1} = (\omega_0, \mu_{-1})$, $\omega_0 = \omega = \bar{y}_{k^*}^{(k^*)}$ and $\mu = \widehat{u}_{k^*}$, with state equations given by*

$$\dot{\widetilde{x}}_{-1} = f_{-1}(t, \widetilde{x}_{-1}) + \bar{g}_{-1}(t, \widetilde{x}_{-1})\omega_0 + \tag{5a}$$
$$+\widehat{g}_{-1}(t, \widetilde{x}_{-1})\mu_{-1} \tag{5b}$$
$$z^{(0)} = a_0(t, \widetilde{x}_{-1}) + b_0(t, \widetilde{x}_{-1})\omega_0 + \tag{5c}$$
$$+c_0(t, \widetilde{x}_{-1})\mu_{-1} \tag{5d}$$

Then execute the steps k = 0, 1, 2, . . . :
Step k.
In the step $k-1$ we have constructed a state representation

$$\dot{\widetilde{x}}_{k-1} = f_{k-1}(t, \widetilde{x}_{k-1}) + \bar{g}_{k-1}(t, \widetilde{x}_{k-1})\omega_k + \tag{6a}$$
$$+\widehat{g}_{k-1}(t, \widetilde{x}_{k-1})\mu_{k-1} \tag{6b}$$
$$z^{(k)} = a_k(t, \widetilde{x}_{k-1}) + b_k(t, \widetilde{x}_{k-1})\omega_k + \tag{6c}$$
$$+c_k(t, \widetilde{x}_{k-1})\mu_{k-1} \tag{6d}$$

where $\tilde{x}_{k-1} = (\tilde{x}_{-1}, \omega_0, \omega_1, \ldots, \omega_{k-1}, \bar{z}_0^{(0)}, \ldots, \bar{z}_{k-1}^{(k-1)})$. Let $\widetilde{\sigma}_k$ = rank *$c_k(t, \widetilde{x}_{k-1})$ and assume that this rank is locally constant around some $(t, \widetilde{x}_{k-1})$. Up to a reordering of the components of z, we may assume that the first $\widetilde{\sigma}_k$ rows of $c_k(t, \widetilde{x}_{k-1})$ are locally independent. Then there exist a partition $z = (\bar{z}_k, \hat{z}_k)$, where* $\dim \bar{z}_k = \widetilde{\sigma}_k$ *and a regular static-state feedback with new input (ω_k, v_k) defined by*

$$\mu_{k-1} = \bar{\alpha}_k(t, \widetilde{x}_{k-1}) + \hat{\alpha}_k(t, \widetilde{x}_{k-1})\omega_k + \beta_k(t, \widetilde{x}_{k-1})v_k$$

where $v_k = (\bar{v}_k, \hat{v}_k)$ is such that[8]

$$\begin{aligned} \bar{z}_k^{(k)} &= \bar{v}_k \\ \hat{z}_k^{(k)} &= \hat{z}_k^{(k)}(t, \widetilde{x}_{k-1}, \omega_k, \bar{v}_k) \end{aligned} \tag{7}$$

Add the following dynamic extension

$$\begin{aligned} \dot{\omega}_k &= \omega_{k+1} \\ \dot{\bar{v}}_k &= \bar{\mu}_k \end{aligned} \tag{8}$$

and let $\hat{\mu}_k = \hat{v}_k$. Hence, one has constructed a new state representation $(\widetilde{x}_k, \widetilde{u}_k)$, with $\widetilde{x}_k = (\widetilde{x}_{k-1}, \omega_k, \bar{z}_k^{(k)})$, $\widetilde{u}_k = (\omega_{k+1}, \mu_k)$, $\mu_k = (\bar{z}_k^{(k+1)}, \widehat{\mu}_k)$ and output $z^{(k)}$ given by:

$$\dot{\widetilde{x}}_k = f_k(t, \widetilde{x}_k) + \bar{g}_k(t, \widetilde{x}_k)\omega_{k+1} + \widehat{g}_k(t, \widetilde{x}_k)\mu_k \tag{9a}$$
$$z^{(k)} = \phi_k(t, \widetilde{x}_k) \tag{9b}$$

Compute

$$z^{(k+1)} = a_{k+1}(t, \widetilde{x}_k) + b_{k+1}(t, \widetilde{x}_k)\omega_{k+1} + c_{k+1}(t, \widetilde{x}_k)\mu_k \tag{9c}$$

The following result summarizes the main geometric properties of the *Relative Dynamic Extension Algorithm* for time-varying nonlinear systems. We stress that the list of integers $\{\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n\}$, where $n = \dim x$, is strongly related to the *algebraic structure at infinity* (see [4]) and is called *relative structure at infinity*. The integer $\widetilde{\rho}(z) = \widetilde{\sigma}_n$ is called *relative output rank* at a point $\xi \in S$.

Lemma 1 *Let S be the system given by* (1a) *with classical state representation (x, u) and classical output y. Let $V_k \subset S$ be the open and dense set of regular points of the codistributions $Y_i, \mathcal{Y}_i$, for $i = 0, \ldots, n$ and of $\mathcal{L}_j, L_j$ for $j \in \{0, \ldots, k\}$, where $L_k = Y_{k^*+k} + \mathbb{Z}_k$ and $\mathcal{L}_k = \mathcal{Y}_{k^*+k} + \mathbb{Z}_k$ for $k = -1, 0, 1, 2, \ldots$ (see* (3)*). Assume that the output rank of the explicit system S is given by $\rho(y)$.*

In the kth step of the relative dynamic extension algorithm, one may construct around $\xi \in V_k$, a new local classical state representation $(\tilde{x}_k, \tilde{u}_k)$ of the system S with state $\tilde{x}_k = (\tilde{x}_{-1}, \omega, \ldots, \omega^{(k)}, \bar{z}_1^{(1)}, \ldots, \bar{z}_k^{(k)})$, input $\tilde{u}_k = (\omega^{(k+1)}, \mu_k)$, where $\mu_k = (\dot{\bar{z}}_k^{(k+1)}, \hat{\mu}_k)$, and output $z^{(k)} = \phi_k(t, \tilde{x}_k)$ defined in an open neighborhood U_k of ξ, such that

1. span $\{d\tilde{x}_k\} = \mathcal{L}_k, k = -1, 0, 1, 2, \ldots$.
2. span $\{d\tilde{x}_k, d\tilde{u}_k\} = \mathcal{L}_{k+1} +$ span $\{du\}, k = -1, 0, 1, 2, \ldots$.
3. *It is always possible to choose $\bar{z}_{k+1}^{(k+1)}$ in a way that $\dot{\bar{z}}_k^{(k)} \subset \bar{z}_{k+1}^{(k+1)}$*
4. *When $\dot{\bar{z}}_k^{(k)} \subset \bar{z}_{k+1}^{(k+1)}$, it is always possible to choose $\hat{\mu}_{k+1} \subset \hat{\mu}_k$.*
5. *Let $\xi \in V_n$. The sequence $\widetilde{\sigma}_k = \dim(\mathcal{L}_k|_\xi)) - \dim(\mathcal{L}_{k-1}|_\xi) - \rho(y)$ is nondecreasing, the sequence $\widetilde{\rho}_k = \dim(L_k|_\xi) - \dim(L_{k-1}|_\xi) - \rho(y)$ is nonincreasing, and both sequences converge to the same integer $\widetilde{\rho}(z)$, called the* relative output rank at ξ, *for some $\widetilde{k}^* \leq n = \dim x$.*
6. *Let $S_k \subset V_n$ be the open neighborhood of a given $\xi \in V_n$, such that the dimensions of $\mathcal{L}_j, L_j$ $j \in \{0, \ldots, k\}$ are constant inside S_k. We have $S_k = S_{\widetilde{k}^*}$ for $k \geq \widetilde{k}^*$.*
7. *Furthermore, $L_k \cap$ span $\{dx\}|_\nu = L_{\widetilde{k}^*-1} \cap$ span $\{dx\}\Big|_\nu$ for every $\nu \in S_{\widetilde{k}^*}$ and $k \geq \widetilde{k}^*$.*
8. *For $k \geq \widetilde{k}^*$, one may choose $\bar{z}_k = \bar{z}_{\widetilde{k}^*}$ in $U_{\widetilde{k}^*}$. Furthermore, $L_{k+1} = L_k +$ span $\left\{\omega^{(k+1)}, \bar{z}_k^{(k+1)}\right\}$ for $k \geq \widetilde{k}^*$.*
9. *Let $\mathcal{Y}$ =* span $\left\{dt, dy^{(k)} | k \in I\!N\right\}$. *Then $\widetilde{\sigma}_k = \dim \frac{\mathcal{L}_k + \mathcal{Y}}{\mathcal{L}_{k-1} + \mathcal{Y}}$. In particular we have $\widetilde{\rho}(z) = \dim \frac{\mathcal{L}_n + \mathcal{Y}}{\mathcal{L}_{n-1} + \mathcal{Y}} = \dim \frac{\mathcal{L}_{\widetilde{k}^*} + \mathcal{Y}}{\mathcal{L}_{\widetilde{k}^*-1} + \mathcal{Y}}$.*

[7] In the last step one does not make the dynamic extension but only the static state feedback.

[8] We stress that $\beta_k(t, \widetilde{x}_{k-1})$ is locally nonsingular.

4 Relative decoupling

4.1 System decompositions

Definition 4 *[13] (Output Subsystem) Given a system S with output y, a (local) output subsystem is a subsystem*[9] *Y with corresponding submersion $\pi : U \subset S \to Y$ such that $\pi^* T^* Y = \text{span}\,\{dt, (dy^{(k)} : k \in I\!\!N)\}$.*

In [13, Theo. 4.3], given a classic state representation (x, u) and a classic output y of S, then the nonsingularity[10] of codistributions (3a) and (3c) for $k = 0, \ldots, n$, where $n = \dim x$, assures the existence and uniqueness[11] of a local output subsystem Y.

One may state the following notion of input-output subsystem.

Definition 5 *Given a system S with (local) state representation (x, u) and output y. Consider the output $w = (y, u)$. The* input-output subsystem *is the output subsystem*[12] *W corresponding to the output w.*

The next definition generalizes this concept a notion of decomposition of [13]

Definition 6 *(i-decomposition and decomposition of systems) Let S be a system and let $\mathcal{F} = \{S_i, i \in \lfloor h \rceil\}$ be a family of subsystems with local coordinates respectively $(t, x_i), i \in \lfloor h \rceil$. The system S is said to be (locally) incompletetely decomposed, or simply,* i-decomposed *by $\mathcal{F}$ if there exist a (local) coordinate system $(t, x_1, \ldots, x_h, x_{h+1})$ defined in $U \subset S$ and a family of Lie-Bäcklund submersions $\{\pi_i : U \to S_i, i \in \lfloor h \rceil\}$ such that the local expression of π_i in these coordinates is given by $\pi_i(t, x_1, \ldots, x_h, x_{h+1}) = (t, x_i), i = 1, \ldots, h$. The system S is (locally)* decomposed *by $\mathcal{F}$ if it is i-decomposed by $\mathcal{F}$ and $x_{h+1} = \emptyset$.*

4.2 Relative-decoupling with respect to a subsystem

Definition 7 *(Relative decoupling problem) Let S be a system with output (y, z), where $z = (z_1, \ldots, z_p)$. Let Y be the (local) output subsystem corresponding to the output y. A (local) state representation (x, u), where $u = (u_1, \ldots, u_m)$ of S is said to be (locally)* relatively decoupled *with respect to Y if system S is (locally) i-decomposed by the family $\mathcal{F} = \{Y, S_1, \ldots, S_p\}$ where S_i is the input-output subsystem corresponding to input u_i and output $z_i, i = 1, \ldots, p$ and the output rank $\rho(z_i)$ of subsystem S_i is one.*

Theorem 1 *Given a system S with (local) state representation (x, u) and output (y, z),* the Relative Decoupling Problem (RDP) *is the question of finding an endogenous feedback,* i. e., *a new state representation $(\bar{x}, \bar{u})$, in a way that the output z is relatively decoupled with respect to output subsystem Y. Then the RDP is solvable around a r-regular point $\xi \in S$ (see Def. 2) if and only if the relative output rank $\widetilde{\rho}(z)$ is equal to the number of components of z.*

5 Relative flatness and relatively flat outputs

Definition 8 *[13] A system S is said to be (locally) relatively-flat with respect to a subsystem S_a if there exists a flat subsystem Z such that S is (locally) decomposed by the family $\mathcal{F} = \{S_a, Z\}$. A flat output z of Z is said to be a relatively-flat output of S (with respect to S_a).*

Theorem 2 *Let S be a system* (2) *with state representation (x, u) and output (y, z) and assume that $\xi \in S$ is a r-regular point of S (see definition 2). Assume that the system S is well formed,* i. e., span $\{dt, dx, du\}$ = span $\{dt, dx, d\dot{x}\}$[13]. *Let Y be the local output subsystem corresponding to the output y*[14]. *The following affirmations are equivalent:*

(i) *The system S is relatively flat around ξ with respect to subsystem Y and the output z is a (local) relatively flat output around ξ.*

(ii) *Around ξ we have $\widetilde{\rho}(z) = \text{card}\, z$ and $\text{span}\,\{dx\} \subset L_{\widetilde{k}^*-1}$.*

(iii) *Around ξ we have $\widetilde{\rho}(z) = \text{card}\, z$ and $n - \dim Y_{k^*-1} + \sum_{i=0}^{k^*-1} \sigma_i - \widetilde{\rho}(z)(\widetilde{k}^* + 1) + \sum_{j=0}^{\widetilde{k}^*} \widetilde{\sigma}_j = 0$.*

6 DAEs

6.1 Relative decoupling and DAEs

Let S be a system defined by (2) with state representation (x, u) and output (y, z). Let $(\widehat{x}, \widehat{u}) = ((x_a, x_b), (u_a, u_b)$ be a state representation adapted to the output subsystem Y with $(\widehat{x}, \widehat{u})$ related to (x, u) by relative static-state feedback (see [13]). Assume that the system (1) defines a regular DAE.

Then the following theorem holds (see [14] for a similar result)

Theorem 3 *Let $\iota : \Gamma \to S$ be the Lie-Bäcklund immersion of proposition [13, Prop 6.1] (see also [16]). Let $\widetilde{z} = z \circ \iota$, $\widetilde{x}_b = x_b \circ \iota$ and $\widetilde{u}_b = u_b \circ \iota$. Then $(\widetilde{x}_b, \widetilde{u}_b)$ is a classic local state representation for Γ, and $\widetilde{z}$ is a classic output. Furthermore, the relative-structure at infinity of the output z of S with respect to output subsystem Y coincides with the structure at infinity of Γ with output z considering the state representation $(\widetilde{x}_b, \widetilde{u}_b)$. In particular the relative output rank $\widetilde{\rho}(z)$ of system* (2) *coincides with the output rank $\rho(z)$ of the DAE* (1).

[9] See section for the [13] for the definition of subsytem.

[10] In particular the existence of Y is generically assured.

[11] The uniqueness is implied by the existence.

[12] By [13, Theo. 4.3], this subsystem exists (generically) and is unique.

[13] This is equivalent to say that $g(x)$ of (2) has independent columns [18]. Note that, if the system is not well formed one my apply the theorem 2 to system S with state (x, u) and input $\dot{u}$, which is well formed.

[14] The output subsystem Y exists and is unique according [13, Thorem 4.3].

6.2 Relative flatness and DAEs

In [13, Prop. 7.1 and Theo. 7.2] it is shown that relative flatness implies flatness of the corresponding DAE. The following result is a direct consequence of those results and of theorem 2.

Corollary 1 *If one of the equivalent conditions of theorem 2 holds for system* (2)*, then z is a local flat output of the regular DAE* (1)*.*

7 Conclusions

The results of this paper may be useful for studying flatness and the dynamic decoupling problem of implicit systems. It is important to point out that our results shows effective ways for computing the output rank and control laws for dynamic feedback linearization and/or decoupling of an implicit system Γ, without the need of transforming Γ into an explicit system. In fact, note that the relative dynamic extension algorithm for affine systems relies only on sums, multiplications and matrix inversions. Given an output z, one may *test* if z is a flat output for the implicit system using the results of this paper. Nevertheless, it must be stressed out that a method for constructing flat outputs is not presented here. An extended version of this paper is available in [15] and it will be submitted to the EJC.

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