THE SURVEY FOR THE EXACT AND OPTIMAL STATE OBSERVERS IN HILBERT SPACES

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Abstract

The problems of design of state observers for exact (non-asymptotic) reconstruction of the state with finite observation window are surveyed. The general optimal exact observer theory in function Hilbert spaces is recalled. The new discrete exact observer is presented. The importance of consideration of both output and input disturbances for optimal observation is pointed.

1 Introduction

In the classic control theory and its applications a common practice in the estimation of inaccessible for measurement state vector of a linear system was the use of Luenberger type observers. D.G.Luenberger in his PhD dissertation (1963) and next in [1] proposed the pole placement technique for calculation of the observer gain matrix. The structure of such observer was derived directly from the differential form of Kalman Filter (1959) [2], [3]. The design technique for optimal gain matrices in KF was based on stochastic properties of disturbances and least-squares approach. In both types of observers their structures were given by an ordinary linear differential equations. Hence under assumption that real initial state is unknown the solution of estimation problem could give only an estimate which tends to real state asymptotically. Current measurement sample of input and output vectors enables fast recursive calculation of current state estimate.

The power of modern computers makes application of the other on-line observation algorithms possible. They reconstruct the value of the current state vector **exactly** but under condition that the calculations are based on finite time history of measurement samples of input and output.

In 1996 Gilchrist [4] proposed an exact state observer based on so called n-observability condition. The exact state $x \in R^n$ was calculated from a knowledge of the system's output at **n** ***discrete*** measurement points which form discrete history output window and from the system's ***continuous*** input measurements which form continuous input history window. The structure of the observer was derived directly from system output equation.

In 1985 Wonham [11] discussed an integral observer which originated directly from definition of observability and used continuous measurements of system input and output. This observer has least-squares properties of state reconstruction error but if disturbances only in output measurements occur.

In the 1992 Medvedev and Toivonen [5] proposed another version of the above continuous state observer with the use of discrete measurement of the output and start to call such type of the observer 'finite memory deadbeat observer' and used them in different application [6]. The structure of such discrete output deadbeat observer guarantees the minimum of reconstruction error (in LS sense) but also if disturbances only in output measurements occur.

Some other least squares version of 'finite memory deadbeat observer was presented in [7].

The exactness in reconstruction of the state is valid only under assumption of perfect input-output measurements i.e. no input-output noise or disturbances occur. In the practical case of noisy measurements the use of above-mentioned observers gives also reconstruction error. In these versions of observers it was assumed that the input function (control) is perfectly known and for calculation there is no need for its measurement. The last assumption however in practical application is not always proper because the input signal produced by actuator differs from computer control signal.

All the above versions of exact state observers are only special subclasses, which may by derived from general exact and optimal observer theory. This theory was formulated and presented by Byrski and Fuksa in 1984. In [8], [9] the problems of general structure and optimality of the exact continuous state observers were solved. This theory also originates from definition of observability and uses functional analysis technique similar to presented in [10]. The authors proposed a deterministic approach to disturbances and to exact and optimal state observation for which the relations were formulated generally in function Hilbert spaces U, Y. Such type of the observer must have the structure of two linear continuous functionals. It is because based on two continuous pieces of functions u and y given on finite time interval, the observer should provide the real unknown number (vector) $x \in R^n$. On the other hand by the Riesz Representation Theorem every linear continuous functional in Hilbert space can be expressed as inner product. Hence the structure of the observer has to be given by two inner products: one product of continuous output function $y \in Y$ and special observation (filtering) function $G_1(\tau) \in Y$ and the second one with the input function $u \in U$ and special observation (filtering) function $G_2(\tau) \in U$ on interval [0,T]. After the first observation interval [0,T] the observer reconstructs the exact value of x(t) for $\forall t \geq T$. Choosing different input-output Hilbert spaces one can obtain different formula for finite state exact observer. As the second problem - the optimal structure of the above observer was derived. This optimal structure was based on optimal shape of functions $G_1(.)$, $G_2(.)$, which should be chosen in such a way that they fulfill observability requirements and minimize the norm of the observer in chosen Hilbert spaces Y and U.

In this norm the weight factor α is connected with the norm of G_1 and represents the norm of output disturbances and the weight factor β is connected with the norm of G_2 and represents the norm of input disturbances. For different α, β different optimal observers are derived. The optimal observer with minimal norm guarantees minimal state reconstruction error for the worst case (from unit balls) disturbances of y and u measurements.

The very special case (if absence of input measurement noises is assumed) is represented by the weight factor β equal to zero, $\beta=0$. For this case the optimal integral observer gives continuous version of observer as in [11] or as in discrete *deadbaet* observer.

As it was pointed before in different approaches to optimal observation problem many authors assume that system input function (control) is known and it dos not have to be measured. Hence one can calculate that part of the system output, which depends on these inputs and not from initial conditions, and subtract it from observed data. This approach especially in industrial applications is not always proper. Theoretically the control signal is known but it represents only the information, which is sending to actuator and control valve (for instance the number of impulses). What is the real e.g. flow of heating steam, which is the real process input signal, one can check only by measurement. This measurement can be also affected by disturbances. Hence measurement errors concern both output and input signals. This fact motivates the need of more general statement of optimization problem for observer structure than e.g. in [11]. The optimal observer should guaranteed exact state observation under perfect input-output measurements and the minimal reconstruction error under noisy input-output measurements.

In [8] and [9] it was assumed that the disturbances will have bounded norm (belong to the unit balls). If the spaces Y and U are chosen as L^2 [0,T] the inner products are represented by an integral operator. Hence in all authors publications the name 'integral observers' was used to underline its contrary type to differential structure of Kalman Filter or Luenberger observer. The extended results of the on-line exact observation and application were presented in [12]–[14]. In paper [12], the integral observers with Expanding and Moving Observation Window (sliding window) and their differential versions were given. In paper [13] a generalization to disturbances from an ellipsoid was derived and in [14] the application to stabilization system of distillation column was presented. The problem of current and fast calculation of integrals does not belong to observation theory and is just numerical problem.

The most important properties of the integral state observers are:

- Continuous measurements of input and output of the system,
- admission of noisy measurements in both u(t) and y(t) signals,
- integral description of the on-line observer (in $L^2[0,T]$),
- the optimal formulas of the observer are obtained by minimization of its norm and depend on input-output disturbance norm,
- independence on state initial conditions,
- fixed finite continuous observation time interval [0,T].

The most important properties of deadbeat observers are:

- Continuous function of input and discrete measurement of output of the system,
- the input is not measure or measurement is perfect,
- the description of the observer is by the use of sum,
- the observer is not optimal in general case,
- independence on state initial conditions,
- fixed finite number of output measurements.

It will be easy to see that 'finite memory deadbeat observer' is special case of general integral observer with delta distribution as the function of G_1 and G_2 which optimal properties result from the assumption of only output noisy measurements.

In this paper also the version of discrete exact and optimal observer is presented, which optimal properties result from the assumption of input and output noisy measurements [16].

2 Exact observation of the state

2.1 The integral observer for x_0 of ODE

Consider the LTI observable system given by ODE

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^r$ and $y(t) \in R^m$ for $\forall t \geq 0$. Matrices A, B, C are of compatible dimensions, m<n and the initial state x(0) is unknown. Assume that we measure the control u and the output y on the interval [0,T] where T is the fixed observation time. Our purpose is to determine the state x(0).

The output of system for $t \in [0,T]$ has the form

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds \tag{2}$$

We will derive exemplary integral observer, which originates directly from observability definition as in [11]. For x(0) calculation one should multiply both side of the equation (2) by transposition $[Ce^{At}]'$ and next integrate equation over interval [0,T]. If system is observable the real Gram matrix M is nonsingular for any T.

$$M = \int_0^T e^{A't}C'Ce^{At}\,dt$$

Then one of possible formula for x(0) reconstruction is

$$x(0) = \int_0^T M^{-1}e^{A't}C'\,y(t)\,dt - \int_0^T\left[\int_s^T M^{-1}\,e^{A't}C'Ce^{A(t-s)}dt\right]B\,u(s)ds,$$

or

$$x(0) = \int_0^T \overline{G}_1(t)\,y(t)\,dt + \int_0^T \overline{G}_2(s)\,u(s)ds \tag{3}$$

The exact state is reconstructed by the integral observer, which is given by two inner products in $L^2[0,T]$ function spaces with matrices $\overline{G}_1$ and $\overline{G}_2$.

2.2 The integral observer for x(T) of ODE

Consider the LTI observable system as in (1). Assume that we measure continuously the control u and the output y on the interval [0,T] where T is the fixed observation time. Our purpose is to determine the state x(T) which is more convenient for on-line control than x(0). The output of system (1) for $t\in[0,T]$ has the form

$$y(t) = Ce^{-A(T-t)}x(T) - C\int_t^T e^{A(t-s)}Bu(s)ds \qquad (4)$$

We will derive the same type of exemplary integral observer. For x(T) calculation one should multiply both side of the equation (4) by $e^{-A'(T-t)}C'$ and integrate it on [0, T].

For observable system the real Gram matrix M is nonsingular for any T.

$$M = \int_0^T e^{-A'(T-t)}C'Ce^{-A(T-t)}\,dt$$

Then one of possible formula for x(T) reconstruction is

$$M\,x(T) = \int_0^T e^{-A'(T-t)}C'\,y(t)\,dt + \int_0^T\left[\int_0^S e^{-A'(T-t)}C'Ce^{A(t-s)}dt\right]B\,u(s)ds$$

or
$$x(T) = \int_0^T G_1(t)\,y(t)\,dt + \int_0^T G_2(s)\,u(s)ds \qquad (5)$$

The exact state x(T) is reconstructed by the integral observer, which is given by two inner products in $L^2[0,T]$ function spaces with matrices G_1 and G_2.

2.3 Finite memory deadbeat observer

Assume that we measure the output y of (1) in interval [0,T] at finite number of discrete moment of time $t_i = T - h_i$, i=0,...,k. The control u is measured continuously in [0,T]. Our purpose is to determine the state x(T). If system is observable then there exist such $k\geq n$ that real Gram matrix M_k is nonsingular

$$M_k = \sum_{i=0}^{k} e^{-A'h_i}C'Ce^{-Ah_i}$$

$$x(T) = M_k^{-1}\sum_{i=0}^{k} e^{-A'h_i}C'y(T-h_i) + \\ + M_k^{-1}\sum_{i=0}^{k} e^{-A'h_i}C'C\int_{T-h_i}^{T} e^{A(T-h_i-s)}Bu(s)ds \qquad (6)$$

After substitution $T - h_i = t_i = i\Delta$, $T = k\Delta$ and integer part of division $s/\Delta = \mathrm{Fix}(s/\Delta)$ the final form is

$$x(T) = M_k^{-1}\sum_{i=0}^{k}\left[Ce^{-A(k-i)\Delta}\right]^T y(i\Delta) + \\ + M_k^{-1}\int_0^T e^{-A^T k\Delta}\sum_{i=0}^{\mathrm{Fix}\left(\frac{s}{\Delta}\right)}\left[e^{A^T i\Delta}C^T Ce^{Ai\Delta}\right]e^{-As}Bu(s)ds$$

The similarity of final formula for x(T) to results of Section 2.2 is quite visible.

2.4 The observation problem in heat process

Consider the LTI system given by PDE. One-dimensional parabolic heat equation is assumed for description of heat transfer by conduction in a homogeneous rod or thin wire of L length under assumption that the surface of the rod is insulated. By italic letter *T*(t,z) the temperature at the time t in the point z is denoted,

$$k^2 \cdot \frac{\partial^2 T(t,z)}{\partial z^2} = \frac{\partial T(t,z)}{\partial t}, \qquad (7)$$

Constant k^2 is called thermal diffusivity and z-denotes a spatial dimension. The initial condition *T*(0,z) is unknown

$T(0,z)=\varphi(z)$, for $0< z <L$,

and Dirichlet boundary conditions are given as a control

$\boldsymbol{T}(t, 0) = \Psi_1(t)$; $:\boldsymbol{T}(t, L) = 0$.

Let us assume that initial condition is a finite sum of n sinusoids with unknown amplitudes x_i. The reconstruction of infinite dimensional initial state was transformed to reconstruction of finite dimensional vector of unknown parameters $x\in R^n$.

$$T(0,z) = \varphi(z) = \sum_{j=1}^{n} x_j \cdot \sin(\frac{j\cdot\pi}{L}z) \qquad (8)$$

For temperature measurement in some point the ideal sensor located in point α, 0<α<L is used. It gives the equation for the system output: $y(t,\alpha) = \boldsymbol{T}(t,\alpha)$. For the Dirichlet boundary conditions: $u(t)=\boldsymbol{T}(t,0) = \Psi_1(t)$ $\boldsymbol{T}(t,L)=0$, for t>0 (the right hand side end is held at temperature zero) the solution of (7) is given by

$$T(t,z) = \sum_{i=1}^{\infty} e^{-(\frac{i\pi k}{L})^2 t}\cdot T_i(0)\cdot\sin(\frac{i\cdot\pi}{L}z) + \\ + \sum_{i=1}^{\infty}\frac{2i\pi k^2}{L^2}\int_0^t e^{-(\frac{i\pi k}{L})^2(t-\tau)}\psi_1(\tau)\,d\tau\cdot\sin(\frac{i\cdot\pi}{L}z) \qquad (9)$$

This equation can be written briefly in the form:

$$T(t,z) = T_0(t,z) + T_u(t,z) \qquad (10)$$

where $T_0(t,z)$ depends on initial condition and $T_u(t,z)$ on the boundary control u. Substituting initial condition (8) to $T_0(t,z)$ only the sum of finite number of elements occurs,

$$T_0(t,z) = \sum_{j=1}^{N} x_j\cdot e^{-(\frac{j\cdot\pi k}{L})^2 t}\cdot\sin(\frac{j\cdot\pi}{L}z)$$

$$T_0(t,\alpha)=\sum_{j=1}^{N} h_1^j x_j = \left[h_1^1, \quad h_1^2, \quad \dots \quad h_1^N\right]\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

where: $h_1^j = e^{-(\frac{j\cdot\pi k}{L})^2 t}\sin(\frac{j\cdot\pi}{L}\alpha)$.

Formula $T_u(t,z)$ will be a sum of infinite series

$$T_u(t,\alpha)=\int_0^t\left[\frac{2\pi k^2}{L^2}\sum_{i=1}^{\infty} i e^{-(\frac{i\pi k}{L})^2(t-\tau)}\sin(\frac{i\cdot\pi}{L}\alpha)\right]\psi_1(\tau)d\tau$$

The solution to optimal observer of x is given in [15].

It is characteristic that in all the examples of Sections 2.1-2.4 that general description of linear system with unknown parameter x and known output y and control u has a form of sum of two operators

$$y = H_1 x + H_2 u .$$

The exact observer has a form of two inner products

$$x = \langle G_1 | y \rangle_Y + \langle G_2 | u \rangle_U$$

In the next sections we will briefly recall the main formulas describing general, integral optimal observers theory in Hilbert spaces [8], [9].

3 Statement of the general observer problem

Consider a linear time invariant LTI system whose output is given in a general operator form as the sum

$$y = H_1 x + H_2 u \tag{11}$$

where the output y and control u belong to Hilbert function spaces Y and U, respectively and the unknown parameter $x \in X = \mathbf{R^n}$. The maps are linear and continuous and the map H_1 is defined as:

$\mathbf{H_1 : X \rightarrow Y}$, $\quad H_1 x = \sum_{i=1}^{n} h_1^i x_i$; $\quad \mathbf{H_2 : U \rightarrow Y}$,

The measurements of function y and u on interval [0,T] are given. The observer for system (11) should reconstruct $x \in X = \mathbf{R^n}$ hence, in general it should be determined by n-dimensional linear continuous functionals on Y and U. By the Riesz Theorem every linear continuous functional in Hilbert spaces can be expressed as inner product. Hence the observer is assumed to have the general form

$$x = \langle G_1 | y \rangle_Y + \langle G_2 | u \rangle_U \tag{12}$$

where the maps $\mathbf{G_1} : Y \rightarrow X$, $\mathbf{G_2} : U \rightarrow X$ are linear, continuous and the operator $H_1 \in Y^n$:

$$G_1 = \begin{bmatrix} g_1^1 \\ \vdots \\ g_1^n \end{bmatrix} \in Y^n, \quad G_2 = \begin{bmatrix} g_2^1 \\ \vdots \\ g_2^n \end{bmatrix} \in U^n, \quad H_1 = \begin{bmatrix} h_1^1 \\ \vdots \\ h_1^n \end{bmatrix} \in Y^n$$

In order to obtain the necessary and sufficient conditions for the relation (12) to be an observer for system (11) we substitute (11) to (12) and use the adjoint operator to H_2

$$x = \langle G_1 | H_1 x \rangle_Y + \langle H_2^* g_1 | u \rangle_U + \langle G_2 | u \rangle_U$$

Hence for equality of left and right hand side we have the following conditions for G_1, G_2 :

- observability condition - $\ker H_1 = 0$,
- identity matrix constrain for G_1 –

$$\langle G_1 | H_1 \rangle = \mathbf{I_{n\times n}} , \tag{13}$$

- the formula for G_2: $\quad G_2' = -H_2^* G_1'; \quad .$ (14)

Condition (13) takes the form of a matrix inner product in Y^n and represents the constraint for G_1 .

There is an infinite number of exact observer pairs (G_1, G_2) which fulfill formula (12), (14) and constraint (13).

4 The general form of optimal observer

Assume we have an LTI observable system (11) and the observer (12) of the unknown parameter x. In the space S of all observer pairs (G_1, G_2), $S = Y^n \times U^n$ we define the norm of the observer

$$\|(G_1, G_2)\|_S^2 \overset{df}{=} \sum_{i=1}^{n} \alpha_i \langle g_1^i, g_1^i \rangle + \sum_{i=1}^{n} \beta_i \langle g_2^i, g_2^i \rangle = J , \tag{15}$$

which represents also the performance index of observation. The norm of the observer contains two parts connected with output and input measurements. Hence the value of the observer norm gives upper estimation of observation error under assumption that measurement noises occur, are bounded, are normalized to unit ball $\|z_1\| \le 1, \ \|z_2\| \le 1$ and are most dangerous.

The error estimation for $\alpha=\beta=1$ has the form:

$$\max_{(z_1,z_2)} \|\varepsilon\|_{R^n}^2 = \max_{(z_1,z_2)} \|\langle G_1 | z_1 \rangle + \langle G_2 | z_2 \rangle\|^2 \le 2\|(G_1, G_2)\|_{Y\times U}^2$$

Hence the norm of the observer can represent performance index of observation which should be minimized.

The constraint (3) and the performance index J give the Lagrangian functional where vectors $\lambda_i \in R^n$ are Lagrange multipliers and e'_i are transpositions of the basis vectors in R^n.

$$L = J + 2\sum_{i=1}^{n}\left(e_i' - [\langle g_1^i, h_1^1 \rangle, \dots, \langle g_1^i, h_1^n \rangle]\right)\lambda_i$$

Hence conditions of optimality for the operators G_1 and G_2 is given by

$$\alpha_i g_1^i + \beta_i H_2 H_2^* g_1^i - \sum_{j=1}^{n} h_1^j \lambda_i^j = 0$$

The final compact solution for $\alpha=\beta=1$ is given by formula

$$G_1^{T\,o} = \mathbf{F}^{-1} H_1^T \langle H_1 \,|\, \mathbf{F}^{-1} H_1^T \rangle^{-1} \tag{16}$$

$$G_2^{T\,o} = -\mathbf{H}_2^* G_1^{T\,o}. \quad \text{where} \quad , \mathbf{F} = \left(I + \mathbf{H}_2 \mathbf{H}_2^*\right).$$

Although the reconstructed state is the vector of finite dimension the general final formulae can be applied also to systems with delays and distributed parameters. The application of this theory to the systems described by ordinary differential equations ODE will be now shown.

5 The optimal integral observer applied to x(T) for ODE in space $L^2[0,T]$

Let a linear system be given

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) \tag{17}$$

where $x(t) \in R^n$, $u(t) \in R^r$ and $y(t) \in R^m$ for $\forall t \geq 0$, m<n. Matrices A, B, C are of compatible dimensions.

Assume that we measure the control u and the output y on the interval [0,T] where T is the fixed observation time. Our purpose is to determine the state x(T). We assume:
state space $X = R^n$, output space $Y=(L^2(0,T))^m$, control space $U = (L^2(0,T))^r$. The output of system (17) has the form (4).

The optimal observer equation is

$$x(T) = \int_0^T G_1^o(T,\tau)y(\tau)d\tau + \int_0^T G_2^o(T,\tau)u(\tau)d\tau \tag{18}$$

where the dimensions of function matrices $G_1^o(T,\tau)$, $G_2^o(T,\tau)$ are (n × m) and (n × r), respectively and they rows represent transpositions of elements g_1^i, g_2^i. The optimal matrices $G_{1,2}$ are functions of two parameters - the fixed observation time T and the time $\tau \in [0,T]$. In the sequel we will omit the first argument in writing. The constraint (13) has a form

$$\int_0^T G_1(\tau)Ce^{-A(T-\tau)}d\tau = I \tag{19}$$

where I is an n× n identity matrix, and relation (14) is

$$G_2(\tau) = \int_0^\tau G_1(s)Ce^{-A(\tau-s)}B\,ds \cdot \tag{20}$$

The squared norm of the observer is:

$$J = \int_0^T \left[\sum_{i=1}^n \alpha_i \sum_{j=1}^m (g_1^{ij}(\tau))^2 + \sum_{i=1}^n \beta_i \sum_{j=1}^r (g_1^{ij}(\tau))^2 \right] d\tau \tag{21}$$

Optimal matrices are [8]

$$G_1^o(t) = P_1(t)C', \qquad G_2^o(t) = P_2(t)B \tag{22}$$

where i-th columns of $P_1(t)$ and $P_2(t)$ are given by

$$p_1^i(t) = \Phi_{11}^i(t)\left[M_i'\right]^{-1}e_i \qquad p_2^i(t) = \Phi_{21}^i(t)\left[M_i'\right]^{-1}e_i \tag{23}$$

and fundamental i-th matrices for i=1,...n,

$$\Phi^i(t) = e^{W_i t} = \begin{bmatrix} \Phi_{11}^i(t) & \Phi_{12}^i(t) \\ \Phi_{21}^i(t) & \Phi_{22}^i(t) \end{bmatrix} \tag{24}$$

and Hamiltonian Matrix

$$W^i = \begin{bmatrix} A & \gamma_i BB^T \\ C^T C & -A^T \end{bmatrix} \tag{25}$$

where $\gamma_i = \beta_i/\alpha_i$ and Gram matrices M_i

$$M_i' = \int_0^T e^{-A'(T-\tau)}C'C\Phi_{11}^i(\tau)d\tau$$

For weight factors **α=1**, **β=0** from the above formula one can obtain the same observer derived by another way in Section 2.2. As it was said before its discrete version (deadbeat observer (6)) one can obtain by the use of suitable delta distribution as G_1 and G_2 functions and **α=1**, **β=0**. The solution of the optimization task for **α=β=1** has simpler form

$$G_1^o(T,\tau) = M^{-1}\Phi_{11}'(\tau)C' \qquad G_2^o(T,\tau) = M^{-1}\Phi_{21}'(\tau)B. \tag{26}$$

The real Gram matrix M^{-1} is of the form:

$$M^{-1} = e^{AT}\left[\int_0^T \Phi_{11}'(\tau)C'Ce^{A\tau}d\tau\right], \tag{27}$$

and two elements of the fundamental matrix $\Phi(\tau)$ are obtained from:

$$W = \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix}, \quad \Phi(t) = e^{W\tau} = \begin{bmatrix} \Phi_{11}(\tau) & \Phi_{12}(\tau) \\ \Phi_{21}(\tau) & \Phi_{22}(\tau) \end{bmatrix}$$

Matrices (22) for the given observation time T can be calculated off-line in interval [0,T] and then applied on-line in optimal filtering moving window [12]. The norm (21) of the observer is the function of observation time T.

6 The importance of weight factors α, β

The importance of consideration of weight factors α, β in (15), (21) and their influence to optimal solution is visible even in very simple example of first order system.
In this case the optimal state observer can be interpreted as integral state filter for optimal noise filtration.

$$\dot{x}(t) = -a\,x(t) + u(t); \qquad y(t) = x(t)$$

Let us assume α=1. From (22), (24), (25) one can obtain

$$W = \begin{bmatrix} -a & \beta \\ 1 & a \end{bmatrix}, \qquad \mu = \sqrt{a^2 + \beta}$$

$$e^{Wt} = \begin{bmatrix} \mu\,ch(\mu t) - a\,sh(\mu t), & \beta\,sh(\mu t) \\ sh(\mu t), & \mu\,ch(\mu t) + a\,sh(\mu t) \end{bmatrix} \cdot \frac{1}{\mu}$$

$$G_1^o(t) = \left[\mu\,ch(\mu t) - a\,sh(\mu t)\right]/sh(\mu T)$$
$$G_2^o(t) = sh(\mu t)/sh(\mu T) \tag{28}$$

The norm of the observer is the function of β:

$$\|(G_1, G_2)\| = \sqrt{\int_0^T \left\{[G_1^o(t)]^2 + \beta\,[G_2^o(t)]^2\right\}dt} = \sqrt{\mu\,ctgh(\mu T) - a}$$

It means that for assumed norm of input disturbance - β, the estimation of maximum error is given by

$$\max_{(z_1,z_2)}\|\varepsilon\| \leq \sqrt{2(\mu\,ctgh(\mu T) - a)} \tag{29}$$

However if we use such an observer and the real norm (power) of input disturbance will be $\hat{\beta} \neq \beta$, then different estimation formula for error should be used with (28):

$$\max_{(z_1,z_2)}\|\varepsilon\| \leq \sqrt{2\int_0^T \left\{[G_1^o(t)]^2 + \hat{\beta}\,[G_2^o(t)]^2\right\}dt} =$$

$$\sqrt{[2a^2 + (\beta + \hat{\beta})]ctgh(\mu T)/\mu - 2a + T(\beta - \hat{\beta})/sh^2(\mu T)}$$

In the Table one can see the values of the above formula.

$\hat{\beta}$ / β	0	1	5	10	20
0	0.052	0.707	1.578	2.231	3.155
1	0.164	0.688	1.503	2.119	2.992
5	0.578	0.817	1.414	1.915	2.646
10	0.900	1.038	1.465	1.866	2.481
20	1.310	1.385	1.654	1.938	2.408

Especially the first row is important for our considerations. It shows the values of errors which gives the observer from Section 2.2 or 2.3 ($\beta = 0, \quad \hat{\beta} = 0$) working in the case when the real input disturbance with norm $0 < \hat{\beta} = 1, 5, 10, 20$ will happen. Column's minimal values (on diagonal) are the same as obtained from (29).

7 The discrete version of exact and optimal observer in R^N Hilbert space [16]

In this Section the general formula (16) will be used for derivation of exact and optimal observer for SISO discrete system. The suitable spaces will be: $Y=R^{N+1}$, $U=R^{R+1}$. $X=R^n$.

$$\begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix} = H_1 \begin{bmatrix} x_1(N) \\ \vdots \\ x_n(N) \end{bmatrix} + H_2 \begin{bmatrix} u_o \\ \vdots \\ u_N \end{bmatrix} \tag{30}$$

The vectors y and u are N+1 samples of scalar y(t) and u(t) in interval [0,T]. The final state x(N) is unknown.

$$H_2 u = -Ce^{Ai\Delta}\left(\sum_{j=i}^{N-1} e^{-Aj\Delta} \int_0^{\Delta} e^{-As} Bu(s + j\Delta) ds \right).$$

Assumption of ZOH use and u(jΔ)=const gives

$$H_2 = \begin{bmatrix} -CB_D & -Ce^{-A\Delta}B_D & \cdots & -Ce^{-A(N-1)\Delta}B_D \\ 0 & -CB_D & \cdots & -Ce^{-A(N-2)\Delta}B_D \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & -CB_D \end{bmatrix}_{N\times N}$$

where $B_D = \int_0^{\Delta} e^{-As} B ds$. The discrete observer is

$$x(N\Delta) = G_1 y + G_2 u \tag{31}$$

The norm of the observer

$$\left\| \left(G_1^T, G_2^T \right) \right\| = \frac{1}{\Delta} \sqrt{\left[\sum_{i=1}^{n} \left\langle g_1^i, g_1^i \right\rangle_Y + \sum_{i=1}^{n} \left\langle g_2^i, g_2^i \right\rangle_U \right]}$$

is minimized by the optimal matrices (16) which give the exact discrete observer $G_1[n\times(N+1)]$, $G_2[n\times(N+1)]$:

$$G_1^o = \left(H_1^T F^{-1} H_1 \right)^{-1} H_1^T F^{-1}; \qquad G_2^o = -G_1^o H_2,$$

Many tests in [16] were done for comparison of optimal discrete observer general formula (31) with α=β=1 and deadbeat observer (6) with α=1, β=0 (i.e. no disturbances in measurement of u is assumed) and the norm of the observer (6) is of the form

$$\left\| \left(G_1^T, G_2^T \right) \right\| = \frac{1}{\Delta} \sqrt{\sum_{i=1}^{n} \left\langle g_1^i, g_1^i \right\rangle_Y}.$$

In the presents of disturbances in control u the observer (31) gives better results then observer (6).

8 Conclusions

In the paper the survey of exact observers in Hilbert spaces is presented. The new version of discrete optimal observer (31) is presented. It was proved that information about the input and output disturbances (or noise) are very important for optimal designing of exact state observers and minimization of observation error. The estimation of the input disturbance norm (in $L^2[0,T]$ sense) and use the general observer formulae (16) enables designing of observers which guarantee minimal observation error and are better than observers of type (3), (5), (6).

References

[1] Luenberger D.G.,Observers for multivariable systems. *IEEE TAC*-11. 1966, p.190

[2] Kalman R.E., Contribution to the theory of optimal control. *Conf. on Ordinary Differential Equation.* Mexico City, Bol.Soc.Mat.Mex. 1959.

[3] Kalman R.E., R.C.Bucy New results in linear filtering and prediction theory. *Trans.of ASME. Journal of Basic Eng.* 83D.p.95.,1961.

[4] Gilchrist J.D., n-Observability for Linear Systems. *IEEE TAC-11*.no 3, 1966,p.388.

[5] Medvedev A., H.T. Toivonen , A continuous finite-memory deadbeat observer. *American Control Conference,* Chicago, 1992, p.1800.

[6] Medvedev A., Fault detection and isolation by functional continuous deadbeat observers, *Int.J.Control, vol.64,* (3), 1996., p.425,

[7] Nuninger W., F. Kratz, J.Ragot, Finite Memory Generalised State Observer for failure Detection in Dynamic Systems, *IEEE Conf. on Decision&Control*, 1998, Tampa, USA.

[8] Byrski W., S.Fuksa, "Optimal finite parameter observer. An application to synthesis of stabilizing feedback for a linear system", *Control and Cybernetics, vol. 13*,(1),1984.

[9] Fuksa S., W. Byrski, General approach to linear optimal estimator of finite number of parameters. *IEEE TAC-29,*1984,p.470.

[10] Rolewicz S., On optimal observability of linear systems in infinite-dimensional states. *Studia Math., no.44,* 1972, p.411.

[11] Wonham W., *Linear Multivariable Control: A Geometric Approach.* (Springer, New York, 1985).

[12] Byrski W., "Optimal State Observers with Moving and Expanding Observation Window", *Procc.of IASTED, XII Intern.Conference on Model.&Simul.*,Innsbruck, 1993,p.68.

[13] Byrski W. "Integral Description of the Optimal State Observers", *Proc. of II European Control Conference,* **vol.4**.1993,p.1832, Groningen.

[14] Byrski W. Theory and Application of the Optimal Integral State Observers. *Proc. of III European Control Conference. ECC'95*, Roma, 1995, p.52

[15] Byrski W., P.Kubik: Integral Observer of Finite State in Heat Equation Approximated by ODE, *IFAC/IMACS Large Scale Systems Symposium LSS'98,* Patras Greece, 1998.

[16] Pelc M., An integral observers in computer control systems. PhD dissertation, 2002, Univ.of Mining&Metallurgy(in Polish).