BOUNDARY STABILIZATION OF THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

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Abstract

This paper establishes the exponential decay of the energy of the solution of the wave equation with variable coefficients in the principal part subject to dissipative feedback acting in the Neumann boundary condition. The approach adopted uses Riemannian geometry combined with classical differential multipliers.

1 Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary Γ of class C^2. It is assumed that Γ consists of two parts Γ_0 and Γ_1 such that

$$\Gamma_1 \neq \emptyset \text{ and } \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \tag{1}$$

Let

$$\mathcal{A}y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) \tag{2}$$

be a second order differential operator with real coefficients $a_{ij} = a_{ji}$ of class C^1satsifying the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} |\xi_i|^2, \quad x \in \Omega \tag{3}$$

for some positive constant α. Assume further that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0,$$

$$\forall x \in \mathbb{R}^n,\ \xi = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n,\ \xi \neq 0.$$

Let k be an $L^\infty(\Gamma_0)$ function satisfying $k(x) \geq 0$ almost everywhere on Γ_0.

In Ω we consider the Neumann mixed second order hyperbolic problem in $y(t,x)$

$$y_{tt} + \mathcal{A}y = 0 \qquad \text{in } (0,+\infty)\times\Omega \tag{4}$$

$$y(0,x) = y_0,\ y_t(0,x) = y_1 \quad \text{in } \Omega \tag{5}$$

$$y = 0 \qquad \text{on } (0,+\infty)\times\Gamma_1 \tag{6}$$

$$\frac{\partial y}{\partial \nu_A} = -k(x)y_t \qquad \text{on } (0,+\infty)\times\Gamma_0 \tag{7}$$

where $y_{tt} = \frac{\partial^2 y}{\partial t^2}$,

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial y}{\partial x_j}\nu_i \tag{8}$$

is the co-normal derivative with respect to $\mathcal{A}$ and $\nu = (\nu_1, ..., \nu_n)$ is the unit normal on Γ pointing towards the exterior of Ω.

We define the energy of a solution $y(t,x)$ as follows

$$E(t) = \frac{1}{2}\int_\Omega [y^2 + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}]d\Omega \tag{9}$$

There has been an extensive work over the last two decades centered on the question of energy decay as $t \to +\infty$ for problem (4)-(7). In the case where the coefficients a_{ij} are constant, energy decay rates were obtained by Chen [1], Lagnese [4],[5] and Komornik and Zuazua [3]. Their work uses a Lyapunov method, based on Lyapunov type functions which contain among other terms, differential multipliers. As it was mentionned in Lagnese [4],[5], this method can be adapted to obtain decay estimates for problem (4)-(7) provided the coefficients a_{ij} satisfy further assumption (Condition (d), p. 167 in [5]).

In this paper, we use Riemann geometric methods combined with classical differential multipliers to study this stabilization question for problem (4)-(7). This approach was introduced by Yao in ([6]) to establish some observability inequalities for the wave equation (4).We prove that the energy decays exponentially without any strict assumption on the coefficients a_{ij}.

The paper is organized as follows. In §2, we present a few preliminary results and some of the machinery needed for proving the main result. Section 3 contains the statement and the proof of the main result.

2 Preliminary results

Let $A(x)$ and $G(x)$ be, repectively, the coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x))$$
$$i,j = 1, ..., n;\ x \in \mathbb{R}^n \tag{10}$$

Euclidean metric

Let $[x_1, ..., x_n]$ be the natural coordinate system in $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, denote by

$$X.Y = \sum_{i=1}^{n} \alpha_i \beta_i, |X|_0^2 = X.X,$$
$$\forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i}\ \in T_x\mathbb{R}^n$$

the Euclidean metric on $\mathbb{R}^n$.

For $f \in C^1(\overline{\Omega})$ and $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$, denote by

$$\nabla_0 f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad div_0(X) = \sum_{i=1}^{n} \frac{\partial \alpha_i(x)}{\partial x_i} \tag{11}$$

the gradient of f and the divergence of X in the Euclidean metric.

Riemannian metric

For each $x \in \mathbb{R}^n$, define the inner product and the corresponding norm on the tangent space $T_x\mathbb{R}^n$ by

$$g(X,Y) = \langle X, Y \rangle_g = X.G(x)Y = \sum_{i=1}^{n} g_{ij}(x) \alpha_i \beta_i$$
$$|X|_g^2 = \langle X, Y \rangle_g,$$
$$\forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n \tag{12}$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with a Riemannian metric g. Denote the Levi-Civita connection in metric g by D. Let H be a vector field on $(\mathbb{R}^n, g)$. The covariant differential DH of H determines a bilinear form on $T_x\mathbb{R}^n \times T_x\mathbb{R}^n$, for each $x \in \mathbb{R}^n$, by

$$DH(X,Y) = \langle D_X H, Y \rangle_g, \quad \forall X, Y \in T_x\mathbb{R}^n$$

where $D_X H$ is the covariant derivative of H with respect to X.

The following lemma provides some useful identities.([6], Lemma 2.1)

Lemma 1 *Let $f, h \in C^1(\overline{\Omega})$ and let H, X be vector fields on $\mathbb{R}^n$. Then with reference to the above notation, we have*
(i)

$$\langle H(x), A(x)X(x) \rangle_g = H(x).X(x), \qquad x \in \mathbb{R}^n \tag{13}$$

(ii) The gradient $\nabla_g f$ of f in the Riemannian metric g is given by

$$\nabla_g f(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i}$$
$$= A(x)\nabla_0 f, \qquad x \in \mathbb{R}^n \tag{14}$$

(iii)

$$\frac{\partial y}{\partial \nu_A} = (A(x)\nabla_0 y).\nu = \nabla_g y.\nu \tag{15}$$

(iv)

$$\begin{aligned} \langle \nabla_g f, \nabla_g h \rangle_g &= \nabla_g f(h) \\ &= \nabla_0 f.A(x)\nabla_0 h, \quad x \in \mathbb{R}^n \end{aligned} \tag{16}$$

(v)

$$\begin{aligned} \langle \nabla_g f, \nabla_g H(f) \rangle_g = DH(\nabla_g f, \nabla_g f) \\ + \frac{1}{2} div_0(|\nabla_g f|_g^2 H)(x) \\ - \frac{1}{2} |\nabla_g f|_g^2 div_0(H)(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{17}$$

(vi)

$$\begin{aligned} \mathcal{A}y &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) \\ &= -div_0(A(x)\nabla_0 y) \\ &= -div_0(\nabla_g y), \quad y \in C^2(\Omega) \end{aligned} \tag{18}$$

(vii)

$$E(t) = \frac{1}{2} \int_{\Omega} [y^2 + |\nabla_g y|_g^2] d\Omega \tag{19}$$

(v) ***Green's formula.*** *Let $z \in C^1(\overline{\Omega})$.Then*

$$\int_{\Omega} (\mathcal{A}y) z d\Omega = \int_{\Omega} \langle \nabla_g y, \nabla_g z \rangle_g d\Omega - \int_{\Gamma} z \frac{\partial y}{\partial \nu_A} d\Gamma \tag{20}$$

Before going on to the main result, we state three preliminary results that we will need in the proof of Theorem.3.1.

Theorem 2 *Assume (1) and set*

$$V = \{\varphi \in H^1(\Omega) : \varphi = 0 \textit{ on } \Gamma_1\}$$

Then for any initial data $\{y_0, y_1\} \in V \times L^2(\Omega)$ there exists a unique solution $y = y(t, x)$ of (4)-(7) such that

$$y \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, L^2(\Omega))$$

If in addition we assume $\{y_0, y_1\} \in W$ where

$$W = \{(y_0, y_1) \in V \times V : \mathcal{A}y_0 \in L^2(\Omega) \textit{ and } \frac{\partial y_0}{\partial \nu_{\mathcal{A}}} = -g(x) y_1 \textit{ on } \Gamma_0\}$$

then we have

$$y \in C^1(\mathbb{R}^+, V);\ \mathcal{A}y \in C(\mathbb{R}^+, L^2(\Omega)$$

This theorem is well known (see [2], Theorem 7.4).

Lemma 3 *Let $\{y_0, y_1\} \in W$. Then the solution of (4)-(7) satisfies*

$$E(S) - E(T) = \int_S^T \int_{\Gamma_0} (y_t)^2 k(x) d\Gamma dt \qquad (21)$$

for all $0 \leq S < T < +\infty$.

Proof. Consider the expression

$$\int_S^T \int_\Omega (y_{tt} + \mathcal{A}y) y_t d\Omega dt = 0$$

If we apply integration by parts in t and Green's formula (19) in x we are led to

$$\left[\frac{1}{2}\int_\Omega [y^2 + |\nabla_g y|_g^2] d\Omega\right]_S^T - \int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A} y_t = 0$$

and (21) follows from (6), (7) and (9). ■

Lemma 4 *Let $\{y_0, y_1\} \in W$, H a vector field on $\overline{\Omega}$ and a some positive constant. Then the solution of (4)-(7) satisfies the following identity for all* $0 \leq S < T < +\infty$.

$$\int_S^T \int_{\Gamma_1} (\frac{\partial y}{\partial \nu_A})^2 \frac{1}{|\nu_A(x)|_g^2} H.\nu d\Gamma dt + \int_S^T \int_{\Gamma_0} {y_t}^2 H.\nu d\Gamma dt$$

$$+ 2\int_S^T \int_{\Gamma_0} \frac{\partial y}{\partial \nu_A} H(y) d\Gamma dt - \int_S^T \int_{\Gamma_0} |\nabla_g y|_g^2 H.\nu d\Gamma dt$$

$$+ \int_S^T \int_{\Gamma_0} (\frac{\partial y}{\partial \nu_A})(div_0(H) - a) y d\Gamma dt$$

$$= \left[\int_\Omega [2y_t H(y) + (div_0(H) - a) y_t y] d\Omega\right]_S^T$$
$$+ a\int_S^T \int_\Omega [y_t^2 - |\nabla_g y|_g^2] d\Omega dt$$
$$+ \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (div_0(H) - a)\rangle_g y d\Omega dt$$
$$+ 2\int_S^T \int_\Omega DH(\nabla_g y, \nabla_g y) d\Omega dt \qquad (22)$$

Proof. We multiply (4) by $2H(y) + (div_0(H) - a)y$ and integrate by parts over $(S,T) \times \Omega$. Using Lemma 1, we find that

$$\int_S^T \int_\Omega \mathcal{A}y[2H(y) + (div_0(H) - a)y] d\Omega dt$$

$$= \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g [2H(y) + (div_0(H) - a)y]\rangle_g d\Omega dt$$

$$- \int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A}[2H(y) + (div_0(H) - a)y] d\Gamma dt$$

$$= 2\int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (H(y))\rangle_g d\Omega dt$$

$$+ \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (div_0(H) - a)_g y\rangle d\Omega dt$$
$$- \int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A}[2H(y) + (div_0(H) - a)y] d\Gamma dt$$

$$= 2\int_S^T \int_\Omega DH(\nabla_g y, \nabla_g y) d\Omega dt + \int_S^T \int_\Gamma |\nabla_g y|_g^2 H.\nu d\Gamma dt$$
$$- \int_S^T \int_\Omega |\nabla_g y|_g^2 div_0(H) d\Omega dt$$
$$+ \int_S^T \int_\Omega |\nabla_g y|_g^2 (div_0(H) - a) d\Omega dt$$
$$+ \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (div_0(H) - a)\rangle_g y d\Omega dt$$

$$- \int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A}[2H(y) + (div_0(H) - a)y] d\Gamma dt \qquad (23)$$

On the other hand, we obtain from integration by parts in t and Lemma 1

$$\int_S^T \int_\Omega y_{tt}[2H(y) + (div_0(H) - a)y] d\Omega dt =$$
$$\left[\int_\Omega y_t [2H(y) + (div_0(H) - a)y] d\Omega\right]_S^T$$
$$- \int_S^T \int_\Omega y_t [2H(y_t) + (div_0(H) - a)y_t] d\Omega dt$$
$$= \left[\int_\Omega y_t [2H(y_t) + (div_0(H) - a)y] d\Omega\right]_S^T$$
$$- \int_S^T \int_\Gamma y_t^2 H.\nu d\Gamma dt +$$
$$\int_S^T \int_\Omega \left[y_t^2 div_0(H) - (div_0(H) - a) y_t^2\right] d\Omega dt \qquad (24)$$

Summing up (23) and (24), we find the identity

$$\int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A}[2H(y) + (div_0(H) - a)y]d\Gamma dt$$
$$- \int_S^T \int_\Gamma |\nabla_g y|_g^2 H.\nu d\Gamma dt + \int_S^T \int_\Gamma y_t^2 H.\nu d\Gamma dt =$$
$$\left[\int_\Omega y_t[2H(y_t) + (div_0(H) - a)y]d\Omega\right]_S^T$$
$$+ a \int_S^T \int_\Omega [y_t^2 - |\nabla_g y|_g^2] d\Omega dt$$
$$+ 2 \int_S^T \int_\Omega DH(\nabla_g y, \nabla_g y) d\Omega dt$$
$$+ \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (div_0(H) - a)\rangle_g y d\Omega dt \qquad (25)$$

We now use the boundary condition (6). Thus

$$\text{on } \Gamma_1: \qquad y = y_t = 0; \quad |\nabla_g y|_g^2 = \frac{1}{|\nu_A(x)|_g^2}(\frac{\partial y}{\partial \nu_A})^2;$$
$$H(y) = \frac{H.\nu}{|\nu_A(x)|_g^2} \frac{\partial y}{\partial \nu_A} \qquad (26)$$

Hence substitution of (6) and (26) into (25) yields the desired identity. ■

3 Main result

Theorem 5 *Assume there is a vector field H on the Riemannian manifold $(\mathbb{R}^n, g)$ such that*

$$\langle D_X H, Y\rangle_g \geq a |X|_g^2$$
$$\forall X, Y \in T_x\mathbb{R}^n, x \in \Omega, \text{ for some constant } a > 0 \qquad (27)$$
$$H.\nu \leq 0 \text{ on } \Gamma_1 \qquad (28)$$
$$H.\nu > 0 \text{ on } \Gamma_0 \qquad (29)$$

Choose

$$k(x) = H(x)\nu(x) \qquad (30)$$

Then there exist $M \geq 1$ and $\omega > 0$ such that

$$E(t) \leq Me^{-\omega t}E(0), \qquad t \geq 0 \qquad (31)$$

for every solution of (4)-(7) for which $E(0) < +\infty$.

Proof. It is sufficient to prove the estimate (31) for smooth $\{y_0, y_1\} \in W$. The general case then follows by an easy density argument.
Let μ_0, μ_1 be the smallest constants such that

$$\int_{\Gamma_0} v^2 d\Gamma \leq \mu_0 \int_\Omega |\nabla_g y|_g^2 d\Omega$$
$$\int_\Omega v^2 d\Omega \leq \mu_1 \int_\Omega |\nabla_g y|_g^2 d\Omega$$

Set

$$C_1 = \sup_{x\in\overline{\Omega}} |H(x)|_g ,$$
$$C_2 = \sup_{x\in\overline{\Omega}} \left|(div_0(H))^2 - a^2 + 2H(div_0(H) - a)\right| ,$$
$$C_3 = \sup_{x\in\overline{\Omega}} |div_0(H) - a| , \; C_4 = \sup_{x\in\overline{\Gamma_0}} |H.\nu| ,$$
$$C_5 = \sup_{x\in\overline{\Omega}} |\nabla_g(div_0(H) - a)|$$

From Lemma 4 and assumptions (27) and (28), we have

$$a \int_S^T \int_\Omega \left[y^2 + |\nabla_g y|_g^2\right] d\Omega dt$$
$$\leq - \left[\int_\Omega [2y_t H(y) + (div_0(H) - a)y_t y]d\Omega\right]_S^T$$
$$- \int_S^T \int_\Omega \langle \nabla_g y, \nabla_g (div_0(H) - a)\rangle_g y d\Omega dt$$
$$- 2\int_S^T \int_{\Gamma_0} y_t H(y) H.\nu d\Gamma dt - \int_S^T \int_{\Gamma_0} |\nabla_g y|_g^2 H.\nu d\Gamma dt$$
$$+ \int_S^T \int_{\Gamma_0} {y_t}^2 H.\nu d\Gamma dt$$
$$- \int_S^T \int_{\Gamma_0} y_t(div_0(H) - a)y H.\nu d\Gamma dt \qquad (32)$$

Now we estimate the terms on the right-hand side of (32).

Term $\left[\int_\Omega [2y_t H(y) + (div_0(H) - a)y_t y]d\Omega\right]_S^T$.
We proceed as in [2]. Application of divergence theorem yields

$$\int_\Omega [2H(y) + (div_0(H) - a)y]^2 d\Omega$$
$$= \int_\Omega [4(H(y))^2 + (div_0(H) - a)^2 y^2] d\Omega$$
$$+ 4 \int_\Omega (div_0(H) - a) y H(y) d\Omega$$
$$= \int_\Omega [4(H(y))^2 + (div_0(H) - a)^2 y^2] d\Omega$$
$$+ 2 \int_\Omega (div_0(H) - a) H(y^2) d\Omega = 4 \int_\Omega (H(y))^2 d\Omega$$
$$- \int_\Omega \left[(div_0(H))^2 + a^2 - 2H(div_0(H) - a)\right] y^2 d\Omega$$
$$+ 2 \int_{\Gamma_0} (div_0(H) - a) y^2 H.\nu d\Gamma$$
$$\leq (4C_1^2 + C_2\mu_1 + 2C_3C_4\mu_0) \int_\Omega |\nabla_g y|_g^2 d\Omega \qquad (33)$$

It follows from (33) that

$$\left|\int_\Omega [2y_t H(y) + (div_0(H) - a)y_t y]d\Omega\right|$$
$$\leq \|y_t\|_{L^2(\Omega)} \|2H(y) + (div_0(H) - a)y\|_{L^2(\Omega)}$$
$$\leq \gamma E(t)$$

where

$$\gamma^2 = 4C_1^2 + C_2\mu_1 + 2C_3C_4\mu_0$$

Hence

$$\left|\left[\int_\Omega [2y_tH(y) + (div_0(H) - a)y_ty]d\Omega\right]_S^T\right| \leq 2\gamma E(S) \quad (34)$$

Term $\int_S^T \int_\Omega \langle \nabla_g y, \nabla_g(div_0(H) - a)\rangle_g \, yd\Omega dt$

We have by the Cauchy-Schwarz inequality

$$\left|\int_S^T \int_\Omega \langle \nabla_g y, \nabla_g(div_0(H) - a)\rangle_g \, yd\Omega dt\right| \leq \frac{\delta}{2}\int_S^T \int_\Omega |\nabla_g y|_g^2 \, d\Omega dt + \frac{C_5^2}{2\delta}\int_S^T \int_\Omega y^2 d\Omega dt \quad (35)$$

where $\delta > 0$ to be chosen below.

Term $2\int_S^T \int_{\Gamma_0} y_tH(y)H.\nu d\Gamma dt$

$$\left|2\int_S^T \int_{\Gamma_0} y_tH(y)H.\nu d\Gamma dt\right| \leq 2\int_S^T \int_{\Gamma_0} \left|y_t \langle H(x), \nabla_g y\rangle_g\right| H.\nu d\Gamma dt \leq C_1^2 \int_S^T \int_{\Gamma_0} y_t^2 H.\nu d\Gamma dt + \int_S^T \int_{\Gamma_0} |\nabla_g y|_g^2 \, H.\nu d\Gamma dt \quad (36)$$

Term $\int_S^T \int_{\Gamma_0} y_t(div_0(H) - a)yH.\nu d\Gamma dt$

$$\left|\int_S^T \int_{\Gamma_0} y_t(div_0(H) - a)yH.\nu d\Gamma dt\right| \leq \frac{C_4C_3^2}{2\delta}\int_S^T \int_{\Gamma_0} y_t^2 H.\nu d\Gamma dt + \frac{\delta}{2}\int_S^T \int_{\Gamma_0} y^2 d\Gamma dt \leq \frac{C_4C_3^2}{2\delta}\int_S^T \int_{\Gamma_0} y_t^2 H.\nu d\Gamma dt + \frac{\delta}{2}\mu_0 \int_S^T \int_\Omega |\nabla_g y|_g^2 \, d\Omega dt \quad (37)$$

Use of (34)-(37) in (32) yields

$$a\int_S^T \int_\Omega \left[y^2 + |\nabla_g y|_g^2\right] d\Omega dt \leq 2\gamma E(S) + \frac{\delta}{2}(\mu_0 + 1)\int_S^T \int_\Omega |\nabla_g y|_g^2 \, d\Omega dt + (C_1^2 + 1 + \frac{C_4C_3^2}{2\delta})\int_S^T \int_{\Gamma_0} y_t^2 H.\nu d\Gamma dt + \frac{C_5^2}{2\delta}\int_S^T \int_\Omega y^2 d\Omega dt$$

Chosing $\delta = \frac{a}{\mu_0+1}$ we obtain

$$\int_S^T E(t)dt \leq C_6E(S) + C_7\int_S^T \int_{\Gamma_0} y_t^2 H.\nu d\Gamma dt + C_8\int_S^T \int_\Omega y^2 d\Omega dt \quad (38)$$

where

$$C_6 = \frac{2\gamma}{a}, \; C_7 = \frac{1}{a}(C_1^2 + 1 + \frac{C_4C_3^2(\mu_0 + 1)}{2a}), \quad C_8 = \frac{C_5^2(\mu_0 + 1)}{2a}$$

Recalling Lemma 3, we obtain from (38)

$$\int_S^T E(t)dt \leq (C_6 + C_7)E(S) + C_8\int_S^T \int_\Omega y^2 d\Omega dt \quad (39)$$

From Theorem 2 of [5], we deduce the existence of an integer $n > 1$ such that

$$\int_S^T \int_\Omega y^2 d\Omega dt \leq C_\eta^* E(S) + \eta \int_S^{nT} E(t)dt \quad (40)$$

where $\eta > 0$ is arbitrary and $C_\eta^* > 0$ depends on η.
Insertion of (40) into (39) yields

$$\int_S^T E(t)dt \leq (C_6 + C_7 + C_8C_\eta^*)E(S) + C_8\eta\int_S^{nT} E(t)dt$$

Let $T \to +\infty$ and choose η so that $1 - \eta > 0$, to obtain

$$\int_S^{+\infty} E(t)dt \leq \frac{(C_6 + C_7 + C_8C_\eta^*)}{1 - C_8\eta}E(S)$$

The conclusions of the theorem with $\omega = \frac{1-C_8\eta}{C_6+C_7+C_\eta^*}$ and $M = e$ follow from Theorem 8.1 of [2]. ■

Remark 6 *In general, it is not easy to find a vector field H satisfying assumption (27). Some sufficient conditions for the existence of a such vector field with a number of nontrivial examples are presented in* [*6*] .

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