BOUNDARY CONTROL OF NON-LINEAR HYPERBOLIC SYSTEMS – MIGRATING OBJECT APPROACH

A. D. Bogobowicz

Department of Electrical and Computer Engineering
Dalhousie University
P.O. Box 1000
Halifax, NS B3J 2X4
A.Bogobowicz@Dal.ca
Fax: +1 902 422 7535

Keywords: non-linear hyperbolic systems, boundary control, entropy production, local semantics.

Abstract

The paper presents part of the created systematic methodology of solving optimal control problems formulated for systems with subsystems (possibly analog) that are changing scale, dimension or symmetry. In the paper we are interested in controlled processes that can be described by hyperbolic equations. The system described by such equation can locally change its qualitative properties. In the considered case the local behaviour is approximated by a parabolic system. New model (semantics) is introduced to model the emerging subsystem. The introduced invariant measure chosen for migrating object is used to identify the local properties and choose a unique local value.

1. Introduction

The work presented here is a part of larger concept concerning the necessary storage and complexity of using the stored data in solving complex problems. In the concepts we consider parameter distributed systems. In such systems, as the parameters change, boundaries assumed for the model may become invalid, order, values and dimension changes must be considered. The reasoning of this paper is based on the assumption that the local properties and parameters are not known if the model was formulated for global behaviour. It is therefore important to observe that changes of function spaces is to be considered and emerging arguments as well as the de-differentiating properties at the boundaries must be learned before decision- control can be applied.

In the paper de-differentiation is proposed to be performed by new entities – migrating objects. Migrating objects work as cognitive lenses and let complex local contexts emerge. The important observation is that local context may not be properly described by global model domain, thus newly introduced spaces and notions must be formulated in order to build appropriate data structure. In theoretical considerations of distributed parameter systems we think about communication, information and flow of information. The point of view of any analog element or human must be literal, material or technical; it is built or developed, located and specific and as such must be incorporated in the recognition and control [10]. If the element is controlled the controls can be in "equilibrium" whereas the state may develop. In other cases, a specific state is only found by deriving complex formulas for the control variables. Time needed for such derivation or numerical computation excludes the approach or implementation applicability. In many numerical methods correct and applicable results can be only obtained after long computation time necessary to satisfy the time and space approximation conditions. The present state of technological advancement makes the design particularly time and parameter sensitive. Thus, the appropriate measures need to be defined for the recognition of dynamics that possibly changes its qualitative properties. The principles based on relevant measures of uncertainty are fundamental to reasoning [8] when the global and local properties are considered and when it is possible that symmetries are locally broken. These can be the principles of Minimum Uncertainty, Maximum Uncertainty, and Uncertainty Invariance. In the work we focus our attention by choosing the reasoning based on the Principle of Uncertainty Invariance which is formulated as follows: Principle of Uncertainty Invariance: This principle requires that the amount of uncertainty be preserved when we transfer uncertainty formalised in one category into an equivalent formalisation in another category. The representation of the above principle considered in this paper has the form of minimum information loss In this paper we are interested in non-linear problems. In highly non-linear problems the convergence property of approximation may not be shown. But for some specifically chosen approximating probe functions their properties can be used in local considerations. Thus, after having the global goals formulated, the core concepts of the operational function are chosen. Dynamic means of a phenomena interpretation and learning of the system are chosen for operational model description. There are several basic questions that must be considered. First off, what are the elements of modelling concepts that will be used for describing associative interactions? The structure of derived equations indicates that neural network concepts can be applied to model associativity and configuration of canonical objects. The operators in the canonical object which represent the concepts of association and configuration are formulated in a direct way and use The reasoning and formal derivation is assumed to prove some computational results that lead to the specific meaning of the novum. The method of recognising a novum is crucial in the recognition process. In our approach the concept new is considered to represent:

- The time openness: the specific properties of time approximations are considered; new results in analysing ergodic properties can be included.
- Open boundaries: e.g. in diffusion processes intermediate distance is preferred instead of the closest; in language related recognition the strong open boundary might be explored by testing dynamics of natural languages;
- The properties of merged dynamic type hierarchies that describe local phenomena; the importance of representing complete information merged in local phenomena model can be observed in geometry-mechanics relationships exploration in data retrieval methods.

If the subsystem novum is incorporated in the global system the further consideration can follow classical approach to controlled systems.

2. The non-linear hyperbolic systems.

As an example the non-linear hyperbolic is considered. The example is interesting in considering highly non-linear problems and spatially distributed systems. It can be viewed as a model that changes locally in space and in time its qualitative properties. In a strict mathematical sense the considered example phenomenon is supposed to be governed by a set of non-linear partial differential equations:

(minimum of entropy production) as the principle used by the author in many earlier computational research and applications.
basic methodology of neural networks. However, the results from neural networks theory are not used here, as they are not confirmed by some models known to the author. The next question is what higher level structures can be unfolded due to complementarity? The operators, which make complementarity, are assumed to be derived from detailed investigation of the particular phenomena considered, its limit properties and sensitivity analyses. Complementary distributed properties of the system can be in many cases incorporated by qualitative solutions of subsystems; by introducing a discrete recursive algorithm, by considering the system (subsystem) in its limit form or by solving optimisation problem. Complementarity in the set of directions and problems introduced is essential in recognition. Similarity in categorisation for database retrieval methodology was the typical approach.

$$\mathbf{u}_t + \nabla.\, \mathbf{f}(\mathbf{u}) = \mathbf{0} \qquad (1)$$

where $\mathbf{u} \in R^m$, $\mathbf{x} \in R^n$, $t \geq 0$ $\mathbf{f} : R^m \rightarrow R^m \otimes R^n$, $\mathbf{f}$ is assumed to be known .
Examples of the phenomena described by above equation come from physics, engineering, economy and biology. The construction of extracted object and the results of applied methodology can be easily understood while the above equation is compared with the equation

$$\mathbf{u}_t + \nabla.\, \mathbf{f}(\mathbf{u}) = \varepsilon\ \mathbf{D}\, \Delta\mathbf{u} \qquad (2)$$

considered in the limit $\varepsilon \to 0$ (D is a positive diagonal matrix).
In physical problems the diffusion term **—.f(u)** may arrive in equations by considering particle taking random walks biased by a convective velocity term. In fluids diffusion terms arise via the consideration of viscous forces.
The conditions for which a global unique solution of non-linear hyperbolic equations exists are derived in [9]. In our approach it is assumed that conditions are not met. We are then thinking about solutions, where new definitions must be introduced in order to prove approximation in non-linear terms for chosen probe functions.
The point about such equations as Equation (1) is that even for smooth initial data the solution surfaces may fold up and develop an infinite derivative in finite time. In order to continue the solutions through the development of such singularities methods of making a

proper local choice must be constructed. Also, in particular cases one might be interested in understanding the behaviour in the neighbourhood of singularity. An instantiation of a migrating object is then derived and a new semantics are chosen for modelling the locally changed phenomena.

3. Boundary control of one-dimensional hyperbolic system.

Let us consider one dimensional case [8]. The example is chosen for its parameter and boundary conditions sensitive behaviour. One dimensional case can be viewed as a communication line model with e.g. velocity proportional to the packets density in the line. In this particular example the function f is derived to be f(u) = u(1-u). There are many other real life problems that use or should use careful analysis of locally changing behaviour of this non trivial model.
The equation (1) is solved with the initial condition

$$u(x,0) = u_0(x)$$

and the boundary condition

$$u_x(1,t) = v(t),$$

where v(t) is a control function, $t \in (0,1)$, $x \in (0,1)$. In this paper the flux boundary conditions are chosen. However, it can be shown that we can derive equivalent forms of entropy productions for different boundary conditions: periodic or absorbing.
Consider (1) along with the initial data $u(x,0) = u_0(x)$ in parameterised form, for $s \in R$.
We write the equation

$$u_t + f(u)_x = 0 \qquad (3)$$

in the following form, along the characteristics where we must have u = constant:

$$x = s, \text{ when } t = 0;$$
$$u = u_0(s); \qquad (4)$$
$$dx/dt = f_u(u_0(s)) \text{ when } t > 0$$

or alternatively, for a given $u_0(s)$

$$u(x,t) = u_0(s), \qquad (5)$$

where

$$x = t\, f_u(u_0(s)) + s \qquad (6)$$

The equation (6) is a solution of (5) provided that u_t and u_x remain finite. When the u_t and u_x are not finite the condition for the instantiation of a singularity tracking migrating canonical object is applied. The condition for instantiating the migrating object for which the new measure is required is of the form:

$$t\frac{d[f_u(u_0(s))]}{dt} + 1 = 0 \qquad (6)$$

where $u_0(s)$ is a parameterised solution of (3). The derived formula is a powerful step in analysis of the emerging local phenomenon.
In order to continue the solution through the discontinuity we construct weak solution in a domain $Q_T \subset (x,t)$ – space by writing for such solution u

$$\int (\Phi_t\, u + \Phi_x\, f(u))\, dx\, dt = 0, \qquad (7)$$

for $\Phi(x,t)$ from C^1 defined on Q_T and $\Phi \equiv 0$ on ∂Q_T.
Solution u is a classical solution everywhere in Q_T except along curve Γ where it possesses a finite discontinuity. The idea is that the weak solution should be open to new properties which are not observed in the space of global solutions. Along Γ, for any Φ we write the equality

$$\int \Phi\, ([u]dx - [f(u)]\, dt) = 0 \qquad (8)$$

where

$$[u] = u_+ - u_-, \quad [f(u)] = f(u_+) - f(u_-) \qquad (9)$$

As Φ was arbitrary then if x = s(t)

$$[u]\, ds/dt = [f(u)] \qquad (10)$$

If the equation (3) is considered for new variable $z=(x-s(t))/\varepsilon$ and solved in a neighbourhood of the shock for $|z \to \pm\infty|$ by considering new variable defined as $u = U(z, t)$ it can be written as

$$U_t = 1/\varepsilon\, (U_{zz} + s_t U_z - f(U)_z)$$

Up to $O(\varepsilon)$ the equation becomes

$$U_{zz} + s_t U_z - f(U)_z = 0$$

which implies

$$U_z + s_t U - f(U) = const$$

For the limit case when $z \to \mp\infty$ above equation can be written as

$$s_t[u] = [f(u)]$$

The equation agrees with (10).

The standard approach indicates that in the neighbourhood of the developed shock qualitatively new entity can be considered. New entity satisfies complementarity definition required in our system. New entity(object) should approximate diffusion phenomenon. Thus, when the condition (6) is met the reasoning must be based on new measure of the phenomena. Density distribution in the new object is considered. Chaotic map is introduced as modelling the diffusion in the system and the entropy change is derived with this assumption.

4. Boundary control problem

The following optimization problem is being considered as a control problem for our system described by the equation (3).
For the functions f_0, f_1 that are continuous, real-valued functions on R there are constants a, b >0 so that

$f_0(u,w,x,t)$ § $a|u| + b\,||w||_E$,

(u,w,x,t) e $A \times B \times Q_T$,

where A, B bounded sets in R, $Q_T = (0,1)\times(0,T)$ and E refers to euclidean norm in R.
Then we are seeking for the pair (u,v) that minimize the following functional:

$$J(u,v) = \int_{Q_T} f_0(u(x,t), \nabla u(x,t), x, t)\,dxdt$$
$$+ \int_{\Gamma_T} f_1(v(s,t), s, t)\,dsdt,$$

where Γ_T denotes boundary of our domain.

In our approach the minimization of the functional over the space of admissible pairs must be replaced by the minimization considered in spaces where for local minimization a new measure is introduced.
Thus the minimization must be carried for the functional:

$$I(\nu, \mu) = \nu(f_0) + \mu(f_1)$$

where new measures ν and μ are to be derived, given the used principle of minimum information loss in transition to local phenomenon.

5. Approximation of the local choice

In the following reasoning it will be shown how the measure for the local specific phenomenon is derived. In order to find the unique local solution of the model itself the criterion of minimum loss of information is assumed. For the extracted object a special measure is construed. The constructing procedure considered here follows the works of Gaspard [4, 5]. These works consider physical phenomena but the mathematics of derivations for flux boundary conditions is of generic value. In our controlled system we assume flux boundary condition. The measure of information loss in the local model of the phenomenon is the entropy production measure. Thus our system can be viewed as an locally open system. Local is understood in a sense of extracted (filtered) object with complementarity conditions applied to the object. This object must be considered in a new space, possibly constructed for the local phenomena. The implementation might be developed as a complementarity condition in a numerical method applied to solve the system or a parameterised system call. In particular, in our hyperbolic example extracted diffusive object can be described with the use of new measure formulated for systems at singularity which develops as was shown in section 3. The measure construed in [4] is not normalisable because it becomes singular in some direction. However, in order to derive the entropy production we use the property that a stationary gradient can be maintained in the system.

The solution of equations modelling the local system system requires boundary conditions at the borders of the phase space. In our particular, controlled case we take into account flux boundary conditions. Such boundary conditions allow us to express the non-equilibrium conditions in the extracted object in the following formulation: if the flow changes at the input and the output of the system, the corresponding boundary of phase space of the subsystem must be represented as a distribution with varying parameter resulting from complementarity condition. From mathematical point of view some rigorous derivations are still needed to show how the developed boundary conditions of the global system should be projected on the local object domain.
The system is assumed to be controlled. The flux boundary conditions give us a proper derivation to identify and use the entropy production as a measure of information loss in the locally changed computational model. As it is shown in [4] the entropy production is caused only by applied boundary conditions.
If we think about some statistical ensembles of copies of the system defined in the phase space by a vector field

an invariant measure is established after some time for continuous flux.

$$\mathbf{U_t = F(U)}$$

The time evolution of the statistical ensemble is governed by Liouville equation.
The construction of the measure is based on the probability measure for the Poisson suspension over the system of measure μ_t, which evolves in time according to the formula

$$\mu_{t+1}(B)=\mu_t(\Phi^{-1} B)$$

for the extracted region B, where Φ is defined in the equation describing system dynamics:

$$\mathbf{U_t = \Phi^t U_0}$$

We consider a region Q, $\{B_i\}$ is a square partition of Q and $C_{Bk} = \{ Y_{Bi}\}$ - subsets of the phase space such that the number of copies U_i in B_i is k.
Definition for entropy production in the square B_i is as follows:

$$\Delta_i S_\varepsilon = \Delta S_e - \Delta_e S_\varepsilon$$

where $\Delta_e S_\varepsilon$ is entropy flow resulting from boundary conditions and ΔS_e is the difference of entropy between two successive instants of time. The ε - entropy S derivation that makes use of the relationship (14) is based on the definition:

$$S_\varepsilon = S(\{B_i\}) = -\sum_i \mu(C_{\{BiKi\}}) \ln \mu(C_{\{BiKi\}})$$

where k_i is the number of copies in the set B_{ki}
The map used in derivation of the entropy production is a baker type map which is area preserving and of hyperbolic type. The map is volume preserving. With the use of this map we consider a time evolution of a finite chain of squares $\varepsilon=(\Delta x, \Delta y)$. In a limit singularity must be approximated thus a singular measure is needed. As it was shown the local evolution of (1) is approximated by diffusive model thus the baker type map is applicable here. The cumulative functions are defined for the chain of squares. The cumulative functions exist even if the singularity occurs. These functions represent the average flow in the object. It was shown that Takagi incomplete functions which are differentiable almost everywhere define the cumulative functions. The incomplete differentiable Takagi functions converge to the Takagi function, which is non-differentiable. Thus the results which are needed can be obtained due to the construction of functions which converge to non-differentiable function. The ε - entropy production has the behaviour expected for diffusion. Its analytical resulting form the derivations form can be easily implemented and tested e.g. computational system can generate ε-entropy production e.g. by implementing the Takagi function:

$$T_n(y)= 1/2\, T_{n-1}(2y) +y \quad \text{if} \quad 0\leq y<1/2,$$
$$T_n(y)= 1/2T_{n-1}(2y-1) +1-y \quad \text{if} \quad 1/2 \leq y \leq 1,$$

$$T_{-1}(y) = T_{L+1}(y) = 0.$$

In [3] it was shown that the final formula expressing the entropy production measure derived for the chosen map takes the following form:

$$\Delta_i S_\varepsilon = (\nabla u)^2/(2\rho_n)+ (\nabla u)^4(1/6+\log_2(1/(\Delta y))/2\, \rho_n^3 + \Theta[(\nabla u)^6/\rho_n^5], \qquad (16)$$

The fourth order terms are positive definite outside very narrow transition section. The minimum of derived entropy production criterion can be found and used as a unique local choice in our control problem.

The result

$$\lim_{\varepsilon\to 0}\lim_{L\to\infty}\lim_{(\nabla\rho)/\rho\to 0}\frac{\rho}{(\nabla\rho)^2}\Delta_i S_\varepsilon = \mathrm{D} \geq 0, \qquad (17)$$

needs special attention. For large chain the limit $L \to\infty$ is taken before the limit $\varepsilon \to 0$. Thus, the result shows an emerging property in a system where entropy production does not vanish even in the fine scale limit. There is an interesting property of Takagi function that is used in further development of the methodology. The derivative of the Takagi function can be computed using symbolic sequences of ω_i =0 or 1 in the assignment

$$y_{\omega 1\ldots\omega t} = \frac{\omega_1}{2}+\frac{\omega_2}{4}+\ldots+\frac{\omega_t}{2^t}$$

and the following properties are derived:

$$2T(y_{\omega 1\ldots\omega t}+\frac{1}{2^{t+1}})-T(y_{\omega 1\ldots\omega t}+\frac{1}{2^t})$$
$$-T(y_{\omega 1\ldots\omega t})=\frac{1}{2^t}$$

and

$$\sum_{\omega 1\ldots\omega t}\left[T(y_{\omega 1\ldots\omega t}+\frac{1}{2^t})-T(y_{\omega 1\ldots\omega t})\right]^2=\frac{1}{2^t}$$

The properties of the Takagi function [7] indicate new possible form of approximation of time dependence in the system. There are mathematical proofs still needed for the approximation but the properties derived for the qualitatively correct model of local phenomena provide a good information for analytical and numerical considerations of the phase space.

5. Conclusions.

The idea conserning the methodology development is based on the theoretical and computational work of the author [1,12]. The results existing in the newest literature in the area of statistical mechanics, theoretical computer science and physics suggest new approaches to control problems considered in different scales. In such systems the rigorous decomposition of the system is assumed. The decomposition of the system is based on the flow of information carried along eikonals. The change of qualitative behaviour of the system is utilized in the methodology derivation in order to decrease computational time necessary to make decision (control) about the local phenomena. In the analysed system the approximation of local phenomenon is assumed to represent an exact model of the phenomenon in the extracted object. The minimum entropy production criterion calculated for the diffusive system extracted from hyperbolic system can be used as a an example of complementarity condition to ensure unique choice of a weak solution in a locally emerging diffusion. In our considerations the criterion is applied for local information – retrieval when the result as one value only is needed. The theory can predict local future state only as probabilities. The real system or computational system that mimics the local behaviour may generate the "unique local choice" of a variable to satisfy the complementarity condition. The idea of deriving the approach arose based on observations of non-equilibrium local behaviour in boundary controlled systems and observations of new qualitative phenomena occurring in computational results [12].

References

[1] Bogobowicz, A.D, L. Rothenburg, M.B. Dusseault, (1991), Solution for Non-Newtonian Flow into Elliptical Opening", Journal of Applied Mechanics, ASME, Vol. 58, No.3.

[2] Follard, G.B., (1976) Introduction to Partial Differential Equations, Princeton University Press,

[4] Gaspard P., Chaos 3(1993), 427.

[5] Gaspard P., (1996), Entropy Production in Open Volume-Preserving Systems, preprint of ESI , 382, Vienna.

[6] Grindrod, P.,(1991), Patterns and Waves, Clarendon Press, Oxford.

[7] Hata, M.,(1986) Fractals in Mathematics, in Patterns and Wves, 1986, pp.259-278.

[8]Klir,G. (1991), Measures and Principles of Uncertainty and Information: Recent Development, in: Information Dynamics, ed. H. Atmanspacher, pp. 1-14, Plenum Press, NY.

[9] Ladyzenskaya,O., (1991), Attractors for Semigroups and Evolution Equations, Cambridge University Press.

[10]Liebl, F., (1996), Strategische Fruhaufklanung: Trends, Issues, Stakeholders; Munchen.

[11]Mulder,W., S. Osher, J.A. Sethian,(1992), Computing Interface Motion in Compressible Gas Dynamics, Journal of Computational Physics, 100, 209-228.

[12]Rothenburg,L., A.D.Bogobowicz, Flow Through Anisotropic Network, Journal of Stochastic Hydrology and Hydraulics, Vol.4.,No.3.