RIESZ BASIS AND EXACT CONTROLLABILITY OF C_0-GROUPS WITH ONE-DIMENSIONAL INPUT OPERATORS

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Abstract

This paper considers the controllability and Riesz basis generation property of linear infinite dimensional systems with C_0-group generators and one-dimensional admissible input operators. The corresponding results of **[Advances in Mathematical Systems Theory, 2000, pp. 221-242]** under the assumption of algebraic simplicity for eigenvalues of the generator are generalized to the case in which the eigenvalues are allowed to be algebraically multiple but with uniform boundedness of multiplicity.

1 Introduction

Consider the infinite dimensional systems of the following kind:

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

where $A : D(A) \to H$ is the generator of a C_0-group $T(t)$ on the complex separable Hilbert space H and b is an admissible ([5]) one-dimensional control operator. The input function u is assumed to be in $L^2_{loc}(0, \infty)$. Moreover, we assume that $-A$ generates an exponentially stable C_0-semigroup. This assumption, however, is not restrictive because both admissibility and exact controllability are invariant with respect to a scalar shift of A. In the sequel, we also use $\sum(A, b)$ to refer to system (1).

We say that system (1) is *exactly controllable* in $[0, t_0]$ if for any $x_0 \in H$ there exists an input function $u \in L^2(0, t_0)$ such that

$$0 = T(t_0)x_0 + \int_0^{t_0} T(t_0 - s)bu(s)ds. \tag{2}$$

When the system (1) is exactly controllable, it has been shown in [3] and [4] that the spectrum of A is of a very special form. We summarize these results as follows:

Theorem 1. *Assume that system (1) is exactly controllable. Then*

(i) the spectrum of A consists of isolated eigenvalues: $\sigma(A) = \{\lambda_n\}_1^\infty, 0 < \inf_n Re\lambda_n \le \sup_n Re\lambda_n < \infty$;

(ii) each eigenvalue has finite algebraic multiplicity and geometric multiplicity one;

(iii) $\sigma(A^*) = \{\overline{\lambda}_n\}_1^\infty$ and every $\overline{\lambda}_n$ is an isolated eigenvalue of A^* with finite algebraic multiplicity and geometric multiplicity one;

(iv) both $\{E(\lambda_n, A)H, n \ge 1\}$ and $\{E(\overline{\lambda}_n, A^*)H, n \ge 1\}$ are dense in H, where $E(\sigma, \cdot)$ denotes the eigen projection with respect to the spectral set σ.

In the sequel, we still use A and $E(\lambda, A)$ to denote their extensions in $H_{-1} = [D(A^*)]'$ without diffusion ([5]).

2 (ξ, ω) representation

Lemma 1. *Suppose that $\sum(A, b)$ is exactly controllable with isolated separated eigenvalues $\{\lambda_n\}_{n=1}^\infty$. Then for any $\delta > 0, R(\lambda, A)$ is uniformly bounded in $G = \mathbb{C} - \cup_{n=1}^\infty S_{n,\delta}$, $S_{n,\delta} = \{\lambda \in \mathbb{C}, \mid |\lambda - \lambda_n| < \delta\}$.*

Proof. It follows from the Hille-Yosida theorem and Lemma 5.12 of [4].

Recall that an entire function $f(z)$ is said to be *exponential type* if the inequality

$$| f(z) | \le Ce^{L|z|} \tag{3}$$

holds for some positive constants C and L and all complex values of z. The smallest of constants L is said to be the *exponential type* of $f(z)$ ([6]).

Lemma 2. *If $\Sigma(A, b)$ is exactly controllable in $[0, t_0]$, then for any $x \in H$, there exist entire functions of exponential type $\xi_x(\lambda)$ and $\omega_x(\lambda)$ such that*

$$x = (\lambda - A)\xi_x(\lambda) - b\omega_x(\lambda), \quad \forall \lambda \in \mathbb{C}$$

where both the exponential type of ξ_x and ω_x are at most t_0. Moreover $\xi_x(\cdot) \in H_2(\mathbb{C}^+; H), \omega_x(\cdot) \in H_2(\mathbb{C}^+)$.

Proof. Since $\Sigma(A, b)$ is exactly controllable in $[0, t_0]$, for any $x \in H$, there exists $u_x \in L^2(0, t_0)$ such that

$$x = -\int_0^{t_0} T(-s)bu_x(s)ds.$$

Define operator $B_{t_0} : L^2(0, t_0) \to H$:

$$B_{t_0}u = \int_0^{t_0} T(-s)bu(s)ds. \tag{4}$$

Since b is admissible, B_{t_0} is a linear bounded operator from $L^2(0, t_0)$ to H. Set

$$U = ker(B_{t_0})^\perp. \tag{5}$$

Then B_{t_0} is a 1-1 mapping from U to H: For any $x \in H$, there exists a unique $u_x \in U$ such that $-x = B_{t_0} u_x$.

Now for $x \in H$, define function $\hat{u}_x$ in $L^2_{loc}(0, \infty)$ as

$$\hat{u}_x = \begin{cases} u_x(t), & 0 \le t \le t_0, \\ 0, & t \ge t_0. \end{cases}$$

and function in H:

$$x(t) = T(t)x + \int_0^t T(t-s) b \hat{u}_x(s) ds.$$

Then $x(t)$ is continuous in H and $x(0) = x, x(t) = 0, t \ge t_0$. Define entire functions

$$\xi_x(\lambda) = \int_0^{t_0} e^{-\lambda t} x(t) dt, \quad \omega_x(\lambda) = \int_0^{t_0} e^{-\lambda t} u_x(t) dt. \tag{6}$$

Then under the restriction $u_x \in U$, both ξ_x and ω_x are uniquely determined by $x \in H$. It is obvious that both ξ_x and ω_x are entire functions of exponential type at most t_0. Elementary arguments show that ξ_x, ω_x are functions required. Since $x(t), u_x(t)$ are square integrable functions, it follows from the Paley-Wiener theorem that $\xi_x(\cdot) \in H_2(\mathbb{C}^+; H), \omega_x(\cdot) \in H_2(\mathbb{C}^+)$.

In the representation of $\omega_x(\lambda)$ of (6), the function $-u_x \in U$ and t_0 are nothing but control which drives x into the zero at time t_0. Moreover, The Plancherel's theorem shows that

$$\int_{-\infty}^{\infty} |\omega_x(i\tau)|^2 d\tau = 2\pi \int_0^{t_0} |u_x(t)|^2 dt. \tag{7}$$

And from Theorem 17, and its Remark of [6] from page 96-98, for any separated sequence $\{\lambda_n\}_{n=1}^{\infty}$ (i.e., $\inf_{n \ne m} |\lambda_n - \lambda_m| > 0$), there exists a constant $C > 0$ such that

$$\begin{aligned} \sum_{n=1}^{\infty} |\omega_x(\lambda_n)|^2 &\le C \int_{-\infty}^{\infty} |\omega_x(i\tau)|^2 d\tau \\ &= 2\pi C \int_0^{t_0} |u_x(t)|^2 dt. \end{aligned} \tag{8}$$

Let λ_n be an eigenvalue of A with algebraic multiplicity m_n. We say that $\phi_{n,1}$ is an highest order generalized eigenvector of A if

$$(A - \lambda_n)^{m_n} \phi_{n,1} = 0, \ (A - \lambda_n)^{m_n - 1} \phi_{n,1} \ne 0.$$

3 Equivalent conditions for multiple eigenvalues

First, we introduce some notations. *We always assume that A satisfies parts (i) through (iv) of Theorem 1.* For each eigenvalue λ_n with algebraic multiplicity m_n, let $\phi_{n,1}$ be a highest order generalized eigenvector of A associated with λ_n. Then other linear independent generalized eigenvectors can be found through $\phi_{n,j} = (A - \lambda_n)^{j-1} \phi_{n,1}, j = 2, 3, \cdots, m_n$. Let $\{\{\psi_{n,j}\}_{j=1}^{m_n}\}_{n=1}^{\infty}$ be the bi-orthogonal sequence of $\{\{\phi_{n,j}\}_{j=1}^{m_n}\}_{n=1}^{\infty}$. Then $(A^* - \overline{\lambda_n})\psi_{n,1} = 0, \psi_{n,j} = (A^* - \overline{\lambda_n})\psi_{n,j+1}, j = 2, 3, \cdots, m_n - 1$. We can always make $\{\|\psi_{n,m_n}\|\}$ be uniformly bounded with respect to n. Denote $b_j^n = < b, \psi_{n,j} >$ for each j and n and

$$B_n = \begin{pmatrix} b_1^n & 0 & 0 & 0 & \cdots & 0 \\ b_2^n & b_1^n & 0 & 0 & \cdots & 0 \\ b_3^n & b_2^n & b_1^n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{m_n}^n & b_{m_n-1}^n & b_{m_n-2}^n & b_{m_n-3}^n & \cdots & b_1^n. \end{pmatrix}, n \ge 1. \tag{9}$$

$$\begin{aligned} \Phi_n &= (\phi_{n,1}, \phi_{n,2}, \cdots, \phi_{n,m_n})^T, \\ \Psi_n &= (\psi_{n,1}, \psi_{n,2}, \cdots, \psi_{n,m_n})^T, n \ge 1. \end{aligned} \tag{10}$$

Let $\Psi_n(x)$ denote the scalar vector

$$\begin{aligned} \Psi_n(x) = \ & (< x, \psi_{n,1} >, < x, \psi_{n,2} >, \cdots, \\ & < x, \psi_{n,m_n} >)^T, x \in H, n \ge 1. \end{aligned} \tag{11}$$

Same definition to $\Phi_n(x)$ by replacing $\psi_{n,j}$ in the definition of $\Psi(x)$ by $\phi_{n,j}$.

Theorem 3. *The following conditions are equivalent:*

(i). $\Sigma(A, b)$ is exactly controllable in $[0, t_0]$.

(ii). For every $x \in H$, there exist entire functions of exponential type at most t_0, $\xi_x(\lambda)$ and $\omega_x(\lambda)$, such that

$$x = (\lambda - A)\xi_x(\lambda) - b\omega_x(\lambda), \quad \forall \lambda \in \mathbb{C}$$

where also $\xi_x \in H_2(\mathbb{C}^+; H), \omega_x \in H_2(\mathbb{C}^+)$.

(iii). For every $x \in H$, there exists an entire function of exponential type at most t_0, $\omega_x(\lambda) \in H_2(\mathbb{C}^+)$ such that

$$\begin{aligned} \Omega_x(\lambda_n) = \Bigg(& \omega_x(\lambda_n), \omega_x'(\lambda_n), \frac{\omega_x''(\lambda_n)}{2!}, \cdots, \\ & \frac{\omega_x^{(m_n-1)}(\lambda_n)}{(m_n - 1)!} \Bigg)^T = -B_n^{-1} \Psi_n(x), \forall n \ge 1. \end{aligned} \tag{12}$$

Proof. $(i) \Longleftrightarrow (ii)$ follow from Lemma 2 and Proposition 12.5 of [3].

$(i) \Longrightarrow (iii)$. Let $\omega_x(\lambda)$ be the function defined in (6). ω_x is an entire function of exponential type at most t_0 and belongs to $H_2(\mathbb{C}^+)$. It is found directly that

$$\omega_x^{(k)}(\lambda) = \int_0^{t_0} e^{-\lambda s} (-s)^k u_x(s) ds.$$

For each n, note that $E(\lambda_n, A)b = \sum_{j=1}^{m_n} < b, \psi_{n,j} > \phi_{n,j}$

and hence

$$E(\lambda_n, A)x = -\int_0^{t_0} T(-s)E(\lambda_n, A)bu_x(s)ds$$
$$= -\sum_{k=1}^{m_n}\int_0^{t_0} e^{-\lambda_n s}\frac{(-s)^{k-1}}{(k-1)!}(A-\lambda_n)^{k-1}E(\lambda_n, A)bu_x(s)ds$$
$$= -\sum_{k=1}^{m_n}\sum_{j=k}^{m_n} < b, \psi_{n,j-k+1} > \frac{\omega_x^{(k-1)}(\lambda_n)}{(k-1)!}\phi_{n,j}$$
$$= -\sum_{j=1}^{m_n}\left[\sum_{k=1}^{j} b_{j-k+1}^n \frac{\omega_x^{(k-1)}(\lambda_n)}{(k-1)!}\right]\phi_{n,j}.$$

On the other hand,

$$E(\lambda_n, A)x = \sum_{j=1}^{m_n} < x, \psi_{n,j} > \phi_{n,j}.$$

Comparing these two expressions, we obtain

$$B_n\Omega_x(\lambda_n) = -\Psi_n(x), \forall n \geq 1. \qquad (13)$$

By the exact controllability assumption, $b_1^n \neq 0$. So B_n^{-1} exists, proving the conclusion.

(iii)$\Longrightarrow$(i). Suppose such an entire function of exponential type $\omega_x(\lambda)$ does exist. Then since $\omega_x(\lambda)$ is square integrable along the imaginary axis, by the Paley-Wiener Theorem [Theorem 18, [6], pp.101] there exists an $u_x(s) \in L^2(0, t_0)$ such that

$$\omega_x(\lambda) = \int_0^{t_0} e^{-\lambda s}u_x(s)ds. \qquad (14)$$

For this function u_x, we compute $B_{t_0}u_x + x$ as follows:

$$< B_{t_0}u_x + x, \psi_{n,j} >=$$
$$\int_0^{t_0} < T(-s)E(\lambda_n, A)b, \psi_{n,j} > u_x(s)ds+ < x, \psi_{n,j} >$$
$$= \sum_{k=1}^{m_n}\int_0^{t_0} e^{-\lambda_n s}\frac{(-s)^{k-1}}{(k-1)!}u_x(s)ds < E(\lambda_n, A)b,$$
$$(A^* - \overline{\lambda_n})^{k-1}\psi_{n,j} > + < x, \psi_{n,j} >$$
$$= \sum_{k=1}^{j} b_{j-k+1}^n\frac{\omega_x^{(k-1)}(\lambda_n)}{(k-1)!} + < x, \psi_{n,j} >= 0$$

Since $\{\psi_{n,j} \mid j = 1, 2\cdots m_n\}_{n\geq 1}$ is complete in H, the above implies that $B_{t_0}u_x = -x$. The proof is complete.

Remark 1. *Take $x = 0$ in (12), that is $B_{t_0}u_x = 0$, we have $\Omega_x(\lambda_n) = 0$ for all $n \geq 1$. By (14), we see that*

$$\int_0^{t_0} t^j e^{-\lambda_n t}u_x(t)dt = 0, \ \forall\ 0 \leq j \leq m_n - 1, n \geq 1.$$

Now we make the following assumption: for some $h > 0$

$$\begin{array}{l} 0 < \inf_n Re\lambda_n \leq \sup_n Re\lambda_n \leq h, \forall\ n \geq 1 \\ \inf_{n\neq m}|\lambda_n - \lambda_m| > 0, \quad \sup_{n\geq 1} m_n < \infty. \end{array} \qquad (15)$$

In the sequel, we need some properties of family of exponential functions

$$\begin{array}{rl} \varepsilon & = \{G_n(t)\}_{n=1}^{\infty} = \{(g_{n,1}(t), g_{n,2}(t), \cdots, \\ & g_{n,m_n}(t))^T\}_{n=1}^{\infty},\ g_{n,j}(t) = \frac{(-t)^{j-1}}{(j-1)!}e^{-\lambda_n t}. \end{array} \qquad (16)$$

The following Proposition is key to the proof of our main results of this paper.

Proposition 1. *Under the assumption (15), there exists an $t_0 > 0$ (and hence for all $t > t_0$) such that ε forms a Riesz basis for $\overline{span\,\varepsilon}$ in $L^2(0, t_0)$. In particular, if $F_n(t) = (f_{n,1}(t), f_{n,2}(t), \cdots, f_{n,m_n}(t))^T$ is the bi-orthogonal sequence of ε in $\overline{span\,\varepsilon}$, then for any $u \in \overline{span\,\varepsilon}, u = \sum_{n=1}^{\infty} U_n^T G_n(t), \sum_{n=1}^{\infty}\|U_n\|^2 < \infty$, there are constants C_1 and C_2 so that*

$$C_1\sum_{n=1}^{\infty}\|U_n\|^2 \leq \|\sum_{n=1}^{\infty}U_n^T G_n(t)\|_{L^2(0,t_0)}^2 \leq C_2\sum_{n=1}^{\infty}\|U_n\|^2 \qquad (17)$$

where $U_n = (< u, f_{n,1} >, \cdots, < u, f_{n,m_n} >)^T$.

Proof. Suppose that we have arranged λ_n so that $Im\lambda_{n+1} \geq Im\lambda_n$ for all $n \geq 1$. Let $\mu_{nj} = \lambda_n, 1 \leq j \leq m_n, n \geq 1, \Lambda = \{\mu_{nj} | 1 \leq j \leq m_n, n \geq 1\}$. We use the same notations of [1]:

$$n^+(r) = \sup_{x\in\mathbb{R}}\#\{Im\Lambda \cap [x, x + r)\}, D^+(\Lambda) = \lim_{r\to\infty}\frac{n^+(r)}{r}.$$

Let $\delta = \inf_{n\neq m}|\lambda_n - \lambda_m|$. For any $x \in \mathbb{R}$, suppose there are M number of balls with radius $\delta/2$, which covers the compact region $\Omega(x) = \{|Re\lambda| \leq h, Im\lambda \in [x, x + 1]\}$ of $\mathbb{R}^2$. Note that M is independent of x by unit shift. Then there are at most kM number of Λ inside $\Omega(x), k = \sup_n m_n$. Hence for any $r > 0$, we have

$$n^+(r) = \sup_{x\in\mathbb{R}}\#\{Im\Lambda \cap [x, x + r)\}$$
$$\leq \sup_{x\in\mathbb{R}}\#\{Im\Lambda \cap [x, x + ([r] + 1))\} \leq ([r] + 1)kM$$

where $[r]$ denotes the maximal integer not exceeding r. Therefore, $D^+(\Lambda) \leq kM$. The result then follows from the Theorem 3 of [1] by taking any $t_0 > 2\pi D^+(\Lambda)$.

Remark 2. *Assume that t_0 is taken as that in Proposition 1 which makes $\sum(A, b)$ be exactly controllable in $[0, t_0]$. Then from Remark 1, we have the explicit representation of U defined by (5), $U = \overline{span\,\varepsilon}$.*

Remark 3. *Assume that t_0 is taken as that in Proposition 1 which makes $\sum(A, b)$ be exactly controllable in $[0, t_0]$. Then for any $x \in H$, motivated from (13), we define*

$$u_x(t) = -\sum_{n=1}^{\infty}(B_n^{-1}\Psi_n(x))^T\overline{F_n(t)}.$$

By (12) and (8), $\sum_{n=1}^{\infty}\|B_n^{-1}\Psi_n(x)\|^2 < \infty$ and hence $u_x \in L^2(0,t_0)$. Define

$$\omega_x(\lambda) = \int_0^{t_0} e^{-\lambda t}u_x(t)dt.$$

Then (12) is satisfied. As we mentioned after the proof of Lemma 2 that such a $u_x(t)$ is nothing but the control which drives x into zero.

Remark 4. *Assume that t_0 is taken as that in Proposition 1 which makes $\sum(A,b)$ be exactly controllable in $[0,t_0]$. Let B_{t_0} is defined by (4). Then a direct computation shows that*

$$\begin{array}{l} B_{t_0}(f_{n,1},f_{n,2},\cdots,f_{n,m_n})^T = B_{t_0}F_n(t) = B_n^T\Phi_n, \\ B_{t_0}^*\Psi_n = B_n\overline{G_n(t)},\ n\geq 1. \end{array} \tag{18}$$

Lemma 3. *Assume that $\Sigma(A,b)$ is exactly controllable in $[0,t_0]$ and condition (15) is satisfied. Then*

$$0<\inf_n\Big|\frac{<b,\psi_{n,1}>}{\|\psi_{n,1}\|}\Big| \leq \inf_n\|B_n\| \leq \sup_n\|B_n\| < \infty.$$

Proof. The first inequality comes from [4]. The second one is trivial. For the third inequality, we first show that there exists a $M>0$, such that for any λ_n, it holds

$$\|\frac{(A-\lambda_n)^k}{k!}E(\lambda_n,A)b\| \leq M \quad \forall\ k\geq 0, n\geq 1. \tag{19}$$

Indeed, by Lemma 1,

$$E(\lambda_n,A) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_n|=\varepsilon} R(\lambda,A)d\lambda$$

is uniformly bounded for some small $\epsilon>0$. Since b is admissible, $B_{t_0}u = \int_0^{t_0}T(-s)bu(s)ds$ is bounded from $L^2(0,t_0)$ to H. So is

$$\begin{array}{l} E(\lambda_n,A)B_{t_0}u = \displaystyle\int_0^{t_0} T(-s)E(\lambda_n,A)bu(s)ds \\ = \displaystyle\sum_{j=1}^{m_n}\int_0^{t_0} e^{-\lambda_n s}\frac{(-s)^{j-1}}{(j-1)!}u(s)ds(A-\lambda_n)^{j-1}E(\lambda_n,A)b. \end{array}$$

Set $k=\sup_n m_n$. Since

$$\{1,(-s),(-s)^2,(-s)^3,\cdots,(-s)^{k-1}\}$$

is linearly independent in $L^2(0,t_0)$, there exists its bi-orthogonal sequence

$$\{u_1,u_2,u_3\cdots,u_k\}$$

such that

$$\int_0^{t_0}(-t)^{j-1}u_i(t)dt = \delta_{ij}, \quad 1\leq i,j\leq k.$$

Now, we choose function $u_{n,j}(s) = e^{\lambda_n s}u_j(s)$. It has

$$\|u_{n,j}\|^2 = \int_0^{t_0}|u_{n,j}(s)|^2ds = \int_0^{t_0}|e^{\lambda_n s}u_j(s)|^2ds \leq e^{2t_0h}\|u_j\|^2.$$

Hence

$$\sup_{n,j}\|u_{n,j}\| \leq e^{t_0h}\max_{1\leq j\leq k}\|u_j\| < \infty.$$

Under this group of functions

$$\begin{array}{l} \|\frac{(A-\lambda_n)^{j-1}}{(j-1)!}E(\lambda_n,A)b\| = \|E(\lambda_n,A)B_{t_0}u_{n,j}\| \\ \leq e^{t_0h}\|E(\lambda_n,A)\|\|\|B_{t_0}\|\|\|u_j\| < \infty,\ \forall\ n\geq 1, 1\leq j\leq m_n. \end{array}$$

Next, notice that

$$\|B_n\|^2 \leq m_n\sum_{j=1}^{m_n}|<b,\psi_{n,j}>|^2.$$

We need only show that for any n

$$\sum_{j=1}^{m_n}|<b,\psi_{n,j}>|^2$$

is uniformly bounded. Since

$$(A^*-\overline{\lambda_n})^{m_n-j}\psi_{n,m_n} = \psi_{n,j}$$

we have

$$\sum_{j=1}^{m_n}|<b,\psi_{n,j}>|^2 \leq \sum_{j=1}^{m_n}|<b,(A^*-\overline{\lambda_n})^{m_n-j}\psi_{n,m_n}>|^2$$

$$\leq \sum_{j=1}^{m_n}|<(A-\lambda)^{m_n-j}E(\lambda_n,A)b,\psi_{n,m_n}>|^2$$

$$\leq \|m_n!\psi_{n,m_n}\|^2\sum_{j=1}^{m_n}\|\frac{(A-\lambda_n)^{m_n-j}}{(m_n-j)!}E(\lambda_n,A)b\|^2$$

$$\leq e^{2t_0h}\|m_n!\psi_{n,m_n}\|^2\|E(\lambda_n,A)\|^2\|B_{t_0}\|^2\sum_{j=1}^{m_n}\|u_j\|^2$$

which is uniformly bounded since as we mentioned in the beginning of this section that we can choose $\sup_n\|\psi_{n,m_n}\|$ to be uniformly bounded. The proof is complete.

Theorem 3. *Assume that $\sum(A,b)$ is exactly controllable and the multiplicities of eigenvalues of A have finite upper bound: $\sup m_n<\infty$. Then the following conditions are equivalent:*

(i). $\inf_{n\neq m}|\lambda_n-\lambda_m|>0$.

(ii).$\{\phi_{n,j}\ \mid\ j=1,2,3,\cdots,m_n\}_{n\geq 1}$ *forms a Riesz basis for H. That is, for any $x=\sum_{n=1}^{\infty}\Psi(x)_n^T\Phi_n$, is has*

$$\sum_{n=1}^{\infty}\|\Psi_n(x)\|^2 \leq \|\sum_{n=1}^{\infty}\Psi_n(x)^T\Phi_n\|^2 \leq \sum_{n=1}^{\infty}\|\Psi_n(x)\|^2.$$

(iii). $\{\psi_{n,j} \mid j = 1,2,3,\cdots,m_n\}_{n\geq 1}$ *forms a Riesz basis for* H.

Proof. The equivalent between *(ii)* and *(iii)* is from general basis theory (see e.g. [6]).

(i)$\Longrightarrow$(ii). Take t_0 as in Proposition 1 which makes $\sum(A,b)$ be exactly controllable in $[0,t_0]$. Then B_{t_0} is bounded invertible from U to H. By formulae (4.12) of [2], pp.28,

$$\|B_n^{-1}\| \leq \gamma \frac{\|B_n\|^{m_n-1}}{|\det B_n|} \tag{20}$$

where γ is independent of n. From Lemma 3, there exists $M > 0$ such that $\|B_n^{-1}\| \leq M, \|B_n\| \leq M$ for all $n \geq 1$. Furthermore, by Remark 4, $\Phi_n = (B_n^{-1})^T B_{t_0} F_n(t)$ for all $n \geq 1$. Since $(B_n^{-1})^T B_{t_0}$ is uniformly bounded with respect to n, $\{\Phi_n\}_{n=1}^\infty$ forms a Riesz basis for H.

(ii)$\Longrightarrow$(i). Take t_0 as in Proposition 1 which makes $\sum(A,b)$ is exactly controllable in $[0,t_0]$. Then B_{t_0} is bounded invertible from U to H. From Remark 4, we know that $B_{t_0}(B_n^{-1})^T F_n(t)$ forms a Riesz basis for $\overline{\text{span}\,\varepsilon}$ of $L^2(0,t_0)$. Since from (20), $B_{t_0}(B_n^{-1})^T$ is uniformly bounded with respect to n, $F_n(t)$ forms a Riesz basis for $\overline{\text{span}\,\varepsilon}$ of $L^2(0,t_0)$. Therefore $\{e^{-\lambda_n t}\}_{n=1}^\infty$ forms a Riesz basis for the closed subspace $\overline{\text{span}\,\{e^{-\lambda_n t}\}}$ of $L^2(0,t_0)$. Thus, $\{\lambda_n\}_{n=1}^\infty$ is separated by the necessary condition of Riesz basis for the functions of exponentials.

Our final result of following generalizes Theorems 12.1, 12.2 of [3] to the case of multiple eigenvalues.

Theorem 4. *Assume that (15) is satisfied. Then the following conditions are equivalent:*

(i). $\Sigma(A,b)$ *is exactly controllable.*

(ii). $0 < \inf_n \left|\frac{< b, \psi_{n,1} >}{\|\psi_{n,1}\|}\right| \leq \sup_n \|B_n\| < \infty$ *and*

$$\sum_{n=1}^\infty \|B_n^{-1}\Psi_n(x)\|^2 < \infty, \ \forall\, x \in H$$

where $\|B_n^{-1}\Psi_n(x)\|$ *denotes the Euclidean norm of* $\mathbb{C}^{m_n}$.

(iii). $0 < \inf_n \left|\frac{< b, \psi_{n,1} >}{\|\psi_{n,1}\|}\right| \leq \sup_n \|B_n\| < \infty$ *and* $\{\psi_{n,j} \mid j = 1,2,3,\cdots,m_n\}_{n\geq 1}$ *forms a Riesz basis for* H.

(iv). $0 < \inf_n \left|\frac{< b, \psi_{n,1} >}{\|\psi_{n,1}\|}\right| \leq \sup_n \|B_n\| < \infty$ *and* $\{\phi_{n,j} \mid j = 1,2,3,\cdots,m_n\}_{n\geq 1}$ *forms a Riesz basis for* H.

Proof. The equivalence between (iii) and (iv) is ensured by the general Riesz basis theory (see e.g. [6]).

$(i) \Longrightarrow (ii)$. The first part follows from Lemma 3. The second part follows from (12) of Theorem 2 and (8).

$(ii) \Longrightarrow (i)$. Let t_0 is as in Proposition 1. For any $x \in H$, define control u_x and ω_x as in Remark 3, we see that (12) is satisfied. The result then follows from Theorem 2.

$(i) \Longrightarrow (iii)$. The first part follows from Lemma 3. Again let t_0 be that in Proposition 1 which makes $\sum(A,b)$ be exactly controllable in $[0,t_0]$. Then the second part follows from Proposition 1 and Remark 4.

$(iii) \Longrightarrow (i)$. Take t_0 as that in Proposition 1. Then $\{G_n(t)\}_{n=1}^\infty$ forms a Riesz basis for the closed subspace spanned by $\{g_{n,j}(t), 1 \leq \jmath \leq m_n, n \geq 1\}$ in $L^2(0,t_0)$. Since $\{\Phi_n\}_{n=1}^\infty$ forms a Riesz basis for H, it has

$$\sum_{n=1}^\infty \|\Psi_n(x)\|^2 < \infty.$$

This, together with (20), gives

$$\sum_{n=1}^\infty \|B_n^{-1}\Psi_n(x)\|^2 < \infty.$$

For any $x \in H$, define function

$$u_x(t) = -\sum_{n=1}^\infty (B_n^{-1}\Psi_n(x))^T \overline{F_n(t)}$$

and B_{t_0} as before with respect to t_0 and above defined u_x. A direct computation as before, we obtain

$$E(\lambda_n, A) B_{t_0} u_x = -(\Psi_n(x))^T \Phi_n.$$

Therefore,

$$-x = \sum_{n=1}^\infty -(\Psi_n(x))^T \Phi_n = \sum_{n=1}^\infty E(\lambda_n, A) B_{t_0} u_x = B_{t_0} u_x.$$

That is, $\sum(A,b)$ is exactly controllable. The proof is complete.

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