CONTROLLABILITY OF SWITCHED LINEAR DISCRETE-TIME SYSTEMS WITH MULTIPLE TIME DELAYS

Yijing Wang, Guangming Xie, Long Wang

Center for Systems and Control,
Department of Mechanics and Engineering Science,
Peking University, Beijing 100871, China.
E-mail: *yjwang@mech.pku.edu.cn.*

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Abstract

This paper studies the controllability of switched linear discrete-time systems with multiple time delays in the control function. It is proved that there exists a basic switching sequence such that the controllable state set of this basic switching sequence is equal to that of the system. Based on this fact, a necessary and sufficient geometric criterion for the controllability of such systems is presented.

1 Introduction

Switched systems arise in varied contexts in manufacturing, communication networks, auto pilot design, automotive engine control, computer synchronization, traffic control, chemical processes, and so on. Switched systems are a special class of hybrid dynamical systems which consist of a family of subsystems and a switching law specifying the switching between the subsystems. In recent years, there has been increasing interest in the control of switched systems due to their significance both in theory and applications.

Liberzon and Morse discussed the basic problems in the study of switched systems in [5]. In the analysis and design of switched systems, controllability and reachability are two important issues that have been addressed in several references [2,7,9-12,15,16].

Time-delay phenomena are very common in practical systems, for instance, economic, biological and physiological systems and so on. [1,3,6] studied the controllability for general time-delay systems. Necessary and sufficient conditions for controllability of switched linear continuous-time systems with time-delay in the control function were presented in [13].

In this paper, the controllability problems for switched linear discrete-time systems with multiple time delays in control are formulated and investigated in detail. A necessary and sufficient condition for controllability is derived.

This paper is organized as follows. Section 2 formulates the problem. Section 3 presents some mathematical preliminary results. In Section 4, the controllable state set and the geometric criterion for switched linear discrete-time systems with multiple time delays are discussed in detail. Section 5 concludes the whole paper.

2 Problem formulation

Consider a switched linear discrete-time control system with input delay given by

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) + \sum_{l=1}^{L} D_{r(k),l}u(k-h_l) \quad (1)$$

where $x(k) \in \Re^n$ is the state, $u(k) \in \Re^p$ is the input, and the piecewise constant scalar function $r(k) : \{0, 1, \cdots\} \to \{1, 2, \cdots, N\}$ is the switching signal to be designed. $L < \infty$ is the number of time-delays of the system. Moreover, $r(k) = i$ implies that the pair $(A_i, B_i, D_{i,1}, D_{i,2}, \cdots, D_{i,L})$ is chosen as the system realization. The positive integers $h_L > h_{L-1} > \cdots > h_1$ are the fixed step delay in control, and $h_{l+1} - h_l \geq n$, where $l \in \{1, 2, \cdots, L-1\}$.

Remark 1 *We assume that $h_{l+1} - h_l \geq n$ in order to avoid unnecessary complications, where $l \in \{1, 2, \cdots, L-1\}$. This assumption is reasonable when the sample time is small enough.*

For system (1), a switching sequence is to specify when and to which realization the system should be switched at each instant of sample time.

Definition 1 (Switching Sequence) *A switching sequence π is a set with finite scalars*

$$\pi \stackrel{def}{=} \{i_0, \cdots, i_{M-1}\} \quad (2)$$

where $M < \infty$ is the length of π, and $i_m \in \{1, \cdots, N\}$ is the index of the mth realization $(A_{i_m}, B_{i_m}, D_{i_m,1}, D_{i_m,2}, \cdots, D_{i_m,L})$, for $m \in \underline{M}$.

In this paper we assume that system (1) is reversible, i.e., $\forall\ i = 1, \cdots, N$, A_i is nonsingular. For clarity, for any integer $M > 0$, set $\underline{M} = \{0, 1, \cdots, M-1\}$ and $\underline{\infty} = \{0, 1, \cdots\}$.

Given a switching sequence $\pi = \{i_0, \cdots, i_{M-1}\}$, an associated switching signal $r(m) : \underline{M} \to \{1, \cdots, N\}$ can be determined as

$$r(m) = i_m,\ m \in \underline{M} \quad (3)$$

3 Mathematical Preliminaries

Now, we introduce some mathematical preliminaries as the basic tools for the discussion in the rest of the paper.

Definition 2 (Column Space) [4][8]*Given a matrix $B_{n\times p}$= $[b_1, \cdots, b_p]$, the range of B is defined as*

$$\mathcal{R}(B) \overset{def}{=} \text{span}\{b_1, \cdots, b_p\}$$

Definition 3 (Minimal Invariant Subspace) [4][8]*Given a matrix $A \in \mathcal{R}^{n\times n}$ and a linear subspace $\mathcal{W} \subseteq \Re^n$, the minimal invariant subspace $\langle A|\mathcal{W}\rangle$ is defined as*

$$\langle A|\mathcal{W}\rangle \overset{def}{=} \sum_{i=1}^{n} A^{i-1}\mathcal{W}$$

For clarity, let $\langle A|B\rangle = \langle A|\mathcal{R}(B)\rangle$, where $A \in \Re^{n\times n}$ and $B \in \Re^{n\times p}$.

Definition 4 (Generalized Invariant Subspace) *Given matrices $A_1, \cdots, A_N \in \Re^{n\times n}$ and $B_1, \cdots, B_N \in \Re^{n\times p}$, the generalized invariant subspace $\langle A_1|B_1 + \cdots + A_N|B_N\rangle$ is defined as*

$$\langle A_1|B_1 + \cdots + A_N|B_N\rangle \overset{def}{=} \sum_{i=0}^{\infty} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N)$$

Especially, $\langle A_1|B_1\rangle \overset{def}{=} \sum_{i=0}^{\infty} \mathcal{R}(A_1^i B_1)$.

Lemma 1 [13]*Given matrices $A_1, \cdots, A_N \in \Re^{n\times n}$ and $B_1, \cdots, B_N \in \Re^{n\times p}$,*

$$\langle A_1|B_1 + \cdots + A_N|B_N\rangle = \sum_{i=0}^{Nn-1} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N)$$

Especially, $\langle A_1|B_1\rangle = \sum_{i=0}^{n-1} \mathcal{R}(A_1^i B_1)$.

Remark 2 *The subspaces $\langle A_1|B_1 + \cdots + A_N|B_N\rangle$ and $\langle A_1|B_1\rangle + \cdots + \langle A_N|B_N\rangle$ are quite different. It is easy to see that $\langle A_1|B_1 + \cdots + A_N|B_N\rangle \subseteq \langle A_1|B_1\rangle + \cdots + \langle A_N|B_N\rangle$.*

Lemma 2 [13] *Given matrices $A_1, A_2 \in \Re^{n\times n}$, $B_1, B_2 \in \Re^{n\times p}$, we have*

$$\langle A_1|B_1 + A_2|B_2\rangle + \langle A_2|B_2\rangle = \langle A_1|B_1\rangle + \langle A_2|B_2\rangle$$

Lemma 3 [11] *For any matrix $A \in R^{n\times n}$, there must exist $T > 0$, such that for any linear subspace $\mathcal{W} \subseteq \Re^n$, the following equation holds*

$$\langle A|\mathcal{W}\rangle = \langle \exp(-AT)|\mathcal{W}\rangle$$

Lemma 4 *Given a nonsingular matrix $A \in \Re^{n\times n}$, for any subspace $\mathcal{W} \subseteq \Re^n$ and any sufficiently large positive integer k, it follows that*

$$\langle A^k|\mathcal{W}\rangle = \langle A|\mathcal{W}\rangle$$

Proof: Since the matrix A is nonsingular, by Theorem 7.6.1 in [4], there must exist $X \in \Re^n$ such that $exp(X) = A$. It is easy to verify that

$$\langle A|\mathcal{W}\rangle \subseteq \langle X|\mathcal{W}\rangle$$

By Lemma 3, for X and sufficiently large positive integer k, we have $\langle exp(Xk)|\mathcal{W}\rangle = \langle X|\mathcal{W}\rangle$. That is

$$\langle A^k|\mathcal{W}\rangle = \langle X|\mathcal{W}\rangle$$

Thus

$$\langle A^k|\mathcal{W}\rangle \supseteq \langle A|\mathcal{W}\rangle$$

On the other hand, it is obvious that

$$\langle A^k|\mathcal{W}\rangle \subseteq \langle A|\mathcal{W}\rangle$$

Therefore $\langle A^k|\mathcal{W}\rangle = \langle A|\mathcal{W}\rangle$.

4 Controllability

4.1 Controllable State Set

The purpose of this subsection is to introduce the concept of controllable state set for a given switching sequence and reveal its characteristics.

Definition 5 (State Controllability) *For system (1), given initial state x_0 and initial inputs $u_0(-h_L), u_0(-h_L + 1), \cdots, u_0(-1)$, the state x_f is said to be $(x_0, u_0) - controllable$, if there exist a positive integer $M > 0$, a switching signal $r(m) : \underline{M} \to \{1, \cdots, N\}$, and inputs $u(m) : \underline{M} \to \Re^p$, such that $x(0) = x_0$ and $x(M) = x_f$.*

Definition 6 (System Controllability) *System (1) is said to be completely controllable, if for any initial state x_0 and initial inputs $u_0(-h_L), u_0(-h_L + 1), \cdots, u_0(-1)$, any state x_f is $(x_0, u_0) - controllable$.*

Definition 7 (Controllable State Set) *Given a switching sequence $\pi = \{i_0, \cdots, i_{M-1}\}$, for initial state x_0 and initial inputs $u_0(-h_L), u_0(-h_L + 1), \cdots, u_0(-1)$, the set of all the states starting from x_0 and u_0, evolving through π is defined as the controllable state set of the switching sequence π, denoted by $\mathcal{C}(x_0, u_0, \pi)$.*

Given initial state x_0 and initial inputs $u_0(-h_L), u_0(-h_L + 1), \cdots, u_0(-1)$, the system state evolving through the switching sequence $\pi = \{i_0, \cdots, i_{M-1}\}$ can be represented as

$$\begin{aligned} x(M) = & \prod_{j=M-1}^{0} A_{i_j} x_0 \\ & + \sum_{m=0}^{M-2} \prod_{j=M-1}^{m+1} A_{i_j} \left[B_{i_m} u(m) + \sum_{l=1}^{L} D_{i_m,l} u(m - h_l) \right] \\ & + B_{i_{M-1}} u(M-1) + \sum_{l=1}^{L} D_{i_{M-1},l} u(M - 1 - h_l) \end{aligned} \tag{4}$$

where $M \geq h_L$.

For simplicity, here we only consider the case $L = 2$. For $L \geq 2$, the process is similar, even though the final expression is very complex.

When $L = 2$, Equation (4) can be rewritten as

$$
\begin{aligned}
x(M) = & \prod_{j=M-1}^{0} A_{i_j} x_0 + \sum_{m=-h_2}^{-h_1-1} \prod_{j=M-1}^{m+h_2+1} A_{i_j} D_{i_{m+h_2},2} u_{01}(m) \\
& + \sum_{m=-h_1}^{-1} \left[\prod_{j=M-1}^{m+h_1+1} A_{i_j} D_{i_{m+h_1},1} \right. \\
& \qquad \left. + \prod_{j=M-1}^{m+h_2+1} A_{i_j} D_{i_{m+h_2},2} \right] u_{02}(m) \\
& + \sum_{m=0}^{M-h_2-2} \left[\prod_{j=M-1}^{m+1} A_{i_j} B_{i_m} + \prod_{j=M-1}^{m+h_1+1} A_{i_j} D_{i_{m+h_1},1} \right. \\
& \qquad \left. + \prod_{j=M-1}^{m+h_2+1} A_{i_j} D_{i_{m+h_2},2} \right] u(m) \\
& + \left[\prod_{j=M-1}^{M-h_2} A_{i_j} B_{i_{M-h_2-1}} + \prod_{j=M-1}^{M-h_2+h_1} A_{i_j} D_{i_{M-h_2+h_1-1},1} \right. \\
& \qquad \left. + D_{i_{M-1},2} \right] u(M-h_2-1) \\
& + \sum_{m=M-h_2}^{M-h_1-2} \left[\prod_{j=M-1}^{m+1} A_{i_j} B_{i_m} + \prod_{j=M-1}^{m+h_1+1} A_{i_j} D_{i_{m+h_1},1} \right] u(m) \\
& + \left[\prod_{j=M-1}^{M-h_1} A_{i_j} B_{i_{M-h_1-1}} + D_{i_{M-1},1} \right] u(M-h_1-1) \\
& + \sum_{m=M-h_1}^{M-2} \prod_{j=M-1}^{m+1} A_{i_j} B_{i_m} u(m) \\
& + B_{i_{M-1}} u(M-1)
\end{aligned} \tag{5}
$$

Thus, we have

$$
\begin{aligned}
& \mathcal{C}(x_0, u_0, \pi) \\
& = \mathcal{I}n(x_0, u_0, \pi) \\
& + \mathcal{R}\Bigg(\Bigg[\prod_{j=M-1}^{1} A_{i_j} B_{i_0} + \prod_{j=M-1}^{h_1+1} A_{i_j} D_{i_{h_1},1} + \prod_{j=M-1}^{h_2+1} A_{i_j} D_{i_{h_2},2} \ , \\
& \cdots, \\
& \prod_{j=M-1}^{M-h_2-1} A_{i_j} B_{i_{M-h_2-2}} + \prod_{j=M-1}^{M-h_2+h_1-1} A_{i_j} D_{i_{M-h_2+h_1-2},1} \\
& \quad + A_{i_{M-1}} D_{i_{M-2},2} \ , \\
& \prod_{j=M-1}^{M-h_2} A_{i_j} B_{i_{M-h_2-1}} + \prod_{j=M-1}^{M-h_2+h_1} A_{i_j} D_{i_{M-h_2+h_1-1},1} + D_{i_{M-1},2} \ , \\
& \prod_{j=M-1}^{M-h_2+1} A_{i_j} B_{i_{M-h_1-2}} + \prod_{j=M-1}^{M-h_2+h_1+1} A_{i_j} D_{i_{M-h_2+h_1},1} \ , \\
& \cdots, \prod_{j=M-1}^{M-h_1-1} A_{i_j} B_{i_{M-h_1-2}} + A_{i_{M-1}} D_{i_{M-2},1} \ , \\
& \prod_{j=M-1}^{M-h_1} A_{i_j} B_{i_{M-h_1-1}} + D_{i_{M-1},1} \ , \prod_{j=M-1}^{M-h_1+1} A_{i_j} B_{i_{M-h_1}} \ , \\
& \cdots, A_{i_{M-1}} B_{i_{M-2}} \ , B_{i_{M-1}} \Bigg] \Bigg)
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
\mathcal{I}n(x_0, u_0, \pi) = & \prod_{j=M-1}^{0} A_{i_j} x_0 \\
& + \sum_{m=-h_2}^{-h_1-1} \prod_{j=M-1}^{m+h_2+1} A_{i_j} D_{i_{m+h_2},2} u_{01}(m) \\
& + \sum_{m=-h_1}^{-1} \left[\prod_{j=M-1}^{m+h_1+1} A_{i_j} D_{i_{m+h_1},1} \right. \\
& \qquad \left. + \prod_{j=M-1}^{m+h_2+1} A_{i_j} D_{i_{m+h_2},2} \right] u_{02}(m)
\end{aligned} \tag{7}
$$

In particular, when $x_0 = 0$, $u_0 = 0$, denote

$$
\mathcal{C}(0, 0, \pi) = \mathcal{C}(\pi) \tag{8}
$$

for clarity in the following discussion. Obviously, $\mathcal{C}(\pi)$ is a linear subspace in $\Re^n$.

The above analysis is summarized in the following propositions.

Proposition 1 *Given a switching sequence $\pi = \{i_0, i_1, \cdots, i_{M-1}\}$, the controllable state set $\mathcal{C}(\pi)$ is a linear space.*

Proposition 2 *Given a switching sequence* $\pi = \{ \overbrace{i, \cdots\cdots, i}^{(n+h_L)\ \text{times}} \}$*, we have*

$$
\mathcal{C}(\pi) = \langle A_i | [B_i,\ D_{i,1},\ D_{i,2},\ \cdots,\ D_{i,L}] \rangle \tag{9}
$$

Proof: See Appendix A.

Next, we define two operations of the switching sequences and discuss the associated controllable state sets.

Definition 8 (Concatenation of Switching Sequences) *Given two switching sequences $\pi_1 = \{i_0, \cdots, i_{M-1}\}$ and $\pi_2 = \{j_0, \cdots, j_{L-1}\}$, the concatenation of π_1 and π_2 is defined as*

$$
\pi_1 \wedge \pi_2 \stackrel{def}{=} \{i_0, \cdots, i_{M-1}, j_0, \cdots, j_{L-1}\} \tag{10}
$$

Since it is easy to verify that $(\pi_1 \wedge \pi_2) \wedge \pi_3 = \pi_1 \wedge (\pi_2 \wedge \pi_3)$, we just denote it by $\pi_1 \wedge \pi_2 \wedge \pi_3$.

Definition 9 (Power of a switching sequence) *Given a switching sequence π, the nth power of π is defined as*

$$
\pi^{\wedge n} \stackrel{def}{=} \overbrace{\pi \wedge \cdots \wedge \pi}^{n\ \text{times}} \tag{11}
$$

Denote Π_0 the set of all the switching sequences of the form

$$
\pi = \{\overbrace{i, \cdots, i}^{k\ \text{times}}\}
$$

where $k \geq n + h_L$. Denote Π_b the set of all the switching sequences which are generated by finite times of concatenation or power operations of some switching sequences in Π_0.

Given a switching sequence $\pi = \{i_0, \cdots, i_{M-1}\} \in \Pi_b$, denote

$$A_\pi = \prod_{m=M-1}^{0} A_{i_m} \tag{12}$$

Theorem 1 *Given switching sequences $\pi_1, \pi_2 \in \Pi_b$, we have*

$$\mathcal{C}(\pi_1 \wedge \pi_2) = A_{\pi_2}\mathcal{C}(\pi_1) + \mathcal{C}(\pi_2) \tag{13}$$

Proof: See Appendix B.

Theorem 2 *Given a switching sequence $\pi \in \Pi_b$, we have*

$$\mathcal{C}(\pi^{\wedge n}) = \langle A_\pi | \mathcal{C}(\pi)\rangle \tag{14}$$

Proof: By Theorem 1, it follows that $\mathcal{C}(\pi^{\wedge n}) = A_\pi \mathcal{C}(\pi^{\wedge(n-1)}) + \mathcal{C}(\pi) = \cdots = \sum_{j=1}^{n} (A_\pi)^{j-1}\mathcal{C}(\pi) = \langle A_\pi|\mathcal{C}(\pi)\rangle$.

Corollary 1 *For any switching sequence $\pi \in \Pi_b$, we have*

$$A_{\pi^{\wedge n}}\mathcal{C}(\pi^{\wedge n}) = \mathcal{C}(\pi^{\wedge n}) \tag{15}$$

Proof: By Theorem 2 and by the property of invariant subspace, we have

$$A_{\pi^{\wedge n}}\mathcal{C}(\pi^{\wedge n}) = (A_\pi)^n \langle A_\pi|\mathcal{C}(\pi)\rangle \subseteq \langle A_\pi|\mathcal{C}(\pi)\rangle \tag{16}$$

Since A_π is reversible, we have

$$dim((A_\pi)^n \langle A_\pi|\mathcal{C}(\pi)\rangle) = dim(\langle A_\pi|\mathcal{C}(\pi)\rangle) \tag{17}$$

It follows that $A_{\pi^{\wedge n}}\mathcal{C}(\pi^{\wedge n}) = \langle A_\pi|\mathcal{C}(\pi)\rangle = \mathcal{C}(\pi^{\wedge n})$.

4.2 Geometric Criterion

For system (1), we define a subspace sequence as follows

$$\begin{aligned} \mathcal{W}_1 &= \sum_{i=1}^{N} \langle A_i|[B_i, D_{i,1}, D_{i,2}, \cdots, D_{i,L}]\rangle, \\ \mathcal{W}_2 &= \sum_{i=1}^{N} \langle A_i|\mathcal{W}_1\rangle, \\ &\cdots\cdots, \\ \mathcal{W}_n &= \sum_{i=1}^{N} \langle A_i|\mathcal{W}_{n-1}\rangle \end{aligned} \tag{18}$$

If there exists a positive integer m such that $\mathcal{W}_m = \mathcal{W}_{m+1}$, by the above definition, we have $\mathcal{W}_{m+1} = \mathcal{W}_{m+2} = \cdots$. Since $0 \le dim(\mathcal{W}_1) \le \cdots \le dim(\mathcal{W}_n) \le n$, we have $\mathcal{W}_n = \mathcal{W}_{n+1} = \cdots$. This implies that $\mathcal{W}_{n+j} \subseteq \mathcal{W}_n$, for $j = 1, 2, \cdots$.

It is easy to see that for any switching sequence π, $\mathcal{C}(\pi) \subseteq \mathcal{W}_n$.

Theorem 3 *For system (1), there exists a basic switching sequence $\pi_b \in \Pi_b$, such that $\mathcal{C}(\pi_b) = \mathcal{W}_n$.*

Proof: By Lemma 4, for each A_i, there exists $k_i \ge n + h_L$ such that $\langle A_i|\mathcal{W}\rangle = \langle A_i^{k_i}|\mathcal{W}\rangle$ for any linear space $\mathcal{W}$. Thus, we can redefined (18) as follows

$$\begin{aligned} \mathcal{W}_1 &= \sum_{i=1}^{N} \langle A_i|[B_i, D_{i,1}, D_{i,2}, \cdots, D_{i,L}]\rangle, \\ \mathcal{W}_2 &= \sum_{i=1}^{N} \langle A_i^{k_i}|\mathcal{W}_1\rangle, \\ &\cdots\cdots, \\ \mathcal{W}_n &= \sum_{i=1}^{N} \langle A_i^{k_i}|\mathcal{W}_{n-1}\rangle \end{aligned} \tag{19}$$

Suppose $\dim(\mathcal{W}_n) = d$. By (19), there exist subspaces $\mathcal{V}_1, \cdots, \mathcal{V}_d$ such that

$$\mathcal{W}_n = \sum_{m=1}^{d} \mathcal{V}_m$$

and each $\mathcal{V}_m$ has the following form

$$\prod_{m=M}^{1} A_{i_m}^{k_{i_m}} \langle A_j|[B_j, D_{j,1}, D_{j,2}, \cdots, D_{j,L}]\rangle \tag{20}$$

where $i_1, \cdots, i_M, j \in \{1, \cdots, N\}, 0 < M < \infty$.

Consider the subspace which has the form (20), we can choose a switching sequence

$$\pi = \{\overbrace{j, \cdots, j}^{k_j}, \overbrace{i_1, \cdots, i_1}^{k_{i_1}}, \cdots, \overbrace{i_M, \cdots, i_M}^{k_{i_M}}\} \tag{21}$$

such that

$$\prod_{m=M}^{1} A_{i_m}^{k_{i_m}} \langle A_j|[B_j, D_{j,1}, D_{j,2}, \cdots, D_{j,L}]\rangle \subseteq \mathcal{C}(\pi)$$

Thus, we can choose switching sequences $\pi_1, \cdots, \pi_d$ such that $\mathcal{V}_m \subseteq \mathcal{C}(\pi_m)$ for $m = 1, \cdots, d$. Then we have

$$\mathcal{W}_n = \sum_{m=1}^{d} \mathcal{C}(\pi_m) \tag{22}$$

Now let us construct the switching sequence π_b as follows.

First, if $\mathcal{C}(\pi_1^{\wedge n}) = \mathcal{W}_n$, we can take $\pi_b = \pi_1^{\wedge n}$. If not, there must exist $k \in \{2, \cdots, d\}$ (without loss of generality, let $k = 2$) such that

$$\mathcal{C}(\pi_2) \not\subseteq \mathcal{C}(\pi_1^{\wedge n})$$

Since

$$\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n}) = A_{\pi_1^{\wedge n}}\mathcal{C}(\pi_2) + \mathcal{C}(\pi_1^{\wedge n})$$

By Corollary 1, we have

$$\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n}) = A_{\pi_1^{\wedge n}}(\mathcal{C}(\pi_2) + \mathcal{C}(\pi_1^{\wedge n}))$$

This implies that

$$\begin{aligned} \dim(\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n})) &= \dim(\mathcal{C}(\pi_2)) + \mathcal{C}(\pi_1^{\wedge n})) \\ &\ge 1 + \dim(\mathcal{C}(\pi_1^{\wedge n})) \\ &= 2 \end{aligned}$$

Thus, we construct switching sequences

$$\begin{aligned}
&\overline{\pi}_1 = \pi_1, \\
&\overline{\pi}_2 = \pi_2 \wedge \overline{\pi}_1^{\wedge n}, \\
&\cdots, \\
&\overline{\pi}_d = \pi_d \wedge (\overline{\pi}_{d-1})^{\wedge n},
\end{aligned}$$

and

$$\pi_b = \overline{\pi}_d.$$

It is easy to prove that

$$\dim(\mathcal{C}(\pi_b)) \geq d$$

Thus, $\mathcal{C}(\pi_b) = \mathcal{W}_n$.

Corollary 2 *System (1) is controllable if and only if*

$$\mathcal{W}_n = \Re^n \tag{23}$$

Remark 3 [14] *considers a switched linear discrete-time systems with a single time-delay in control described as follows*

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) + D_{r(k)}u(k-h) \tag{24}$$

and presents a necessary and sufficient condition for the controllability of such systems as follows

$$\begin{aligned}
\mathcal{V}_1 &= \sum_{i=1}^{N} \langle A_i | [B_i, D_i] \rangle, \\
\mathcal{V}_2 &= \sum_{i=1}^{N} \langle A_i | \mathcal{V}_1 \rangle, \\
& \cdots\cdots, \\
\mathcal{V}_n &= \sum_{i=1}^{N} \langle A_i | \mathcal{V}_{n-1} \rangle
\end{aligned} \tag{25}$$

If let $L = 1$ in equation (18), we get $\mathcal{W}_n = \mathcal{V}_n$. Therefore, it is a special case of our results.

Remark 4 *We not only prove the existence of the basic switching sequence, but also provide a construction method. By the proof of Theorem 3, it is easy to see that π_b is not unique. One reason is that $k_i \geq n + h_L$ is not unique, and another reason is that the subspaces $\mathcal{V}_1, \cdots, \mathcal{V}_d$ are not unique. For system (1), we can use only one basic switching sequence to realize the controllability.*

5 Conclusion

This paper has studied the controllability of switched linear discrete-time systems with multiple time delays in control. The concept of controllable state set has been introduced as the basic tool. We have proved that there exists a basic switching sequence such that its controllable state set is exactly the controllable state set of the system. In addition, a necessary and sufficient condition for the controllability of such systems has been presented.

Appendix A

Proof: We still consider the case $L = 2$, since other cases are similar. By (8), we have

$$\begin{aligned}
\mathcal{C}(\pi) = {} & \mathcal{R}([A_i^{n+h_2-1}B_i + A_i^{n+h_2-h_1-1}D_{i,1} + A_i^{n-1}D_{i,2}, \\
& \cdots,\ A_i^{h_2+1}B_i + A_i^{h_2-h_1+1}D_{i,1} + A_iD_{i,2}, \\
& A_i^{h_2}B_i + A_i^{h_2-h_1}D_{i,1} + D_{i,2}, \\
& A_i^{h_2-1}B_i + A_i^{h_2-h_1-1}D_{i,1},\ \cdots, \\
& A_i^{h_1+1}B_i + A_iD_{i,1},\quad A_i^{h_1}B_i + D_{i,1}, \\
& A_i^{h_1-1}B_i,\ \cdots,\ A_iB_i,\quad B_i]) \\
= {} & \mathcal{R}\Bigg(\Bigg[A_i^{n-1}[A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2}],\ \cdots, \\
& A_i[A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2}], \\
& A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2}, \\
& A_i^{h_2-h_1-1}(A_i^{h_1}B_i + D_{i,1}),\ \cdots, \\
& A_i(A_i^{h_1}B_i + D_{i,1}),\quad A_i^{h_1}B_i + D_{i,1}, \\
& A_i^{h_1-1}B_i,\ \cdots,\ A_iB_i,\quad B_i\Bigg]\Bigg)
\end{aligned} \tag{26}$$

Since $h_1 - 1 \geq n - 1$ and $h_2 - h_1 - 1 \geq n - 1$, by the definition of minimal invariant subspace and by Lemma 2, it is easy to see that

$$\begin{aligned}
\mathcal{C}(\pi) = {} & \langle A_i | A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2} \rangle \\
& + \langle A_i | A_i^{h_1}B_i + D_{i,1} \rangle + \langle A_i | B_i \rangle \\
= {} & \langle A_i | A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2} \rangle \\
& + \langle A_i | A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) \rangle + \langle A_i | B_i \rangle \\
= {} & \langle A_i | A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) \rangle + \langle A_i | D_{i,2} \rangle + \langle A_i | B_i \rangle \\
= {} & \langle A_i | A_i^{h_1}B_i + D_{i,1} \rangle + \langle A_i | D_{i,2} \rangle + \langle A_i | B_i \rangle \\
= {} & \langle A_i | A_i^{h_1}B_i + D_{i,1} \rangle + \langle A_i | D_{i,2} \rangle + \langle A_i | A_i^{h_1}B_i \rangle \\
= {} & \langle A_i | A_i^{h_1}B_i \rangle + \langle A_i | D_{i,1} \rangle + \langle A_i | D_{i,2} \rangle \\
= {} & \langle A_i | B_i \rangle + \langle A_i | D_{i,1} \rangle + \langle A_i | D_{i,2} \rangle \\
= {} & \langle A_i | [B_i,\ D_{i,1}\ D_{i,2}] \rangle
\end{aligned} \tag{27}$$

On the analogy of this process, for $L \geq 2$, we have

$$\begin{aligned}
\mathcal{C}(\pi) = {} & \langle A_i | A_i^{h_L-h_{L-1}}[A_i^{h_{L-1}-h_{L-2}}[\cdots[A_i^{h_2-h_1}(A_i^{h_1}B_i \\
& + D_{i,1}) + D_{i,2}] + \cdots] + D_{i,L-1}] + D_{i,L} \rangle \\
& + \langle A_i | A_i^{h_{L-1}-h_{L-2}}[A_i^{h_{L-2}-h_{L-3}}[\cdots[A_i^{h_2-h_1}(A_i^{h_1}B_i \\
& + D_{i,1}) + D_{i,2}] + \cdots] + D_{i,L-2}] + D_{i,L-1} \rangle \\
& + \cdots \\
& + \langle A_i | A_i^{h_2-h_1}(A_i^{h_1}B_i + D_{i,1}) + D_{i,2} \rangle \\
& + \langle A_i | A_i^{h_1}B_i + D_{i,1} \rangle + \langle A_i | B_i \rangle \\
= {} & \langle A_i | B_i \rangle + \langle A_i | D_{i,1} \rangle + \langle A_i | D_{i,2} \rangle + \cdots + \langle A_i | D_{i,L} \rangle \\
= {} & \langle A_i | [B_i,\ D_{i,1},\ D_{i,2},\ \cdots,\ D_{i,L}] \rangle
\end{aligned} \tag{28}$$

Appendix B

Proof: We only consider the case π_1, $\pi_2 \in \Pi_0$, as the case π_1, $\pi_2 \in \Pi_b$ is similar. Suppose that

$$\pi_1 = \{\overbrace{i_1, \cdots, i_1}^{k_1 \text{ times}}\}, \quad \pi_2 = \{\overbrace{i_2, \cdots, i_2}^{k_2 \text{ times}}\}$$

where $k_1,\ k_2 \geq n + h_L$.

By (8) and by Proposition 2, we have

$$
\begin{aligned}
&\mathcal{C}(\pi_1 \wedge \pi_2)\\
&= \mathcal{R}\Bigg(\Bigg[A_{i_2}^{k_2} A_{i_1}^{k_1-h_2-1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\ \cdots\ ,\\
&\quad A_{i_2}^{k_2} A_{i_1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1-1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})]+A_{i_2}^{k_2-1}D_{i_2,2}\ ,\ \cdots\ ,\\
&\quad A_{i_2}^{k_2}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+A_{i_2}^{k_2-h_2+h_1}D_{i_2,2}\ ,\\
&\quad A_{i_2}^{k_2}A_{i_1}^{h_1-1}B_{i_1}+A_{i_2}^{k_2-1}D_{i_2,1}+A_{i_2}^{k_2-h_2+h_1-1}D_{i_2,2}\ ,\ \cdots\ ,\\
&\quad A_{i_2}^{k_2}B_{i_1}+A_{i_2}^{k_2-h_1}D_{i_2,1}+A_{i_2}^{k_2-h_2}D_{i_2,2}\ ,\\
&\quad A_{i_2}^{k_1-h_2-1}[A_{i_2}^{h_2-h_1}(A_{i_2}^{h_1}B_{i_2}+D_{i_2,1})+D_{i_2,2}]\ ,\ \cdots\ ,\\
&\quad A_{i_2}[A_{i_2}^{h_2-h_1}(A_{i_2}^{h_1}B_{i_2}+D_{i_2,1})+D_{i_2,2}]\ ,\\
&\quad A_{i_2}^{h_2-h_1}(A_{i_2}^{h_1}B_{i_2}+D_{i_2,1})+D_{i_2,2}\ ,\\
&\quad A_{i_2}^{h_2-h_1-1}(A_{i_2}^{h_1}B_{i_2}+D_{i_2,1})\ ,\ \cdots\ ,\ A_{i_2}(A_{i_2}^{h_1}B_{i_2}+D_{i_2,1})\ ,\\
&\quad A_{i_2}^{h_1}B_{i_2}+D_{i_2,1}\ , A_{i_2}^{h_1-1}B_{i_2}\ ,\ \cdots\ ,\ A_{i_2}B_{i_2}\ ,\ B_{i_2}\Bigg]\Bigg)\\
&= \mathcal{R}\Bigg(\Bigg[A_{i_2}^{k_2} A_{i_1}^{k_1-h_2-1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad \cdots\ ,\ A_{i_2}^{k_2} A_{i_1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\Bigg]\Bigg)\\
&\quad +\mathcal{R}\Bigg(\Bigg[A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1-1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})]+A_{i_2}^{k_2-1}D_{i_2,2}\ ,\\
&\quad \cdots\ ,\ A_{i_2}^{k_2}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+A_{i_2}^{k_2-h_2+h_1}D_{i_2,2}\ ,\\
&\quad A_{i_2}^{k_2}A_{i_1}^{h_1-1}B_{i_1}+A_{i_2}^{k_2-1}D_{i_2,1}+A_{i_2}^{k_2-h_2+h_1-1}D_{i_2,2}\ ,\\
&\quad \cdots\ ,\ A_{i_2}^{k_2}B_{i_1}+A_{i_2}^{k_2-h_1}D_{i_2,1}+A_{i_2}^{k_2-h_2}D_{i_2,2}\Bigg]\Bigg)\\
&\quad +\langle A_{i_2}|[B_{i_2},\ D_{i_2,1}\ D_{i_2,2}]\rangle
\end{aligned}
\tag{29}
$$

For (29), eliminating all the correlation terms between the second part and the third part in the linear space, we have

$$
\begin{aligned}
&\mathcal{C}(\pi_1 \wedge \pi_2)\\
&= \mathcal{R}\Bigg(\Bigg[A_{i_2}^{k_2} A_{i_1}^{k_1-h_2-1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad \cdots\ ,\ A_{i_2}^{k_2} A_{i_1}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\ ,\\
&\quad A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})+D_{i_1,2}]\Bigg]\Bigg)\\
&\quad +\mathcal{R}\Bigg(\Bigg[A_{i_2}^{k_2}[A_{i_1}^{h_2-h_1-1}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})]\ ,\ \cdots\ ,\\
&\quad A_{i_2}^{k_2}(A_{i_1}^{h_1}B_{i_1}+D_{i_1,1})\ , A_{i_2}^{k_2}A_{i_1}^{h_1-1}B_{i_1}\ ,\ \cdots\ , A_{i_2}^{k_2}B_{i_1}\Bigg]\Bigg)\\
&\quad +\langle A_{i_2}|[B_{i_2},\ D_{i_2,1}\ D_{i_2,2}]\rangle\\
&= A_{i_2}^{k_2}\langle A_{i_1}|[B_{i_1}, D_{i_1,1}, D_{i_1,2}]\rangle + \langle A_{i_2}|[B_{i_2}, D_{i_2,1}, D_{i_2,2}]\rangle\\
&= A_{\pi_2}\mathcal{C}(\pi_1)+\mathcal{C}(\pi_2)
\end{aligned}
\tag{30}
$$

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