DISCRETE-TIME STABILITY OF HYBRID SYSTEMS MODELED BY LINEAR IMPULSIVE SYSTEMS

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Abstract

This paper considers stability of hybrid systems described by linear impulsive systems. Stability conditions are derived by transforming the linear impulsive systems to discrete dynamical systems at the occurrences of impulsive effects on the linear systems. This transformation leads to a discrete time system driven by forcing inputs that represent the impulsive effects.

1 Introduction

Hybrid systems can be found in physical systems which are coupled with digital controllers or subsystems modeled as finite automaton. In particular, a hybrid system arises wherever there is a mixture of continuous control laws and logical decision processes. Examples of hybrid control systems are found in automotive engine control, automated highway systems, flexible manufacturing, chemical process control, electric power distribution and computer communication networks. To illustrate, some examples of hybrid control systems are discussed in [1, 2].

Hybrid systems are used for modeling and analyzing systems which have interacting continuous-valued and discrete-valued state variables. The continuous state variable may be the value of the state in continuous time, discrete time or a mixture of the two. The mathematical model of the continuous state is described by a differential or difference equation. The discrete state variable is generally represented by a finite state digital automaton or an input/output transition system. The behaviour of the hybrid system is influenced by state variables which interact at an *event* (or *trigger*) time which occurs whenever the evolution of the system satisfies a particular condition which then initiates changes in the state variables.

Hybrid control systems are control systems where both plant and controller consist of continuous and discrete state variables. Recently, a common framework used for a hybrid control has been developed by separating the system components into three sub components; namely, a plant with a conventional controller, a discrete state variable controller and an interface in the three layered configuration [2]. The plant and the conventional controller are usually modeled by differential or difference equations. The discrete state variable controller is designed via a rule based decision process which supervises the conventional controller. This interface facilitates communication between the plant and the discrete controller and simultaneously converts the continuous state to discrete state variables (C/D), and vice versa (D/C).

Physical systems are often subjected to disturbances, changing operation conditions and component failures, and in many cases, the changes take place in a short space of time. Examples are found for example in biological systems and mechanical systems subjected to shock. Such systems can be modeled by differential and/or difference equations which jump instantaneously from one state to another. If there is no jump over some time interval, then the mathematical model is described by the solution of a differential and/or difference equation. The analysis of an instantaneous change in the state of a system is much more complicated. Mathematical models of systems that undergo instantaneous changes in the state are called *impulsive systems*.

There are two main approaches for studying the behaviour of impulsive differential systems. The first approach uses a *generalized function* to represent a *jump discontinuity* in the state with the help of the Dirac function [5, 13, 18]. In the second approach, the jump discontinuity is represented by an *impulsive vector* [3, 12, 14] and the references cited therein. The impulsive vector representation provides a general characterization of external disturbances, perturbations or even impulsive controls, see [7]. The equations studied here are referred to as linear impulsive differential equations, or simply as *linear impulsive systems (LIS)*.

Several qualitative theories for stability properties of zero solution of impulsive systems in sense of Lyapunov stability has been established in [3, 8, 9, 11, 14] and the references there in. In general, the stability conditions rely on the standard results of Lyapunov stability for linear dynamical systems, see for example [6, 16], with some additional work to account for the effect of the impulsive vectors. The work in [17] has given a more general stability condition than the conditions in [3]. In this paper, Lyapunov stability of the LIS has been analyzed by modeling the state at time instants as a discrete time dynamical system. Within this new representation, impulsive vectors act as forcing inputs. The Lyapunov stability is then derived by using the standard stability of discrete time systems, see for example [4].

2 Lyapunov Stability: Discrete-Time Systems

Consider the following autonomous LIS with state $x(t) \in R^n$ and output $y(t) \in R^l$ subjected to impulsive vectors $\{d(t_k); k \in Z^+\}$ on the plant state as described by

$$\begin{array}{rcll} \dot{x}(t) & = & Ax(t)\ ; & t \neq t_k \\ x(t_k^+) & = & x(t_k) + d(t_k)\ ; & t = t_k \\ x(t_0^+) & = & x_0 & \end{array} \tag{1}$$

and assume that the following condition holds.

Condition 1 *Let Ω be an open set where $\Omega \subset R^n$ and*

1. $0 \equiv t_0 < t_1 < t_2 < \cdots < t_k < \cdots$
2. $d(t_k)$ *are bounded impulsive vectors such that* $x(t_k^+) = x(t_k) + d(t_k) \in \Omega$

Then the nonempty $\mathcal{G}_k$ is defined by

$$\mathcal{G}_k = \{(t, x) \in R \times \Omega : t_k < t \leq t_{k+1}, x \in \Omega\}$$

The solution $x(t)$ for $t > t_0 = 0$ is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_0^t e^{A(t-\sigma)} \sum_k d(t_k)\delta(\sigma - t_k)\, d\sigma \tag{2}$$

The solution of systems with impulsive effects (1) in the extended state space begins from the initial condition (t_0, x_0) and moves along the trajectory $(t, x(t))$. If at time instants $t_k \geq t_0$ there is an impulsive jump $d(t_k)$, then the state is instantaneously changed to the new state $x(t_k^+) = x(t_k) + d(t_k)$. The state then follows the trajectory with the new initial condition $x(t_k^+)$ until the occurrence of the next transition time instant at time t_{k+1}. That is, the solutions of impulsive systems are characterized by three components: the dynamics of ordinary differential equations, the transition time instants $\{t_k\}$ and the impulsive vectors $\{d(t_k)\}$.

The solution at time instants t_{k+1}^+ can therefore be written as

$$x(t_{k+1}^+) = e^{A(t_{k+1}-t_k)}x(t_k^+) + d(t_{k+1}) \tag{3}$$

Now define $h_k \stackrel{\triangle}{=} x(t_k^+)$. Then the solution of the LIS (1) at time instants t_k^+ which is given by (3) for all $k \in Z^+$ can be described as a discrete dynamical system

$$h_{k+1} = \Phi(t_{k+1}, t_k)h_k + d(t_{k+1})\ ; \tag{4}$$

$$\Phi(t_{k+1}, t_k) \stackrel{\triangle}{=} e^{A(t_{k+1}-t_k)}$$

where $\Phi(t_{k+1}, t_k) \stackrel{\triangle}{=} e^{A(t_{k+1}-t_k)}$. The discrete system matrix $\Phi(t_{k+1}, t_k)$ represents the evolution of the continuous solution of the LIS (1) from the initial condition at time t_k to t_{k+1} while the forcing vector $d(t_{k+1})$ is the impulsive vector at time t_{k+1}.

The solution $h_k = h_k(k; 0, h_0)$ of (4) is then given by

$$h_k = \Phi(t_k, t_0)h_0 + \sum_{i=0}^{k-1} \Phi(t_k, t_i)d(t_i) \tag{5}$$

Condition for stability of the LIS (1) will be deduced from the behaviour of the discrete system (4).

Definition 1 *Given $\{(t_k, d(t_k)); k = 1, 2, \cdots\}$, then the solution $h_k(k; 0, h_0)$ of (4) is said to be*

1. *stable: if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|h_0\| < \delta(\epsilon)$ implies*
$$\|h_k(k; 0, h_0)\| < \epsilon$$
for all $k \geq 0$.
2. *asymptotically stable: if the solution $h_k(k; 0, h_0)$ is stable, and there exists $\delta > 0$ such that for all $\|h_0\| < \delta$ implies*
$$\lim_{k\to\infty} h_k(k; 0, h_0) = 0.$$
3. *exponentially stable: if the solution $h_k(k; 0, h_0)$ is stable, and there exist $\delta > 0$ and $0 < \rho < 1$ such that if $\|h_0\| < \delta$ implies*
$$\|h_k(k; 0, h_0)\| \leq \mu\rho^k\|h_0\|$$
for all $k \geq 0$ and some $\mu \geq 1$.

Definition 2 *Given $\{(t_k, d(t_k)); k = 1, 2, \cdots\}$, such that the solution $x(t; t_0^+, x_0)$ of the linear impulsive system (1) exists, then the solution $x(t; t_0^+, x_0)$ is said to be*

1. *stable: if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|x_0\| < \delta(\epsilon)$ implies*
$$\|x(t; t_0^+, x_0)\| < \epsilon$$
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3. *exponentially stable: if the solution $x(t; t_0^+, x_0)$ is stable, and there exist $\delta > 0$ and $0 < \rho < 1$ such that if $\|x_0\| < \delta$ implies*
$$\|x(t; t_0^+, x_0)\| \leq \mu\rho^k\|x_0\|$$
for all $k \geq 0$ and some $\mu \geq 1$.

The following theorem gives necessary and sufficient conditions for the stability of the linear impulsive systems (1) in term of the stability of the discrete dynamical system (4).

Theorem 1

1. *The linear impulsive system (1) is stable if and only if the discrete system (4) is stable.*

2. *The linear impulsive system (1) is asymptotically stable if and only if the discrete system (4) is asymptotically stable.*

Proof. The state of the solution of the linear impulsive system (1) at transition time instants t_k^+ for all k is given by

$$\begin{aligned} x(t_{k+1}^+) &= \Phi(t_{k+1}, t_k)(x(t_k) + d(t_k)) + d(t_{k+1}) \\ &= \Phi(t_{k+1}, t_k)x(t_k^+) + d(t_{k+1}) \end{aligned}$$

The solution, by replacing $x(t_k^+)$ with h_k, is given by (5) which can then be written

$$\|h_k\| = \|\Phi(t_k, t_0)\| \, \|h_0\| + \sum_{i=0}^{k-1} \|\Phi(t_k, t_i)\| \, \|d(t_i)\|$$

From the standard discrete time systems, it then follows that the solution h_k is stable (asymptotically stable) if and only if the matrix A is stable (asymptotically stable).

As preliminaries to proof of the stability theory, some definitions and standard results will be presented.

Definition 3 *[6] A function $\alpha : R_+ \to R_+$ is said to belong to a class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$.*

Lemma 1 *[16] Let $\phi : R_+ \to R_+$ be continuous, non-decreasing and that $\phi(r) > 0, \forall r > 0$, with $\phi(0) = 0$. Then there exists a class $\mathcal{K}$ function α such that $\alpha(r) \le \phi(r), \forall r$. Furthermore, if $\phi(r) \to \infty$ as $r \to \infty$, then α can be chosen to have the same property of ϕ.*

Definition 4 *A function $V : R^n \to R$ is said to be a local positive definite function if:*

(i) V is continuous

(ii) V(0) = 0

(iii) There exists a class $\mathcal{K}$ function α such that

$$\alpha(\|h_k\|) \le V(h_k), \quad \forall k \ge 0, \quad h_k \in \Omega \subset R^n$$

positive definite function if $h_k \in R^n (r \to \infty)$.

V is said to be decrescent if there exists a class $\mathcal{K}$ function β such that

$$V(h_k) \le \beta(\|h_k\|), \quad \forall k \ge 0, \quad h_k \in \Omega \subset R^n$$

where $h_k \in R^n (r \to \infty)$.

A discrete Lyapunov function is then defined as

$$V_k \triangleq V(h_k)$$

where V_k denotes the value of the positive function V at the state $x(t_k^+)$ at time t_k^+. Along the discrete trajectories of (4), the change ΔV in the positive definite function V is defined by

$$\Delta V_{k+1} \triangleq V_{k+1} - V_k$$

The following results follow directly from the discrete time stability results. (See for example [4].)

Theorem 2 *Consider the linear impulsive system (1) and the associated discrete time system (4). Suppose Condition 1 holds and a positive definite function V is defined on Ω. If for all decision vectors $\{d(t_k)\}$ occurring at time instants $\{t_k, k \in Z^+\}$ and for some θ such that $0 < t_{k+1} - t_k < \theta < \infty$ and*

$$\Delta V_k \le 0 \tag{6}$$

then the equilibrium point $\bar{h} = 0$ is stable. Moreover, if $\Omega = R^n$ and $V(h_k) \to \infty$ for $\|h_k\| \to \infty$ then the stability is global.

Theorem 3 *Consider the linear impulsive system (1) and the associated discrete time system (4). Suppose Condition 1 holds and a positive definite function V is defined on Ω. If for all decision vectors $\{d(t_k)\}$ occurring at time instants $\{t_k, k \in Z^+\}$ and for some $\theta > 0$ such that $t_{k+1} - t_k < \theta < \infty$ and*

(i) $\Delta V_k < 0$, for all $k \in Z^+$, or

(ii) $\Delta V_k \le 0$, for all $k \in Z^+$ and $\Delta V_k = 0$ implies $h_k = 0$ for $k \ge 0$

then the equilibrium point $\bar{h} = 0$ is asymptotically stable. Moreover, if $\Omega = R^n$ and $V(h_k) \to \infty$ for $\|h_k\| \to \infty$ then the stability is global.

Theorem 4 *If the conditions in Theorem 3 hold with respect to a continuous function $V(h)$ where*

$$0 \le \alpha\|h\|^p \le V(h) \le \beta\|h\|^p$$

for some $\alpha, \beta, p > 0$ and in addition

$$V(h_{k+1}) \le \gamma V(h_k)\,; \quad 0 \le \gamma < 1$$

then (4) is exponentially asymptotically stable.

Theorem 5 *Consider the linear impulsive system (1) and the associated discrete time system (4). Suppose Condition 1 holds and a positive definite function V is defined on Ω. If for all decision vectors $\{d(t_k)\}$ occurring at time instants $\{t_k, k \in Z^+\}$ and for some $\theta > 0$ such that $t_{k+1} - t_k < \theta < \infty$ and*

$$\Delta V_k > 0$$

then (4) is unstable.

3 Quadratic Lyapunov Function

Consider the continuous time linear system

$$\dot{x}(t) = Ax(t) \tag{7}$$

and consider a quadratic Lyapunov function candidate of the form

$$V(x(t)) = x^T(t)Px(t) \; ; \quad P = P^T > 0 \tag{8}$$

Then $\dot{V}$ is given by

$$\dot{V}(x(t)) = -x^T(t)Qx(t) \tag{9}$$

where the matrix Q satisfies the Lyapunov matrix equation

$$A^T P + PA = -Q \tag{10}$$

Lemma 2 *Consider the discrete dynamical system*

$$x_{k+1} = \Phi(t_{k+1}, t_k)x_k \; ; \qquad \Phi(t_{k+1}, t_k) \triangleq e^{A(t_{k+1}-t_k)}$$

for some θ such that $0 < t_{k+1} - t_k < \theta < \infty$. Then

(i) If the matrix A is asymptotically stable, for a given $Q > 0$, there exists a unique matrix $P > 0$ such that

$$A^T P + PA = -Q$$

(ii) If $V_k = x_k^T P x_k$ then $\Delta V_k \triangleq V_k - V_{k-1} < 0$ implies

$$\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P < 0$$

Furthermore, if the matrix A is stable then $\Delta V_k \triangleq V_k - V_{k-1} \leq 0$ implies $\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P \leq 0$.

Proof. Consider a continuous time linear system (7). If A is an asymptotically stable matrix, then equation (10) is satisfied such that $\dot{V} < 0$. Consequently

$$\Delta V_k = V_k - V_{k-1} = -\int_{t_{k-1}}^{t_k} x^T(\sigma)Qx(\sigma)\, d\sigma < 0$$

which implies $\Delta V_k = V_k - V_{k-1} = x_{k-1}^T[\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P]x_{k-1} < 0$. Similarly, if the matrix A is only stable, then $\dot{V} \leq 0$, then

$$\Delta V_k = V_k - V_{k-1} = -\int_{t_{k-1}}^{t_k} x^T(\sigma)Qx(\sigma)\, d\sigma \leq 0$$

implies $\Delta V_k = V_k - V_{k-1} = x_{k-1}^T[\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P]x_{k-1} \leq 0$.

From this quadratic Lyapunov function, the change ΔV_k for (5) is then given by

$$\begin{aligned}\Delta V_k &= h_{k-1}^T[\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P]h_{k-1} \\ &+2h_{k-1}^T\Phi^T(t_k, t_{k-1})Pd(t_k) + d^T(t_k)Pd(t_k)\end{aligned} \tag{11}$$

Corollary 1

1. If $\Re\lambda(A) \leq 0$ and $2h_{k-1}^T\Phi^T(t_k, t_{k-1})Pd(t_k) + d^T(t_k)Pd(t_k) \leq 0$ then (1) is stable.

2. If $\Re\lambda(A) < 0$ and $2h_{k-1}^T\Phi^T(t_k, t_{k-1})Pd(t_k) + d^T(t_k)Pd(t_k) \leq 0$ then (1) is asymptotically stable.

Proof.

Let $V_k = h_k^T P h_k$. Then

$$\begin{aligned}V_k &= h_{k-1}^T[\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P]h_{k-1} \\ &+2h_{k-1}^T\Phi^T(t_k, t_{k-1})Pd(t_k) + d^T(t_k)Pd(t_k)\end{aligned}$$

It follows that $V_k \leq 0$ if

$$h_{k-1}^T[\Phi^T(t_k, t_{k-1})P\Phi(t_k, t_{k-1}) - P]h_{k-1} \leq 0$$

$$2h_{k-1}^T\Phi^T(t_k, t_{k-1})Pd(t_k) + d^T(t_k)Pd(t_k) \leq 0$$

The results are completed using Lemma 2.

Example 1 *Consider the following PFM feedback system that was studied in [15]. The system representation is described by*

$$\begin{aligned}\begin{bmatrix} \dot{y}(t) \\ \dot{q}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ -1 & -c \end{bmatrix}\begin{bmatrix} y(t) \\ q(t) \end{bmatrix} \; ; |q(t_k)| \neq r \\ \begin{bmatrix} y(t_k^+) \\ q(t_k^+) \end{bmatrix} &= \begin{bmatrix} y(t_k) \\ q(t_k) \end{bmatrix} + \begin{bmatrix} K \\ -r \end{bmatrix} sgn(q(t_k)) \; ; |q(t_k)| = r \\ y(t) &= [1 \quad 0]\begin{bmatrix} y(t) \\ q(t) \end{bmatrix}\end{aligned} \tag{12}$$

where y is the output of the system, and the variable q is the input of the modulator. The pulses are emitted at time t_k whenever $|q(t_k)| = r$ for a fixed threshold r. These impulses then reset q to zero; that is, $q(t_k^+) = 0$. The rate of pulse generation depends on the value of the threshold r, and the dynamics of the system.

The dynamics of this example can therefore be arranged into the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) \\ x(t_k^+) &= x(t_k) + d(t_k)\end{aligned}$$

where

$$x(t) = \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} \; ; d(t_k) = \begin{bmatrix} K \\ -r \end{bmatrix} sgn(\bar{C}^T x(t_k)) \; ; \tag{13}$$

$$\bar{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for some K and r. Since $|sgn(q)| = 1$, the magnitude of the impulsive vectors are identical, but their direction is determined by the sign of the modulator input q.

Consider the PFM system (12) with the impulsive vector $d(t_k)$ given by (13). The stability of this system was investigated in [14] where it was shown that the system is stable in the sense of Lyapunov. The open set Ω for this system, for an infinitesimal $\delta > 0$, is

$$\Omega = \{(y, q) : |q| + \delta < r\}$$

Consider the quadratic Lyapunov function V given by

$$V(y, q) = \begin{bmatrix} y(t) & q(t) \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix}$$

for $\alpha, \beta, \gamma > 0$ and $\alpha\gamma > \beta^2$.

The first condition for stability is that $\dot{V}(x) \leq 0$ for all $x \in \Omega$. This can be obtained from the Lyapunov matrix equation

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -c \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -c \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} -2\beta & -(\gamma + \beta c) \\ -(\gamma + \beta c) & -2\gamma c \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $\dot{V} \leq 0$ if the determinant

$$\begin{aligned} 0 &\geq 4\beta\gamma c - (\gamma + \beta c)^2 \\ &\geq (\gamma - \beta c) \end{aligned}$$

which gives $\beta c \geq \gamma$. Also, the set G defined by $G = \{x : \dot{V}(x) = 0, x \in \Omega\}$ is given by $y + cq = 0$ and $|q| \leq r$.

The condition on the impulsive vectors is computed by using the condition

$$2h_{k-1}^T \Phi^T(t_k, t_{k-1}) P d(t_k) + d^T(t_k) P d(t_k) \leq 0$$

Define $y_k \triangleq y(t_k)$, $q_k \triangleq q(t_k)$, the above condition is equivalent to

$$2 \begin{bmatrix} y_k \\ q_k \end{bmatrix}^T \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} K \\ -r \end{bmatrix} sgn(q_k)$$
$$+ \begin{bmatrix} K \\ -r \end{bmatrix}^T sgn(q_k) \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} K \\ -r \end{bmatrix} sgn(q_k) \leq 0$$

This condition in turn is equivalent to

$$2y_k sgn(q_k)[\alpha K - \beta r] + \alpha K^2 - \gamma r^2 \leq 0$$

Now the dynamic behaviour of the input modulator q in the absence of impulses is given by $\dot{q} = -cq - y$. Then multiplying both sides by q gives

$$\frac{1}{2}\frac{d}{dt}(q^2) = q\dot{q} = -cq^2 - qy\ ; \quad \textit{for } |q| < r \tag{14}$$

Since q is increasing in absolute value, hence $q\dot{q}$ is positive which from (14), then implies that qy must be negative. This in turn leads to the condition that $y_k\, sgn(q_k)$ is negative.

The condition on K is then obtained as follows:

$$\alpha K \geq \beta r \quad \textit{and} \quad \alpha K^2 \geq \gamma r^2$$

These inequalities leads to $0 \leq K \leq \frac{\gamma r}{\beta}$. Since for $c \neq 0$, the maximum value of β is $\beta = \frac{\gamma}{c}$, we conclude that the system is stable if

$$0 \leq K \leq cr$$

which is the same as the result in [14]. However the present approach for deriving the result is more direct since the results are obtained by using the Lyapunov matrix equation.

It has been assumed that impulsive vectors $\{d(t_k)\}$ which lead to Lyapunov stability exist in R^n. Therefore, it is of interest to show the existence of such impulsive vectors. In the following result, an existence condition for impulsive vectors to satisfy the condition

$$2h_{k-1}^T \Phi^T(t_k, t_{k-1}) P d(t_k) + d^T(t_k) P d(t_k) \leq 0 \tag{15}$$

is derived.

Corollary 2 *Given a vector $\tilde{h}_k \triangleq h_{k-1}^T \Phi^T(t_k, t_{k-1})$, suppose for some ϵ_k, there exists a vector γ_k such that*

$$\tilde{h}_k^T P \gamma_k \leq -\epsilon_k < 0$$

Then there exists an $\alpha_k > 0$ such that condition (15) is satisfied with $d(t_k) = \alpha_k \gamma_k$.

Proof. Let $d(t_k) = \alpha_k \gamma_k$. Then

$$\begin{aligned} 2\tilde{h}_k^T P d(t_k) + d^T(t_k) P d(t_k) &= \alpha_k(2\tilde{h}_k^T P \gamma_k + \alpha_k \gamma_k^T P \gamma_k) \\ &\leq \alpha_k(-2\epsilon_k + \alpha_k \gamma_k^T P \gamma_k) \end{aligned}$$

Here, if for any choice $\alpha_k > 0$ such that $\alpha_k \gamma_k^T P \gamma_k < 2\epsilon_k$, condition (15) is satisfied.

Another possible form for the impulsive vectors is given as follows.

Corollary 3 *Let $\tilde{h}_k \triangleq h_{k-1}^T \Phi^T(t_k, t_{k-1})$. Suppose the impulsive vector $d(t_k)$ for all $k \in Z^+$ is given by*

$$d^T(t_k) = -[\beta_1 sgn(p_1^T \tilde{h}_k) \;\; \beta_2 sgn(p_2^T \tilde{h}_k) \;\; \cdots \;\; \beta_n sgn(p_n^T \tilde{h}_k)] P^{-\frac{1}{2}} \tag{16}$$

where $sgn(\alpha)$ is the sign of a scalar α, p_m^T is the m-th row of $P^{\frac{1}{2}}$, and

$$0 \leq \beta_i \leq 2 \qquad \textit{for } 1 \leq i \leq n$$

Then $2\tilde{h}_k^T P d(t_k) + d^T(t_k) P d(t_k) \leq 0$.

Proof. Define $\omega = P^{\frac{1}{2}} \tilde{h}$. Therefore $\dot{w} = P^{\frac{1}{2}} A P^{-\frac{1}{2}} \omega \triangleq F_1 \omega$. Then

$$\begin{aligned} \tfrac{d}{dt}(\omega^T \omega) &= \omega^T (P^{-\frac{1}{2}} A^T P^{\frac{1}{2}} + P^{\frac{1}{2}} A P^{-\frac{1}{2}}) \omega \\ &= -\omega^T P^{-\frac{1}{2}} Q P^{-\frac{1}{2}} \omega < 0, \end{aligned}$$

which implies $A_1^T + A_1 = -Q_1$, $Q_1 \triangleq P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}$. Define

$$\mu_k \triangleq 2\tilde{h}_k^T P d(t_k) + d^T(t_k) P d(t_k) = 2\omega_k^T f_k + f_k^T f_k$$

where $\omega_k \stackrel{\triangle}{=} \omega(t_k)$ and $f_k \stackrel{\triangle}{=} P^{\frac{1}{2}} d(t_k)$. Now let $f_k^T = -[\beta_1\omega_{1k} \quad \beta_2\omega_{2k} \quad \cdots \quad \beta_n\omega_{nk}]$ where $\omega_k^T = [\omega_{1k} \quad \omega_{2k} \quad \cdots \quad \omega_{nk}]$ so that $\mu_k = -2(\beta_1\omega_{1k}^2 + \beta_2\omega_{2k}^2 + \cdots + \beta_n\omega_{nk}^2) + (\beta_1^2\omega_{1k}^2 + \beta_2^2\omega_{2k}^2 + \cdots + \beta_n^2\omega_{nk}^2) \leq 0$ for all k if $0 \leq \beta_i \leq 2$, for all $1 \leq i \leq n$.

The following example shows how the impulsive vectors are computed by using the stability condition (16) in Corollary 3.

Example 2 *Consider a second order plant*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} x(t) \; ; \qquad y \neq 1 \\ x(t_k^+) &= x(t_k) + d(t_k) \; ; \qquad y = 1 \\ y(t) &= [1 \quad 0]\, x(t) \end{aligned} \tag{17}$$

The system matrix has eigenvalues $\lambda_1 = -0.268$ *and* $\lambda_2 = -3.732$ *so the linear system is asymptotically stable. Then from (10),* $Q = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ *implies*

$$P = \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}; \quad P^{\frac{1}{2}} = \begin{bmatrix} 2.5790 & 0.5907 \\ 0.5907 & 0.8069 \end{bmatrix}$$

According to Corollary 3, if the decision vector $d(t_k)$ *is given by*

$$d(t_k) = P^{-\frac{1}{2}} \begin{bmatrix} -\beta_1\, sgn([2.5790 \quad 0.5907]\tilde{h}_k) \\ -\beta_2\, sgn([0.5907 \quad 0.8069]\tilde{h}_k) \end{bmatrix}$$

then (17) will be asymptotically stable. Now suppose $x^T(t_k) = [1 \quad \alpha]$ *which satisfies* $y = 1$. *Using the condition* $2\tilde{h}_k^T P d(t_k) + d^T(t_k) P d(t_k) \leq 0$ *then yields*

$$-2(2.5790\beta_1 + 0.5907\beta_2) - 2(0.5907\beta_1 + 0.8069\beta_2)\alpha \tag{18}$$

$$+\beta_1^2 + \beta_2^2 \leq 0$$

and so choosing β_1 *and* β_2 *which satisfy (18) and* $0 \leq \beta_1, \beta_2 \leq 2$ *leads to the asymptotic stability of the system (17).*

4 Conclusions

In this paper, stability of hybrid systems modeled as linear impulsive systems was developed by transforming the hybrid systems discrete dynamical systems. When the Lyapunov function is of the quadratic form, the stability condition is a function of the impulsive vector. This requirement leads to a condition on inner product between gradient of Lyapunov function discrete states and the impulsive vectors at the occurrences of impulsive effects. This representation improves the stability condition, since the impulsive vector has nonunique representation as a function of the state at that time instant. The developed stability conditions have been applied in a class of receding horizon hybrid reference control (RHHRC) appeared in [10].

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