OPTIMAL $\mathcal{H}_2$ MODEL REDUCTION IN STATE-SPACE: A CASE STUDY

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Abstract

A state-space approach to $\mathcal{H}_2$ model reduction of stable systems is pursued, using techniques from global rational optimization, and aimed at finding global optima. First, a concentrated version of the $\mathcal{H}_2$ criterion is derived, by optimizing analytically with respect to a subset of parameters. Next, a bound is developed, which is shown to be sharp for reduction to models of order 1. The choice of parameterization is such as to keep all these criteria rational.
To illustrate the approach, a nontrivial example of a multivariable stable system of order 4 was designed, together with an approximation of order 2. Detailed analysis shows the approximation to be the global optimum among all stable system of order 2, constituting the first such system described in the literature. The fact that it also yields a global optimum to the bound, which happens to be sharp at the optimum, leads to a couple of questions for further research.

1 Introduction

One of the important problems in the field of modelling and systems identification is the *model reduction problem*. It is not uncommon in practical applications to arrive at models of high complexity, either directly as a result of the identification procedure when using a high order (black-box) model on available input-output data, or indirectly as a result of the interconnection of a number of subsystems, each of which is modelled individually. Models of lower complexity, however, usually lend themselves much better for model analysis and for the application of popular techniques for control system design. Thus, there is a clear practical interest in the problem of finding good lower order approximate models to a given model of high order.

One of the important aspects to deal with this problem lies in the choice of a criterion to measure the difference between two models. Various choices for such criteria are described in the literature. In this paper we focus attention on one of the most well-known criteria for stable systems, the $\mathcal{H}_2$ criterion, which allows for a straightforward interpretation in system theoretic terms. However, one of the unattractive characteristics of the $\mathcal{H}_2$ criterion is that it may well possess a number of local, non-global optima. This makes it a hazardous task to apply numerical techniques only to determine an $\mathcal{H}_2$-optimal reduced-order model to a given system, such as the usual iterative local search techniques for multivariate function minimization (like steepest descent, Newton's method, the Gauss-Newton method, etc.).

Depending on the choice of parameterization, the $\mathcal{H}_2$ model reduction problem can be formulated as a *rational optimization problem*. This opens up the possibility to apply techniques from the field of global rational optimization to determine a *global* optimum to this problem. Such techniques often consist of a mix of methods from (constructive) algebra and numerical analysis, and they may put high demands on the available computer hardware and software. The current state-of-the-art of these methods is such that only examples of a rather limited complexity can be handled satisfactorily in all detail. However, despite such limitations, these methods are still quite valuable e.g. in test situations where exact results are required or when theoretic results and conjectures are being analyzed.

In this paper we pursue a state-space approach along such lines. First, the $\mathcal{H}_2$ model reduction problem is formulated and a 'concentrated version' of it is derived, by optimizing analytically with respect to a number of parameters which happen to enter the problem in a linear least squares manner. Next, a bound on this 'concentrated $\mathcal{H}_2$ criterion' is derived, which is shown to be sharp in the case of approximation by a model of order 1. To illustrate the approach we have designed an example of a stable system of order 4 with two inputs and two outputs, which we intend to approximate by a stable system of order 2. For this particular example, which leads to a rational optimization problem, the available software was unable to handle the concentrated $\mathcal{H}_2$ criterion due to its complexity, but it did return a global optimum for the much simpler bound. Subsequent analysis shows that this global optimum for the bound also happens to constitute a global optimum for the concentrated $\mathcal{H}_2$ criterion. This leads to a number of new research questions by which we conclude the paper.

2 The discrete time $\mathcal{H}_2$ model reduction problem

We consider the space of linear time-invariant systems in discrete time which are stable and have strictly proper rational transfer functions. We deal with the real multivariable situation, where the systems have m inputs and p outputs.

It is supposed that a system Σ of McMillan degree n is given, stemming from this class. The $\mathcal{H}_2$ model reduction problem consists of finding an approximation $\hat{\Sigma}$ which is of a prescribed order $k < n$ and of which the $\mathcal{H}_2$-distance between Σ and $\hat{\Sigma}$ is as small as possible. Here, the $\mathcal{H}_2$ criterion is given by (cf., e.g., [3]):

$$V = \|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_2} = \text{trace}\left\{ \sum_{i=1}^{\infty} (H_i - \hat{H}_i)(H_i - \hat{H}_i)^T \right\} \quad (1)$$

where H_i denotes the i-th Markov matrix of the transfer function $H(z) = \sum_{i=1}^{\infty} H_i z^{-i}$ of Σ (and likewise for $\hat{H}_i$).
For fixed $k \leq n$, it is known that among all systems of order $\leq k$ no approximation of Σ of order $< k$ exists which constitutes a local minimum of the $\mathcal{H}_2$ criterion. See also [1, 2]. Therefore, one may require the order of $\hat{\Sigma}$ to be equal to k or to be $\leq k$, depending on what is convenient; relaxation of this constraint will not lead to different solutions.

3 A state space approach to $\mathcal{H}_2$ model reduction

We will pursue a state space approach. It is assumed that a minimal realization of order n (i.e., state space dimension n) is available for Σ, which is denoted by (A, B, C). It then holds that $H(z) = C(zI_n - A)^{-1}B$ and $H_i = CA^{i-1}B$. Likewise, $\hat{\Sigma}$ will be represented by a (minimal) state space matrix triple $(\hat{A}, \hat{B}, \hat{C})$ of order k. Stability of Σ and $\hat{\Sigma}$ corresponds to asymptotic stability of the dynamical matrices A and $\hat{A}$, which means that all eigenvalues are in the open complex unit disk.
Substitution of the expressions for the Markov matrices in terms of the matrices of the state space realizations (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ makes it possible to rewrite the $\mathcal{H}_2$ criterion as follows:

$$V = \text{trace}\left\{ \begin{pmatrix} C & -\hat{C} \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix} \begin{pmatrix} C^T \\ -\hat{C}^T \end{pmatrix} \right\} \quad (2)$$

where P_1, P_2 and P_3 denote the unique solutions of the following discrete time Lyapunov and Sylvester matrix equations:

$$P_1 - AP_1A^T = BB^T, \quad (3)$$
$$P_2 - AP_2\hat{A}^T = B\hat{B}^T, \quad (4)$$
$$P_3 - \hat{A}P_3\hat{A}^T = \hat{B}\hat{B}^T. \quad (5)$$

(See for instance [4, 13].) An equivalent dual formulation of the $\mathcal{H}_2$ criterion is offered by the expression

$$V = \text{trace}\left\{ \begin{pmatrix} B^T & -\hat{B}^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix} \begin{pmatrix} B \\ -\hat{B} \end{pmatrix} \right\} \quad (6)$$

where Q_1, Q_2 and Q_3 denote the unique solutions of:

$$Q_1 - A^TQ_1A = C^TC, \quad (7)$$
$$Q_2 - A^TQ_2\hat{A} = C^T\hat{C}, \quad (8)$$
$$Q_3 - \hat{A}^TQ_3\hat{A} = \hat{C}^T\hat{C}. \quad (9)$$

It is noted that the entries of the matrix solutions of the Lyapunov and Sylvester equations involved are *rational* in terms of the entries of $\hat{A}$, $\hat{B}$ and $\hat{C}$. Consequently, the $\mathcal{H}_2$ criterion V is rational in these entries too. This shows how the $\mathcal{H}_2$ model reduction problem can be formulated as a rational optimization problem.

4 On the parameterization of $(\hat{A}, \hat{B}, \hat{C})$

The $\mathcal{H}_2$ criterion is obviously insensitive to a change of basis of the state space. This creates parameter redundancy in the formulation of V above, if all the entries of $\hat{A}$, $\hat{B}$ and $\hat{C}$ are left free. To remove this excess freedom, (pseudo-)canonical parameterizations of $\hat{A}$, $\hat{B}$ and $\hat{C}$ can be employed, as described in the literature, for which the entries of these matrices depend in a (usually rather simple) rational way on the parameters. The use of such parameterizations preserves rationality of V.
For most of our purposes attention can be restricted, without loss of generality, to a single *generic* parameter chart for the matrix triple $(\hat{A}, \hat{B}, \hat{C})$. A convenient parameter chart is for instance offered by the following structured matrices (see [11, 13]):

$$\hat{A} = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \star & \star & \cdots & \star \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \star & \cdots & \star \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \star & \cdots & \star \end{pmatrix},$$
$$\hat{C} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & \star & \cdots & \star \\ \vdots & \vdots & & \vdots \\ \star & \star & \cdots & \star \end{pmatrix}. \quad (10)$$

Here the starred entries denote parameters to be chosen freely under the sole constraint that stability of $\hat{A}$ holds. Note that observability is already built in. Controllability (and hence minimality) of $(\hat{A}, \hat{B})$ may be required to hold too, but as we have indicated at the end of Section 2, this is not really an issue. This parameter chart involves $k(m+p)$ parameters, which precisely agrees with the manifold dimension of the space of (strictly proper) systems of order k. An alternative, dual parameter chart which has controllability built in instead of observability, is easily constructed in a likewise fashion.
Other parameterizations may have their virtues too. For instance, one may employ *Schur parameters* to build the stability property of $\hat{A}$ directly into the parameterization in a way which allows one to get rid of any restrictions on the parameter domain. (Cf., e.g., [12] and [9].) Such an approach can be combined with the choice of parameterization described above, and can be achieved in a way which leaves V rational. In that case the $\mathcal{H}_2$ model reduction problem is reformulated as an *unconstrained* rational optimization problem for which one may hope to find a finite number of stationary points.

5 The concentrated $\mathcal{H}_2$ criterion

From Eqn. (6) it is not difficult to see that the criterion V is of the *linear least squares* form with respect to the matrix $\hat{B}$ if $\hat{A}$ and $\hat{C}$ are kept fixed. This makes it possible to optimize for $\hat{B}$

analytically in terms of $\hat{A}$, $\hat{C}$ and (A, B, C). As it happens, a unique minimizing solution for $\hat{B}$ for this restricted linear least squares problem exists, provided that $(\hat{A}, \hat{C})$ is observable. It is given by

$$\hat{B} = Q_3^{-1} Q_2^T B. \tag{11}$$

Note that in view of the choice of parameter chart advocated above, this allows one to eliminate km parameters. Substitution into (6) returns what we will call the 'concentrated $\mathcal{H}_2$ criterion', denoted V_c and given by:

$$V_c = \text{trace}\left\{B^T Q_1 B\right\} - \text{trace}\left\{B^T Q_2 Q_3^{-1} Q_2^T B\right\}. \tag{12}$$

This criterion is to be minimized with respect to $\hat{A}$ and $\hat{C}$, which enter the expression through Q_2 and Q_3. Since the first term of V_c is entirely determined by the given system Σ (it denotes the squared $\mathcal{H}_2$ norm of Σ, viz. $\|\Sigma\|^2_{\mathcal{H}_2}$) we may focus on the second term instead. To this end we define the criterion W_c by:

$$W_c = \text{trace}\left\{B^T Q_2 Q_3^{-1} Q_2^T B\right\}. \tag{13}$$

It is noted that:
(i) W_c is to be *maximized* with respect to $\hat{A}$ and $\hat{C}$;
(ii) $W_c \geq 0$, and since also $V_c \geq 0$, a natural uniform upper bound on W_c is provided by $\|\Sigma\|^2_{\mathcal{H}_2}$;
(iii) $W_c = \|\hat{\Sigma}\|^2_{\mathcal{H}_2}$, so the aim is to maximize the norm of the approximation $(\hat{A}, \hat{B}, \hat{C})$, but notably under the constraint that $\hat{B}$ satisfies Eqn. (11), which involves $\hat{A}$, $\hat{C}$ and (A, B, C);
(iv) W_c can still be constructed to be an unconstrained rational function of the remaining parameters.

Of course, from (2) a similar (dual) approach can be followed to optimize V with respect to only $\hat{C}$, for fixed $\hat{A}$ and $\hat{B}$. In this case the unique minimizing value for $\hat{C}$ is given by

$$\hat{C} = C P_2 P_3^{-1}. \tag{14}$$

Substitution into (2) leads to an alternative concentrated $\mathcal{H}_2$ criterion $\widetilde{V}_c$, which is given by

$$\widetilde{V}_c = \text{trace}\left\{C P_1 C^T\right\} - \text{trace}\left\{C P_2 P_3^{-1} P_2^T C^T\right\}. \tag{15}$$

This criterion is to be minimized with respect to $\hat{A}$ and $\hat{B}$. In this case it is natural to introduce the associated criterion $\widetilde{W}_c$ according to

$$\widetilde{W}_c = \text{trace}\left\{C P_2 P_3^{-1} P_2^T C^T\right\} \tag{16}$$

which is to be *maximized* with respect to $\hat{A}$ and $\hat{B}$. The criterion $\widetilde{V}_c$ involves only kp parameters. A choice between the two alternatives V_c and $\widetilde{V}_c$ may be based on the number of inputs m and the number of outputs p.

6 A doubly concentrated $\mathcal{H}_2$ criterion for $k = 1$

Above it was noted that the $\mathcal{H}_2$ criterion V is a linear least squares criterion with respect to the sets of parameters in $\hat{B}$ or in $\hat{C}$ individually, but *not simultaneously*. Nevertheless it would be nice if V could be optimized analytically for a fixed choice of $\hat{A}$ with respect to *all* of the parameters in $\hat{B}$ and $\hat{C}$, in which case the number of free parameters in the resulting 'doubly concentrated $\mathcal{H}_2$ criterion' would drop to just n. This actually turns out to be possible in case approximation is carried out with systems of order 1 (i.e., $k = 1$). In this context the following theorem is useful (for a proof, cf. [10]).

Theorem 6.1 *Let Q be a fixed $n \times n$ Hermitian matrix. Consider the (real-valued) expression*

$$r(P) = \text{trace}\{QP(P^*P)^{-1}P^*\} \tag{17}$$

where P ranges over the set of $n \times k$ matrices of full column rank $k \leq n$. Then:
(i) The (globally) maximal value of $r(P)$ is equal to the sum of the k largest eigenvalues of Q.
(ii) This maximal value is attained for any matrix P of which the column space is spanned by k independent eigenvectors of Q corresponding to these k largest eigenvalues.

To see how this theorem can be applied, we reconsider the expression $W_c = \text{trace}\left\{B^T Q_2 Q_3^{-1} Q_2^T B\right\}$. In case $k = 1$, it is noted that Q_3 and $\hat{A}$ are actually scalar and Q_2 is of size $n \times 1$. They are given by

$$Q_2 = (I_n - \hat{A}A^T)^{-1} C^T \hat{C}, \tag{18}$$
$$Q_3 = \hat{C}^T \hat{C} / (1 - \hat{A}^2). \tag{19}$$

Therefore $W_c = (1 - \hat{A}^2)\text{trace}\Big\{B^T (I_n - \hat{A}A^T)^{-1} C^T \times$ $\times \hat{C}(\hat{C}^T\hat{C})^{-1}\hat{C}^T C (I_n - \hat{A}A)^{-1} B\Big\} = \text{trace}\Big\{(1 - \hat{A}^2) \times$ $\times C(I_n - \hat{A}A)^{-1} B B^T (I_n - \hat{A}A^T)^{-1} C^T \hat{C} (\hat{C}^T \hat{C})^{-1} \hat{C}^T\Big\}$.
This last expression is in a suitable form to apply the above theorem to. The role of Q is played by the matrix $(1 - \hat{A}^2) C (I_n - \hat{A}A)^{-1} B B^T (I_n - \hat{A}A^T)^{-1} C^T$ and that of P is played by $\hat{C}$. Thus, the doubly concentrated $\mathcal{H}_2$ criterion is given by

$$W_d = \lambda_{\max}\left\{\frac{(1 - \hat{A}^2)}{\hat{A}^2} H(\hat{A}^{-1}) H(\hat{A}^{-1})^T\right\}, \tag{20}$$

which is to be maximized with respect to the scalar real parameter $\hat{A}$ over the open interval $(-1, 1)$.

7 A bound on the $\mathcal{H}_2$ criterion

To apply the previous approach also to the situation with $k > 1$ we first note that Q_2 and Q_3 can be written as products of observability matrices of infinite size:

$$Q_2 = O_\infty^T \hat{O}_\infty, \tag{21}$$
$$Q_3 = \hat{O}_\infty^T \hat{O}_\infty, \tag{22}$$

where

$$O_\infty = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix}, \quad \hat{O}_\infty = \begin{pmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \hat{C}\hat{A}^2 \\ \vdots \end{pmatrix}. \tag{23}$$

Then W_c can be rewritten as

$$W_c = \text{trace}\left\{[O_\infty^T B B^T O_\infty^T]\hat{O}_\infty(\hat{O}_\infty^T \hat{O}_\infty)^{-1}\hat{O}_\infty^T\right\}, \quad (24)$$

which resembles the form of Eqn. (17). There are two problems with this attempt to apply Theorem 6.1.
(1) The matrices playing the role of Q and P are of *infinite* size.
(2) The matrix $\hat{O}_\infty$, which plays the role of P, cannot be chosen freely: it involves the fixed matrix $\hat{A}$ and it has the *structure* of an observability matrix. (A closely related problem is that the matrix playing the role of Q currently does not involve $\hat{A}$ at all.)

It is possible to overcome the first problem, essentially by noting that high matrix powers of A and $\hat{A}$ can be expressed as linear combinations of low matrix powers according to the theorem of Cayley-Hamilton. One way to work this out is offered by the approach based on Faddeev sequences to solve Lyapunov and Sylvester equations, see [7].

We have the following definitions and notation.

Definition 7.1 *Let X be a matrix of size $n \times n$, and y a vector of size $n \times 1$. Then:*
(i) The characteristic polynomial $\det(zI_n - X)$ *of X is denoted by*

$$\chi_X(z) = z^n + \chi_1 z^{n-1} + \ldots + \chi_{n-1} z + \chi_n. \quad (25)$$

(ii) The controller companion form matrix associated with X is denoted by X_c and defined by

$$X_c = \begin{pmatrix} -\chi_1 & \cdots & -\chi_{n-1} & -\chi_n \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}. \quad (26)$$

(iii) The Faddeev sequence of X is denoted by $\{X_0, X_1, \ldots, X_{n-1}\}$ and defined recursively by

$$X_0 := I_n, \quad (27)$$
$$X_k := X X_{k-1} - \frac{\text{trace}\{X X_{k-1}\}}{k} I_n. \quad (28)$$

Equivalently, it holds (for $k = 0, 1, \ldots, n-1$) that

$$X_k = X^k + \chi_1 X^{k-1} + \ldots + \chi_{k-1} X + \chi_k I_n. \quad (29)$$

(iv) The Faddeev reachability matrix of (X, y) is denoted by $F_X(y)$ and defined by

$$F_X(y) = \left(\; X_0 y \;\middle|\; X_1 y \;\middle|\; \cdots \;\middle|\; X_{n-1} y \; \right). \quad (30)$$

It then follows (see again [7]) that the matrices Q_2 and Q_3 are given by

$$Q_2 = \sum_{i=1}^{p} F_{A^T}(c_i)\hat{\Delta}(F_{\hat{A}^T}(\hat{c}_i))^T, \quad (31)$$

$$Q_3 = \sum_{i=1}^{p} F_{\hat{A}^T}(\hat{c}_i)\Delta(F_{\hat{A}^T}(\hat{c}_i))^T, \quad (32)$$

where (for $i = 1, \ldots, p$) the column vectors c_i and $\hat{c}_i$ denote the transposed rows of C and $\hat{C}$, respectively, whence:

$$C = \begin{pmatrix} c_1^T \\ \hline \vdots \\ \hline c_p^T \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} \hat{c}_1^T \\ \hline \vdots \\ \hline \hat{c}_p^T \end{pmatrix}, \quad (33)$$

and where the matrices $\hat{\Delta}$ (of size $n \times k$) and Δ (of size $k \times k$) are the unique solutions of the associated (highly structured) Sylvester and Lyapunov equations in controller companion form:

$$\hat{\Delta} - A_c \hat{\Delta} \hat{A}_c^T = e_1 e_1^T, \quad (34)$$
$$\Delta - \hat{A}_c \Delta \hat{A}_c^T = e_1 e_1^T. \quad (35)$$

It is noted that the expression for Q_3 now involves a finite number of Faddeev reachability matrices of *finite* size, but also the positive definite symmetric matrix Δ. To deal with this, we introduce the following matrices:

$$G_{\hat{A}^T}(\hat{c}_i) := F_{\hat{A}^T}(\hat{c}_i)\Delta^{1/2}, \text{ (of size } k \times k), \quad (36)$$
$$G_{A^T}(c_i) := F_{A^T}(c_i)\hat{\Delta}\Delta^{-1/2}, \text{ (of size } n \times k). \quad (37)$$

Then:

$$Q_2 = \sum_{i=1}^{p} G_{A^T}(c_i)(G_{\hat{A}^T}(\hat{c}_i))^T, \quad (38)$$

$$Q_3 = \sum_{i=1}^{p} G_{\hat{A}^T}(\hat{c}_i)(G_{\hat{A}^T}(\hat{c}_i))^T. \quad (39)$$

The matrices $G_{\hat{A}^T}(\hat{c}_i)$ and $G_{A^T}(c_i)$ depend *linearly* on the entries of $\hat{c}_i$ and c_i, respectively. Therefore, these matrices can also be rewritten as:

$$G_{\hat{A}^T}(\hat{c}_i) = \left(\; \hat{M}_1 \hat{c}_i \;\middle|\; \cdots \;\middle|\; \hat{M}_k \hat{c}_i \; \right), \quad (40)$$
$$G_{A^T}(c_i) = \left(\; M_1 c_i \;\middle|\; \cdots \;\middle|\; M_k c_i \; \right), \quad (41)$$

with $\hat{M}_1, \ldots, \hat{M}_k$ of size $k \times k$ depending only on $\hat{A}$, and $M_1, \ldots, M_k$ of size $n \times k$ depending only on $\hat{A}$ and A. Consequently, we find:

$$Q_2 = \sum_{i=1}^{p} \sum_{j=1}^{k} M_j c_i \hat{c}_i^T \hat{M}_j^T = \sum_{j=1}^{k} M_j C^T \hat{C} \hat{M}_j^T, \quad (42)$$

$$Q_3 = \sum_{i=1}^{p} \sum_{j=1}^{k} \hat{M}_j \hat{c}_i \hat{c}_i^T \hat{M}_j^T = \sum_{j=1}^{k} \hat{M}_j \hat{C}^T \hat{C} \hat{M}_j^T. \quad (43)$$

Finally, we therefore may write:

$$Q_2 = M(C)\hat{M}(\hat{C})^T, \quad (44)$$
$$Q_3 = \hat{M}(\hat{C})\hat{M}(\hat{C})^T, \quad (45)$$

where

$$M(C) = \left(\; M_1 C^T \;\middle|\; \cdots \;\middle|\; M_k C^T \; \right), \text{ (of size } n \times kp), \quad (46)$$
$$\hat{M}(\hat{C}) = \left(\; \hat{M}_1 \hat{C}^T \;\middle|\; \cdots \;\middle|\; \hat{M}_k \hat{C}^T \; \right), \text{ (of size } k \times kp). \quad (47)$$

Substitution of these expressions into the expression for W_c yields:

$$W_c = \text{trace}\left\{[M(C)^T BB^T M(C)]\times \right.$$
$$\left. \times \hat{M}(\hat{C})^T (\hat{M}(\hat{C})\hat{M}(\hat{C})^T)^{-1}\hat{M}(\hat{C})\right\}. \quad (48)$$

This shows how the 'infinite size problem' has been solved. The 'structure problem', however, still exists for the matrix $\hat{M}(\hat{C})$, which apart from $\hat{C}$ also still depends on $\hat{A}$.

In view of Theorem 6.1 we conclude that the sum of the k largest eigenvalues of the $pk \times pk$ matrix $Z = M(C)^T BB^T M(C)$ constitutes an *upper bound* for the maximally achievable value of W_c. It is noted that this matrix Z depends on the given system (A, B, C) and on $\hat{A}$. As we have seen, this bound is actually sharp for $k = 1$.

8 Case study: globally optimal $\mathcal{H}_2$ model reduction of a multivariable system from order 4 to order 2

The set-up above has been applied to study a nontrivial multivariable example with 2 inputs and 2 outputs, where a given system Σ of order 4 is approximated by another system $\hat{\Sigma}$ of order 2. Use has been made of tools from constructive algebra, computer algebra software and numerical software (Maple, Mathematica, Matlab). The methods employed for global rational optimization are described in [5, 8, 9].

The system Σ involved in this example is given by:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{8} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \; B = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{383}{2080} & \frac{279}{1040} \\ \frac{1839}{8320} & -\frac{1317}{4160} \\ \frac{1419}{33280} & \frac{99}{1280} \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \quad (49)$$

The second order approximation $\hat{\Sigma}$ is given by:

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ \frac{4}{9} & 0 \end{pmatrix}, \; \hat{B} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{pmatrix}, \; \hat{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (50)$$

The *design* of this example will be discussed in more detail elsewhere. It is such that the system $\hat{\Sigma}$ is guaranteed to constitute a stationary point (on the manifold of systems of order 2) of the $\mathcal{H}_2$ criterion V associated with Σ, a property which is easily verified. However, no further aspects of the concentrated $\mathcal{H}_2$ criterion W_c nor of its bound W_d have been used in the design of this example.

The Hessian of V at $\hat{\Sigma}$ can be computed to be positive definite, which implies that $\hat{\Sigma}$ yields a *local* minimum of V. A similar statement holds for the concentrated $\mathcal{H}_2$ criterion V_c. To show that $\hat{\Sigma}$ constitutes a *global* minimum of V we intend to employ techniques from global rational optimization.

- Using the generic chart for $(\hat{A}, \hat{B}, \hat{C})$ indicated in Section 4, Eqn. (10), the criterion V is rational and involves 8 free parameters. The numerator polynomial has total degree 8 and consists of 655 terms; the denominator polynomial has total degree 6 and consists of 27 terms.

- After the first concentration step, which eliminates the matrix $\hat{B}$, the concentrated criterion W_c is still rational and involves 4 free parameters. The numerator polynomial has total degree 14 and consists of 705 terms; the denominator polynomial has total degree 12 and consists of 277 terms (but can be factored to become more concise). *This problem is still too complex to be handled by our software, using tools from constructive algebra.*

The upper bound W_d

The second concentration step, which eliminates the matrix $\hat{C}$, produces an upper bound W_d on the concentrated criterion W_c. It turns out that this bound is again rational and involves only 2 free parameters. The numerator polynomial has total degree 8 and consists of 42 terms and the denominator polynomial has total degree 6 and consists of 28 terms (and can be factored to become more concise). It has the following explicit form:

$$\begin{aligned} W_d = {} & ((1 - a_2)(7907854144 + 3296829824a_1 \\ & -6927169920a_1^2 - 2897270720a_1^3 + 818184528a_1^4 \\ & +438089920a_1^5 - 18864576a_1^6 + 18209891616a_2 \\ & +6894389696a_1 a_2 - 5524025360a_1^2 a_2 - 2356001232a_1^3 a_2 \\ & -65091328a_1^4 a_2 - 56513808a_1^5 a_2 + 60932736a_1^6 a_2 \\ & +14986490756a_2^2 + 5732635144a_1 a_2^2 - 1324024308a_1^2 a_2^2 \\ & -583912552a_1^3 a_2^2 - 270351000a_1^4 a_2^2 + 30466368a_1^5 a_2^2 \\ & +5601025568a_2^3 + 2101676108a_1 a_2^3 + 244751432a_1^2 a_2^3 \\ & -85974588a_1^3 a_2^3 - 57124440a_1^4 a_2^3 + 1003427217a_2^4 \\ & +357191240a_1 a_2^4 + 128010888a_1^2 a_2^4 - 30466368a_1^3 a_2^4 \\ & +91843237a_2^5 + 57115746a_1 a_2^5 + 11424888a_1^2 a_2^5 \\ & +5940378a_2^6 + 7616592a_1 a_2^6 + 952074a_2^7))/ \\ & (1081600(16 + 4a_1 + a_2)^2(4 - 2a_1^2 + 4a_2 + a_2^2)^2), \end{aligned} \quad (51)$$

where the characteristic polynomial of $\hat{A}$ is denoted by $\det(zI_2 - \hat{A}) = z^2 + a_1 z + a_2$. Investigation of the upper bound W_d shows a number of interesting properties for this example.

(i) The expression for W_d is *rational*, despite the fact that square roots appear in intermediate calculations and that the two largest eigenvalues of a 4×4 matrix have to be summed.

(ii) The bound W_d is not sharp everywhere on the domain of stable matrices $\hat{A}$, but *it happens to be sharp at the intended approximation* $\hat{\Sigma}$.

(iii) The bound W_d can be handled by our software. We are able to compute all of its stationary points and to determine its *global* maximum. *It turns out that there is a single global maximum for W_d within the stability domain; it is located at the intended approximation $\hat{\Sigma}$. Together with (ii) this proves that $\hat{\Sigma}$ is the unique globally optimal $\mathcal{H}_2$ approximation of order 2 of Σ.*

(iv) Using the techniques of [8, 9], the rational bound W_d is transformed into a *polynomial* expression when investigating global maximality at the approximation $\hat{\Sigma}$. This polynomial should then be nonnegative everywhere on the stability domain. As it happens, the polynomial is actually nonnegative on the whole of $\mathbb{R}^2$, which means that there is no need to employ Schur parameters. (The use of Schur parameters would double the total degrees of the numerator and denominator polynomials involved, and largely increase the number of terms.)

A popular technique to show nonnegativity of a polynomial is by writing it as a sum of squares of polynomials, for which (numerical) semi-definite programming software is available in Matlab. Although not every nonnegative polynomial admits such a decomposition, many polynomials do and counterexamples are not easily constructed. However, the available software breaks down on the polynomial of the present example, indicating no such decomposition to exist! Therefore, other (symbolic, exact) methods from constructive algebra were used to establish nonnegativity of this polynomial.

As it turns out, the polynomial has 7 global minima, of which only one is in the feasible area.

9 Conclusions

1. We have employed a state space approach to derive a bound on the achievable quality of a k-th order $\mathcal{H}_2$ approximation to a given stable system of order n. For $k = 1$ this bound is sharp and reduces the model reduction problem to a rational optimization problem in a single variable.

2. We have presented a nontrivial multivariable example of a stable system with two inputs and two outputs of order 4 and an associated unique globally optimal $\mathcal{H}_2$-approximation of order 2. To the best of our knowledge, this is the first such example reported in the literature. (The cases for which global optimality results of this kind are available are usually limited either to the scalar case, or to the case $k = 1$, or to the case $k = n - 1$, see for instance [6].)

3. We have pursued an *ad hoc* technique, involving exact methods from constructive algebra, to establish global optimality of the approximation of the example. The success of this approach gives rise to a couple of research questions:
   (i) Why is the bound sharp at the approximation $\hat{\Sigma}$?
   (ii) Why does the bound have a global maximum at the approximation $\hat{\Sigma}$?

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