COPRIME FACTOR REDUCTION OF $\mathcal{H}_\infty$ CONTROLLERS

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Abstract

We consider the efficient solution of the coprime factorization based $\mathcal{H}_\infty$ controller approximation problems by using frequency-weighted balancing related model reduction approaches. It is shown that for a class of frequency-weighted performance preserving coprime factor reduction as well as for a relative error coprime factor reduction method, the computation of the frequency-weighted controllability and observability grammians can be done by solving Lyapunov equations of the order of the controller. The new approach can be used in conjunction with accuracy enhancing square-root and balancing-free techniques developed for the balancing related coprime factors based model reduction.

1 Introduction

Using the $\mathcal{H}_\infty$ controller synthesis methodology (see for example [19]) often leads to controllers whose orders are too large for practical use. Therefore, in such cases it is necessary to perform controller reduction by determining a lower order approximation of the original controller. Controller reduction problems are often formulated as special *frequency-weighted model reduction* (FWMR) problems, where the frequency-weights are chosen to enforce closed-loop stability and an acceptable performance degradation when the low order controller is used instead the original high order one [1].

The idea to apply frequency-weighted balancing techniques to reduce the stable coprime factors of the controller has been discussed in several papers [1, 8, 18]. For the reduction of controllers originating from $\mathcal{H}_\infty$ synthesis several methods have been proposed [4, 5, 16, 2]. While the frequency-weights in [1, 8] have been primarily chosen to guarantee closed-loop stability, the $\mathcal{H}_\infty$ controller reduction mainly focusses on preserving the performance bounds achieved by the original controllers. Interestingly, many stability/performance preserving controller reduction problems have very special structure which can be exploited when developing efficient numerical algorithms for controller reduction. For example, it has been shown in [15] that for the frequency-weighted balancing related approaches applied to several controller reduction problems with the special stability/performance enforcing weights proposed in [1], the computation of grammians can be done by solving reduced order Lyapunov equations. Similarly, it was shown recently in [14] that this is also true for a class of frequency-weighted coprime factor controller reduction methods.

In this paper, we address the efficient solution of frequency-weighted balancing-related coprime factor controller reduction problems for the special stability and performance preserving frequency-weights proposed in [4, 5]. We show that for the reduction of the $\mathcal{H}_\infty$ central controller, the computation of frequency-weighted grammians for the coprime factor controller reduction can be done efficiently by solving Lyapunov equations of the order of the controller. The Lyapunov equations can be solved directly for the Cholesky factors of the grammians, thus allowing the application of the balancing-free square-root accuracy enhancing method for coprime factor reduction [13].

Notation. Throughout the paper, the following notational convention is used. The bold-notation $\mathbf{G}$ is used to denote a state-space system $\mathbf{G} := (A,B,C,D)$ with the *transfer-function matrix* (TFM)

$$G(\lambda) = C(\lambda I - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

According to the system type, λ is either the complex variable s appearing in the Laplace transform in the case of a continuous-time system or the variable z appearing in the Z-transform in the case of a discrete-time system. Throughout the paper we denote $G(\lambda)$ simply as G, when the system type is not relevant. The bold-notation is used consistently to denote systems corresponding to particular TFMs: $\mathbf{G}_1\mathbf{G}_2$ denotes the series coupling of two systems having the TFM $G_1(\lambda)G_2(\lambda)$, $\mathbf{G}_1 + \mathbf{G}_2$ represents the (additive) parallel coupling of two systems with TFM $G_1(\lambda)+G_2(\lambda)$, $\mathbf{G^{-1}}$ represents the inverse systems with TFM G^{-1}, $[\,\mathbf{G}_1\;\;\mathbf{G}_2\,]$ represents the realization of the compound TFM with $[\,G_1\;G_2\,]$, etc.

2 Coprime factor controller reduction

Consider $\mathbf{G} := (A,B,C,D)$, an n-th order state-space model and let $\mathbf{K}$ be a stabilizing controller with a stabilizable and detectable n_c-th order state space realization $\mathbf{K} := (A_c, B_c, C_c, D_c)$. The solution of the a frequency-weighted coprime factor controller reduction problem (see for example [1, 8]) consists in computing an approximation of the coprime factors of the controller. Specifically, the *Frequency-Weighted Left Coprime Factor Reduction* (FWLCFR) *Problem* is: given a *left coprime factorization* (LCF) $K = \widetilde{V}^{-1}\widetilde{U}$ of the controller, find $\mathbf{K_r}$, an r_c-th order approximation of $\mathbf{K}$, in a LCF form

$K_r = \widetilde{V}_r^{-1}\widetilde{U}_r$, such that the weighted approximation error

$$\|\widetilde{W}_o[\,\widetilde{U} - \widetilde{U}_r \;\; \widetilde{V} - \widetilde{V}_r\,]\widetilde{W}_i\|_\infty, \qquad (1)$$

is minimized. Similarly, the *Frequency-Weighted Right Coprime Factor Reduction* (FWRCFR) *Problem* is: given a *right coprime factorization* (RCF) $K = UV^{-1}$ of the controller, find $\mathbf{K_r}$, an r_c-th order approximation of $\mathbf{K}$, in the RCF form $K_r = U_r V_r^{-1}$, such that the weighted approximation error

$$\|W_o \begin{bmatrix} U - U_r \\ V - V_r \end{bmatrix} W_i\|_\infty, \qquad (2)$$

is minimized. In (1) and (2), $\widetilde{W}_o$, $\widetilde{W}_i$, W_o and W_i are stable weighting TFMs, which are specially chosen to enforce closed-loop stability and performance.

Balancing related FWMR techniques which attempt to minimize (1) or (2) can be used to determine reduced order controllers. The following procedure to solve the FWLCFR Problem is based on the FWMR approach proposed by Enns in [3]:

FWLCFR Procedure.

1. Compute the controllability grammian of $[\,\widetilde{\mathbf{U}} \;\; \widetilde{\mathbf{V}}\,]\widetilde{\mathbf{W}}_\mathbf{i}$ and the observability grammian of $\widetilde{\mathbf{W}}_\mathbf{o}[\,\widetilde{\mathbf{U}} \;\; \widetilde{\mathbf{V}}\,]$ and define according to [3], appropriate n_c order frequency-weighted controllability and observability grammians P_E and Q_E, respectively.
2. Using P_E and Q_E in place of standard grammians of $[\,\widetilde{\mathbf{U}} \;\; \widetilde{\mathbf{V}}\,]$, determine a reduced order approximation $[\,\widetilde{\mathbf{U}}_\mathbf{r} \;\; \widetilde{\mathbf{V}}_\mathbf{r}\,]$ by applying, for example, the *balanced truncation* (BT) method [9] or the *singular perturbation approximation* (SPA) [7].
3. Form $\mathbf{K_r} = \widetilde{\mathbf{V}}_\mathbf{r}^{-1}\widetilde{\mathbf{U}}_\mathbf{r}$.

A completely similar procedure can be used to solve the FWRCFR Problem:

FWRCFR Procedure.

1. Compute the controllability and observability grammians of $\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \mathbf{W_i}$ and $\mathbf{W_o} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$, respectively, and define according to [3], appropriate n_c order frequency-weighted controllability and observability grammians P_E and Q_E, respectively.
2. Using P_E and Q_E in place of standard grammians of $\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$, determine $\begin{bmatrix} \mathbf{U_r} \\ \mathbf{V_r} \end{bmatrix}$, a reduced order approximation, by applying either the BT method [9] or the SPA [7].
3. Form $\mathbf{K_r} = \mathbf{U_r}\mathbf{V_r}^{-1}$.

In this paper we focus on the efficient and numerically accurate computation of low order controllers by using these procedures to solve the frequency-weighted coprime factorization based $\mathcal{H}_\infty$ controller reduction problems formulated in [4]. Let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad (3)$$

be the TFM used to parameterize all admissible γ-suboptimal controllers [19]. It follows that K can be expressed in terms of a lower *linear fractional transformation* (LFT) in the form

$$K = \mathcal{F}_l(M,Q) := M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21},$$

where Q is a stable and proper rational matrix satisfying $\|Q\|_\infty < \gamma$. Since for standard $\mathcal{H}_\infty$ problems both M_{12} and M_{21} are invertible and minimum-phase [19], a "natural" LCF of the central controller ($Q = 0$) as $K_0 = \widetilde{V}^{-1}\widetilde{U}$ can be obtained with

$$\widetilde{U} = M_{12}^{-1}M_{11}, \quad \widetilde{V} = M_{12}^{-1}$$

while a "natural" RCF of the central controller as $K_0 = UV^{-1}$ can be obtained with

$$U = M_{11}M_{21}^{-1}, \quad V = M_{21}^{-1}$$

These coprime factorizations can be used to perform unweighted coprime factor controller reduction using accuracy enhancing model reduction algorithms [13].

The frequency weighted left coprime factor reduction formulated in [4] is one sided with

$$\widetilde{W}_o = I, \quad \widetilde{W}_i = \widetilde{\Theta}^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \qquad (4)$$

where

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} := \begin{bmatrix} M_{21} - M_{22}M_{12}^{-1}M_{11} & -M_{22}M_{12}^{-1} \\ M_{12}^{-1}M_{11} & M_{12}^{-1} \end{bmatrix}$$

Note that $\widetilde{\Theta}$ is stable, invertible and minimum-phase. With the help of the submatrices of $\widetilde{\Theta}$ it is possible to express K also as

$$K = (\widetilde{\Theta}_{22} + Q\widetilde{\Theta}_{12})^{-1}(\widetilde{\Theta}_{21} + Q\widetilde{\Theta}_{11})$$

and thus the central controller is factorized as $K_0 = \widetilde{\Theta}_{22}^{-1}\widetilde{\Theta}_{21}$.

Similarly, a frequency-weighted right coprime factor reduction can be formulated with the one sided weights

$$W_o = \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}\Theta^{-1}, \quad W_i = I \qquad (5)$$

where

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} := \begin{bmatrix} M_{12} - M_{11}M_{21}^{-1}M_{22} & M_{11}M_{21}^{-1} \\ -M_{21}^{-1}M_{22} & M_{21}^{-1} \end{bmatrix}$$

Note that this time we have

$$K = (\Theta_{12} + \Theta_{11}Q)(\Theta_{22} + \Theta_{21}Q)^{-1}$$

and the central controller is factorized as $K_0 = \Theta_{12}\Theta_{22}^{-1}$.

The importance of the above frequency-weighted coprime factor reduction can be seen from the following result [4].

Theorem 2.1 *Let K_0 be a stabilizing continuous-time γ-suboptimal $\mathcal{H}_\infty$ central controller, and let K_r be an approximation of K_0 computed by applying either the **FWLCFR Procedure** or **FWRCFR Procedure**. Then K_r stabilizes the closed-loop system and preserves the γ-suboptimal performance, provided the weighted approximation error (1) or (2) is less than $1/\sqrt{2}$.*

We conjecture that this result holds also in the discrete-time, and can be proved along the lines of the proof provided in [19].

An alternative approach to $\mathcal{H}_\infty$ controller reduction uses the relative error method as suggested in [17]. Using this approach in conjunction with the LCF reduction we can define the weights as

$$\widetilde{W}_o = [\, \widetilde{U}\ \widetilde{V}\,]^+, \quad \widetilde{W}_i = I \tag{6}$$

where $[\, \widetilde{U}\ \widetilde{V}\,]^+$ denotes a stable right inverse of $[\, \widetilde{U}\ \widetilde{V}\,]$. A variant of this approach (see [19]) is to perform a relative error coprime factor reduction on an invertible augmented minimum-phase system $[\, \widetilde{U}_a\ \widetilde{V}_a\,]$ instead of $[\, \widetilde{U}\ \widetilde{V}\,]$. In our case, $\widetilde{\Theta}$ can be taken as the augmented system. Thus this method essentially consists of determining an approximation $\widetilde{\Theta}_r$ of $\widetilde{\Theta}$ by minimizing the relative error $\widetilde{\Delta}_r = \widetilde{\Theta}^{-1}(\widetilde{\Theta} - \widetilde{\Theta}_r)$. The reduced controller is recovered from the sub-blocks (2,1) and (2,2) of $\widetilde{\Theta}_r$ as $K_r = \widetilde{\Theta}_{r,22}^{-1}\widetilde{\Theta}_{r,21}$.

A relative error RCF reduction can be formulated with the weights

$$W_o = I, \quad W_i = \left[\begin{array}{c} U \\ V \end{array}\right]^+ \tag{7}$$

where $\left[\begin{array}{c} U \\ V \end{array}\right]^+$ denotes a stable left inverse of $\left[\begin{array}{c} U \\ V \end{array}\right]$. Alternatively, an augmented relative error problem can be solved by approximating Θ by a reduced order system Θ_r which minimizes the relative error $\Delta_r = (\Theta - \Theta_r)\Theta^{-1}$. The reduced controller is recovered from the sub-blocks (1,2) and (2,2) of $\widetilde{\Theta}_r$ as $K_r = \Theta_{r,12}\Theta_{r,22}^{-1}$. This method has been also considered in [2] for the case of normalized coprime factor $\mathcal{H}_\infty$ controller reduction.

The main computational burden in applying to these problems either the **FWLCFR** or **FWRCFR** procedure is the computation of the grammians at Step 1. Apparently, the computation of grammians involves the solutions of at least one Lyapunov equation of order $2n_c$. In this paper we show that for the method of [4] as well as for the augmentation based relative error methods, the frequency-weighted grammians can be computed by solving Lyapunov equations each of order n_c. Complete formulas for both continuous- and discrete-time systems are given for both LCF and RCF based approaches.

In a separate section, we discuss shortly the direct computation of the Cholesky factors of the frequency-weighted grammians. This is a prerequisite for the applicability of the balancing-free square-root accuracy-enhancing techniques to coprime factor controller reduction of [13], along the lines of the model reduction methods developed for the BT in [12] and SPA in [11].

3 Efficient solution of frequency-weighted $\mathcal{H}_\infty$ controller reduction problems

3.1 LCF controller reduction

We consider the efficient computation of the frequency-weighted controllability grammian at Step 1 of the **FWLCFR Procedure** for the weights defined in (4). Let consider a realization of the parameterization TFM M (3) in the form

$$\mathbf{M} = \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C}_1 & \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{C}_2 & \widehat{D}_{21} & \widehat{D}_{22} \end{array}\right]$$

Note that for the central controller we have $(A_c, B_c, C_c, D_c) = (\widehat{A}, \widehat{B}_1, \widehat{C}_1, \widehat{D}_{11})$. Since M_{12} and M_{21} are stable, minimum-phase and invertible TFMs, it follows that $\widehat{D}_{12}$ and $\widehat{D}_{21}$ are invertible, $\widehat{A}$, $\widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1$ and $\widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2$ are all stable matrices, i.e., have eigenvalues in the open left half plane for a continuous-time controller and in the interior of the unit circle for a discrete-time controller. The realizations $\widetilde{\mathbf{\Theta}} = (A_{\widetilde{\Theta}}, B_{\widetilde{\Theta}}, C_{\widetilde{\Theta}}, D_{\widetilde{\Theta}})$ and $\widetilde{\mathbf{\Theta}}^{\mathbf{-1}} = (A_{\widetilde{\Theta}^{-1}}, B_{\widetilde{\Theta}^{-1}}, C_{\widetilde{\Theta}^{-1}}, D_{\widetilde{\Theta}^{-1}})$ can be computed as [19]

$$\widetilde{\mathbf{\Theta}} = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{B}_1 - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{D}_{11} & -\widehat{B}_2\widehat{D}_{12}^{-1} \\ \hline \widehat{C}_2 - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{21} - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{D}_{11} & -\widehat{D}_{22}\widehat{D}_{12}^{-1} \\ \widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{12}^{-1}\widehat{D}_{11} & \widehat{D}_{12}^{-1} \end{array}\right]$$

$$\widetilde{\mathbf{\Theta}}^{\mathbf{-1}} = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{B}_1\widehat{D}_{21}^{-1} & \widehat{B}_2 - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} \\ \hline \widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{D}_{21}^{-1} & \widehat{D}_{21}^{-1}\widehat{D}_{22} \\ \widehat{C}_1 - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{D}_{11}\widehat{D}_{21}^{-1} & \widehat{D}_{12} - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{D}_{22} \end{array}\right]$$

Since the realization of $[\, \widetilde{\mathbf{U}}\ \ \widetilde{\mathbf{V}}\,]\widetilde{\mathbf{W}}_{\mathbf{i}}$ has order $2n_c$, it follows that the solution of the controller reduction problem for the special weights defined in (4) involves the solution of a Lyapunov equation of order $2n_c$ to determine the frequency-weighted controllability grammian P_E and a Lyapunov equation of order n_c to compute the observability grammian Q_E. The following result shows that it is always possible to solve two Lyapunov equations of order n_c to compute the frequency-weighted grammians for the special weights in (4).

Lemma 3.1 *The frequency-weighted controllability grammian P_E and observability grammian Q_E according to Enn's choice [3] satisfy, according to the system type: continuous-time (c) or discrete-time (d), the corresponding Lyapunov equations*

$$(c)\left\{\begin{array}{lcl} A_{\widetilde{\Theta}^{-1}}P_E + P_E A_{\widetilde{\Theta}^{-1}}^T + \widetilde{B}_{\widetilde{\Theta}^{-1}}\widetilde{B}_{\widetilde{\Theta}^{-1}}^T & = & 0 \\ A_{\widetilde{\Theta}}^T Q_E + Q_E A_{\widetilde{\Theta}} + \widetilde{C}_{\widetilde{\Theta}}^T\widetilde{C}_{\widetilde{\Theta}} & = & 0 \end{array}\right.$$

$$(d)\left\{\begin{array}{lcl} A_{\widetilde{\Theta}^{-1}}P_E A_{\widetilde{\Theta}^{-1}}^T + \widetilde{B}_{\widetilde{\Theta}^{-1}}\widetilde{B}_{\widetilde{\Theta}^{-1}}^T & = & P_E \\ A_{\widetilde{\Theta}}^T Q_E A_{\widetilde{\Theta}} + \widetilde{C}_{\widetilde{\Theta}}^T\widetilde{C}_{\widetilde{\Theta}} & = & Q_E \end{array}\right.$$

where $\widetilde{B}_{\widetilde{\Theta}^{-1}} = B_{\widetilde{\Theta}^{-1}}\mathit{diag}\,(\gamma^{-1}I, I)$ and $\widetilde{C}_{\widetilde{\Theta}} = \widehat{D}_{12}^{-1}\widehat{C}_1$.

Proof: We can construct immediately the realization of $[\, \widetilde{\mathbf{U}}\ \widetilde{\mathbf{V}}\,]\widetilde{\mathbf{W}}_{\mathbf{i}} := (\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ with

$$
\overline{A}_i = \left[\begin{array}{cc} \widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 \\ 0 & \widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 \end{array} \right] \quad (8)
$$

$$
\overline{B}_i = \left[\begin{array}{cc} \gamma^{-1}\widehat{B}_1\widehat{D}_{21}^{-1} & -\widehat{B}_2 + \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} \\ -\gamma^{-1}\widehat{B}_1\widehat{D}_{21}^{-1} & \widehat{B}_2 - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} \end{array} \right]
$$

and let $\overline{P}_i$ and Q be the controllability grammian of $[\,\widetilde{\mathbf{U}}\ \widetilde{\mathbf{V}}\,]\widetilde{\mathbf{W}}_{\mathbf{i}}$ and the observability grammian of $[\,\widetilde{\mathbf{U}}\ \widetilde{\mathbf{V}}\,]$, respectively. According to the system type, $\overline{P}_i$ and Q satisfy the corresponding Lyapunov equations

$$
(c)\left\{ \begin{array}{rcl} \overline{A}_i\overline{P}_i + \overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T & = & 0 \\ A_{\widetilde{\Theta}}^T Q + QA_{\widetilde{\Theta}} + (\widehat{D}_{12}^{-1}\widehat{C}_1)^T\widehat{D}_{12}^{-1}\widehat{C}_1 & = & 0 \end{array} \right.
$$

$$
(d)\left\{ \begin{array}{rcl} \overline{A}_i\overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T & = & \overline{P}_i \\ A_{\widetilde{\Theta}}^T QA_{\widetilde{\Theta}} + (\widehat{D}_{12}^{-1}\widehat{C}_1)^T\widehat{D}_{12}^{-1}\widehat{C}_1 & = & Q \end{array} \right.
$$

Partition $\overline{P}_i$ in accordance with the structure of $\overline{A}_i$ in (8)

$$
\overline{P}_i = \left[\begin{array}{cc} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{12}^T & \overline{P}_{22} \end{array} \right] \quad (9)
$$

such that $\overline{P}_{11}$ is an $n_c \times n_c$ matrix. Enns defines in [3] $P_E = \overline{P}_{11}$ and $Q_E = Q$ as the frequency-weighted controllability and observability grammians, respectively.

Consider the transformation matrix T

$$
T = \left[\begin{array}{cc} I_{n_c} & -I_{n_c} \\ 0 & I_{n_c} \end{array} \right]
$$

It is easy to see that the controllability grammian $\widetilde{P}_i$ for the transformed pair $(\widetilde{A}_i, \widetilde{B}_i) := (T^{-1}\overline{A}_iT, T^{-1}\overline{B}_i)$ has the form $\widetilde{P}_i = \mathrm{diag}\,(0, P_i)$, where P_i satisfies the appropriate Lyapunov equation

$$
\begin{array}{lll} (c) & A_{\widetilde{\Theta}^{-1}}P_i + P_iA_{\widetilde{\Theta}^{-1}}^T + \widetilde{B}_{\widetilde{\Theta}^{-1}}\widetilde{B}_{\widetilde{\Theta}^{-1}}^T & = 0 \\ (d) & A_{\widetilde{\Theta}^{-1}}P_iA_{\widetilde{\Theta}^{-1}} + \widetilde{B}_{\widetilde{\Theta}^{-1}}\widetilde{B}_{\widetilde{\Theta}^{-1}}^T & = P_i \end{array} \quad (10)
$$

The grammian in the original coordinate basis results as

$$
\overline{P}_i = T\widetilde{P}_iT^T = \left[\begin{array}{cc} P_i & -P_i \\ -P_i & P_i \end{array} \right]
$$

Thus, the frequency-weighted controllability grammian according to Enns' method is $P_E = P_i$, the leading $n_c \times n_c$ block of $\overline{P}_i$. □

Remark. It is easy to see that $[\,\widetilde{U}\ \ \widetilde{V}\,]\widetilde{W}_i = [\,0\ I\,]$, thus complete pole-zero cancellation takes place between the system to be reduced and the input weight. This situation is typical for several frequency-weighted controller reduction problems (see for instance [1, 15, 19]) and can be addressed by using Enns' choice of frequency-weighted grammians.

3.2 Relative error LCF reduction

The relative error approximation of $\widetilde{\Theta}$ is in fact a FWMR problem with the weights $W_o = \widetilde{\Theta}^{-1}$ and $W_i = I$. We have the following straightforward result [19, Theorem 7.5]:

Lemma 3.2 *The frequency-weighted controllability grammian P_E and observability grammian Q_E for Enns' method [3] satisfy, according to the system type, the corresponding Lyapunov equations*

$$
(c)\left\{ \begin{array}{rcl} A_{\widetilde{\Theta}}P_E + P_EA_{\widetilde{\Theta}}^T + B_{\widetilde{\Theta}}B_{\widetilde{\Theta}}^T & = & 0 \\ A_{\widetilde{\Theta}^{-1}}^T Q_E + Q_EA_{\widetilde{\Theta}^{-1}} + C_{\widetilde{\Theta}^{-1}}^T C_{\widetilde{\Theta}^{-1}} & = & 0 \end{array} \right.
$$

$$
(d)\left\{ \begin{array}{rcl} A_{\widetilde{\Theta}}P_EA_{\widetilde{\Theta}} + B_{\widetilde{\Theta}}B_{\widetilde{\Theta}}^T & = & P_E \\ A_{\widetilde{\Theta}^{-1}}^T Q_EA_{\widetilde{\Theta}^{-1}} + C_{\widetilde{\Theta}^{-1}}^T C_{\widetilde{\Theta}^{-1}} & = & Q_E \end{array} \right.
$$

Remark. For the relative error method with the weights given in (6), a right inverse can be immediately constructed as

$$
[\,\widetilde{U}\ \widetilde{V}\,]^+ = \left[\begin{array}{c} M_{21}^{-1}M_{22} \\ M_{12} - M_{11}M_{21}^{-1}M_{22} \end{array} \right]
$$

A realization of the output weight $W_o = [\,\widetilde{U}\ \widetilde{V}\,]^+$ is given by

$$
\mathbf{W_o} = \left[\begin{array}{c|c} A_{\widetilde{\Theta}^{-1}} & \widetilde{B}_{\widetilde{\Theta}^{-1}} \\ \hline C_{\widetilde{\Theta}^{-1}} & \widetilde{D}_{\widetilde{\Theta}^{-1}} \end{array} \right]
$$

where $\widetilde{B}_{\widetilde{\Theta}^{-1}} = B_{\widetilde{\Theta}^{-1}}[\,0\ \ I\,]^T$ and $\widetilde{D}_{\widetilde{\Theta}^{-1}} = D_{\widetilde{\Theta}^{-1}}[\,0\ \ I\,]^T$. Thus, the grammians P_E and Q_E used in Lemma 3.2 are the controllability grammian of $[\,\widetilde{U}\ \widetilde{V}\,]$ and the observability grammian of $[\,\widetilde{U}\ \widetilde{V}\,]^+$, respectively. □

3.3 RCF controller reduction

We consider the efficient computation of the frequency-weighted controllability grammian at Step 1 of the **FWRCFR Procedure** for the weights defined in (5). The realizations $\mathbf{\Theta} = (A_\Theta, B_\Theta, C_\Theta, D_\Theta)$ and $\mathbf{\Theta}^{-1} = (A_{\Theta^{-1}}, B_{\Theta^{-1}}, C_{\Theta^{-1}}, D_{\Theta^{-1}})$ can be computed as [19]

$$
\mathbf{\Theta} = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{B}_2 - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{B}_1\widehat{D}_{21}^{-1} \\ \hline \widehat{C}_1 - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{D}_{12} - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{D}_{11}\widehat{D}_{21}^{-1} \\ -\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{D}_{21}^{-1} \end{array} \right]
$$

$$
\mathbf{\Theta}^{-1} = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{B}_2\widehat{D}_{12}^{-1} & \widehat{B}_1 - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{D}_{11} \\ \hline -\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{12}^{-1} & -\widehat{D}_{12}^{-1}\widehat{D}_{11} \\ \widehat{C}_2 - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{22}\widehat{D}_{12}^{-1} & \widehat{D}_{21} - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{D}_{11} \end{array} \right]
$$

Since the realization of $\mathbf{W_o}\left[\begin{array}{c} \mathbf{U} \\ \mathbf{V} \end{array} \right]$ has order $2n_c$, it follows that the solution of the controller reduction problem for the special weights defined in (5) involves the solution of a Lyapunov equation of order $2n_c$ to determine the frequency-weighted controllability grammian P_E and a Lyapunov equation of order n_c to compute the observability grammian Q_E. The following result shows that it is always possible to solve two Lyapunov equations of order n_c to compute the frequency-weighted grammians for the special weights in (5).

Lemma 3.3 *The frequency-weighted controllability grammian P_E and observability grammian Q_E for Enns' method [3] satisfy, according to the system type, the corresponding Lyapunov* *equations*

$$(c)\begin{cases} A_\Theta P_E + P_E A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T & = 0 \\ A_{\Theta^{-1}}^T Q_E + Q_E A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} & = 0 \end{cases}$$

$$(d)\begin{cases} A_\Theta P_E A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T & = P_E \\ A_{\Theta^{-1}}^T Q_E A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} & = Q_E \end{cases}$$

where $\widetilde{B}_\Theta = B_\Theta \begin{bmatrix} 0 \\ I \end{bmatrix} = \widehat{B}_1 \widehat{D}_{21}^{-1}$ *and* $C_{\Theta^{-1}} =$ *diag* $(\gamma^{-1} I, I) C_{\Theta^{-1}}$.

Proof: We can construct the realization of $\mathbf{W_o}\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} := (\overline{A}_o, \overline{B}_o, \overline{C}_o, \overline{D}_o)$ with the matrices

$$\overline{A}_o = \begin{bmatrix} \widehat{A} - \widehat{B}_2 \widehat{D}_{12}^{-1} \widehat{C}_1 & -\widehat{B}_1 \widehat{D}_{21}^{-1} \widehat{C}_2 + \widehat{B}_2 \widehat{D}_{12}^{-1} \widehat{C}_1 \\ 0 & \widehat{A} - \widehat{B}_1 \widehat{D}_{21}^{-1} \widehat{C}_2 \end{bmatrix} \quad (11)$$

$$\overline{C}_o = \begin{bmatrix} -\gamma^{-1} \widehat{D}_{12}^{-1} \widehat{C}_1 & \gamma^{-1} \widehat{D}_{12}^{-1} \widehat{C}_1 \\ \widehat{C}_2 - \widehat{D}_{22} \widehat{D}_{12}^{-1} \widehat{C}_1 & -\widehat{C}_2 + \widehat{D}_{22} \widehat{D}_{12}^{-1} \widehat{C}_1 \end{bmatrix}$$

Let P and $\overline{Q}_o$ be the controllability and observability grammians of $\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$ and $\mathbf{W_o}\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$, respectively. According to the system type, P and $\overline{Q}_o$ satisfy the corresponding Lyapunov equations

$$(c)\ \begin{cases} A_\Theta P + P A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T = 0 \\ \overline{A}_o^T \overline{Q}_o + \overline{Q}_o \overline{A}_o + \overline{C}_o^T \overline{C}_o = 0 \end{cases}$$

$$(d)\ \begin{cases} A_\Theta P A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T = P \\ \overline{A}_o^T \overline{Q}_o \overline{A}_o + \overline{C}_o^T \overline{C}_o = \overline{Q}_o \end{cases}$$

Partition $\overline{Q}_o$ in accordance with the structure of the matrix $\overline{A}_o$ in (11)

$$\overline{Q}_o = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{12}^T & \overline{Q}_{22} \end{bmatrix} \quad (12)$$

where $\overline{Q}_{22}$ is an $n_c \times n_c$ matrix. The approach proposed by [3] defines

$$P_E = P, \quad Q_E = \overline{Q}_{22} \quad (13)$$

as the frequency-weighted observability grammian.

Consider the transformation matrix T

$$T = \begin{bmatrix} I_{n_c} & I_{n_c} \\ 0 & I_{n_c} \end{bmatrix}$$

It is easy to see that the observability grammian $\widetilde{Q}_o$ for the transformed pair $(\widetilde{A}_o, \widetilde{C}_o) := (T^{-1}\overline{A}_o T, \overline{C}_o T)$ has the form

$$\widetilde{Q}_o = \begin{bmatrix} Q_o & 0 \\ 0 & 0 \end{bmatrix}$$

where Q_o satisfies the appropriate Lyapunov equation

$$\begin{array}{ll} (c) & A_{\Theta^{-1}}^T Q_o + Q_o A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} = 0 \\ (d) & A_{\Theta^{-1}}^T Q_o A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} = Q_o \end{array} \quad (14)$$

The grammian in original coordinates results as

$$\overline{Q}_o = T^{-T} \widetilde{Q}_o T^{-1} = \begin{bmatrix} Q_o & -Q_o \\ -Q_o & Q_o \end{bmatrix}$$

According to Enns' method, the frequency-weighted observability grammian is $Q_E = Q_o$, the trailing $n_c \times n_c$ block of $\overline{Q}_o$ in (12). □

3.4 Relative error RCF reduction

The relative error approximation of Θ is a FWMR problem with the weights $W_o = I$ and $W_i = \Theta^{-1}$. We have the following straightforward result [19, Theorem 7.5]:

Lemma 3.4 *The frequency-weighted controllability grammian P_E and observability grammian Q_E for Enns' method [3] satisfy, according to the system type, the corresponding Lyapunov equations*

$$(c)\begin{cases} A_{\Theta^{-1}} P_E + P_E A_{\Theta^{-1}}^T + B_{\Theta^{-1}} B_{\Theta^{-1}}^T & = 0 \\ A_\Theta^T Q_E + Q_E A_\Theta + C_\Theta^T C_\Theta & = 0 \end{cases}$$

$$(d)\begin{cases} A_{\Theta^{-1}} P_E A_{\Theta^{-1}} + B_{\Theta^{-1}} B_{\Theta^{-1}}^T & = P_E \\ A_\Theta^T Q_E A_\Theta + C_\Theta^T C_\Theta & = Q_E \end{cases}$$

Remark. For the relative error method with the weights given in (6), a left inverse can be immediately constructed as

$$\begin{bmatrix} U \\ V \end{bmatrix}^+ = [\, M_{22} M_{12}^{-1} \quad M_{12} - M_{22} M_{12}^{-1} M_{11} \,]$$

A realization of the input weight $W_i = \begin{bmatrix} U \\ V \end{bmatrix}^+$ is given by

$$\mathbf{W_i} = \left[\begin{array}{c|c} A_{\Theta^{-1}} & B_{\Theta^{-1}} \\ \hline \widetilde{C}_{\Theta^{-1}} & \widetilde{D}_{\Theta^{-1}} \end{array}\right]$$

where $\widetilde{C}_{\Theta^{-1}} = [\,0\ I\,] C_{\Theta^{-1}}$ and $\widetilde{D}_{\Theta^{-1}} = [\,0\ I\,] D_{\Theta^{-1}}$. Thus, the grammians P_E and Q_E used in Lemma 3.4 are the controllability grammian of $\begin{bmatrix} U \\ V \end{bmatrix}^+$ and the observability grammian of $\begin{bmatrix} U \\ V \end{bmatrix}$, respectively. □

4 Square-root techniques

Accuracy enhancing balancing-free square-root techniques for coprime factor model reduction have been proposed in [13] along the lines of similar methods developed for the BT in [12] and SPA in [11]. The key computation in the proposed procedures is the determination of the Cholesky factors of the grammians such that $P_E = S_E S_E^T$ and $Q = R_E^T R_E$. The method of Hammarling [6] can be generally employed to solve the Lyapunov equations in question directly for the Cholesky factors. Having these factors, the reduction of coprime factors can be performed by computing two truncation matrices L and T using the singular value decomposition

$$R_E S_E = [\ U_1 \quad U_2\ ]\, \mathrm{diag}(\Sigma_1, \Sigma_2)\, [\ V_1 \quad V_2\ ]^T$$

with $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_{r_c})$, $\Sigma_2 = \text{diag}(\sigma_{r_c+1}, \ldots, \sigma_{n_c})$ and $\sigma_1 \geq \ldots \geq \sigma_{r_c} > \sigma_{r_c+1} \geq \ldots \geq \sigma_{n_c} \geq 0$. The *square-root* method determines L and T as [10]

$$L = \Sigma_1^{-1/2} U_1^T R_E, \qquad T = S_E V_1 \Sigma_1^{-1/2}.$$

If the original system is highly unbalanced, potential accuracy losses can be induced in the reduced model if either of the truncation matrices L or T is ill-conditioned (i.e., nearly rank deficient). To avoid ill-conditioned truncation matrices, *balancing-free* approaches can be used, as for example, the *balancing-free square-root* algorithm for the BT introduced by [12]. Similar formulas have been developed for the SPA approach in [11].

5 Conclusions

Efficient and numerically reliable balancing related computational approaches have been proposed for the frequency-weighted coprime factors $\mathcal{H}_\infty$ controller reduction with special frequency weights enforcing closed-loop stability and performance. To compute lower order approximations of the coprime factors, ”natural” coprime factorizations of the central $\mathcal{H}_\infty$ controller are used, which result from the parameterization of all suboptimal $\mathcal{H}_\infty$ controllers. We developed complete formulas to compute the frequency-weighted grammians for both LCF and RCF based reductions, which are generally applicable for the reduction of all types of $\mathcal{H}_\infty$ controllers, provided the associated parameterization of all controllers is also available. To compute the grammians, in all cases it is sufficient to solve two Lyapunov equations of the order of the controller. Therefore, the new procedures are sensibly more efficient than the standard frequency-weighted balancing based reduction approach. The frequency weighted grammians can be determined directly in Cholesky factored forms to facilitate the application of square-root and balancing-free accuracy enhancing techniques.

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