POSITIVE REACHABILITY OF DISCRETE-TIME LINEAR SYSTEMS

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Abstract

This paper considers a necessary and sufficient condition for a multiple input discrete-time linear system to be positive reachable based on the Jordan canonical form. It is pointed out that the reachability of a given system can be reduced to those of its subsystems with nonnegative eigenvalues. Because the dimension of the subsystem is much smaller than that of the given system, the reachability test can be simplified considerably.

1 Introduction

While the problem of unconstrained controllability of linear systems is completely solved [3], no more than fragmentary results are available in the constrained cases. As for the controllability under positive input, necessary and sufficient conditions for continuous-time linear systems were obtained which arise frequently in the practical problems, such as antivibration control of pendulums system [6], optimal control of economic system [1], electrically heated oven system [7], and tracer kinetics in medical system [5]. In general, there are two types of controllabilities considering the final state; the first is null-controllability where the final state is the origin, on the contrary, the second is reachability where the final state is arbitrary. It is known that these two types of controllabilities differ in discrete-time systems [4]. Reachability under positive input was investigated by Evans et. al. for single input discrete-time linear systems [2]. On the other hand, null-controllability under positive input was discussed by the authors for multiple input discrete-time linear systems [8]. Although reachability under positive input may be considered to arise frequently in the practical problems, as is mentioned above, it is unfortunate that the generalization of the results of [2] to the multiple input case is still incomplete. The purpose of this paper is to consider necessary and sufficient conditions for the reachability of multiple input discrete-time linear systems with positive controls.

2 Preliminaries

Consider a multiple input discrete-time linear system described by

$$S : \boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)\,;\ k = 0, 1, 2, \ldots \tag{1}$$

where

$$\boldsymbol{A} \in \mathbb{R}^{n\times n},\ \boldsymbol{B} \in \mathbb{R}^{n\times m},\ \boldsymbol{x}(k) \in \mathbb{R}^{n},\ \boldsymbol{u}(k) \in \mathbb{R}^{m} \tag{2}$$

The control input is limited to the following condition

$$C : 0 \le u_i(k) < \infty\,;\ i = 1, 2, \ldots, m \tag{3}$$

where $u_i(k)$ is the i-th component of $\boldsymbol{u}(k)$.

Definition 1 *The control input which satisfies condition C is called a positive control.*

Definition 2 *Let $\boldsymbol{x}_f$ be any final state. Then system S is called positive reachable if there exist some positive integer N and some positive control sequence $\{\boldsymbol{u}(0), \boldsymbol{u}(1), \ldots, \boldsymbol{u}(N-1)\}$ which will bring the system from $\boldsymbol{x}(0) = \boldsymbol{0}$ to $\boldsymbol{x}(N) = \boldsymbol{x}_f$.*

Definition 3 *If $x_i > 0$ or $x_i \ge 0$ for all $i = 1, 2, \ldots, n$, then $\boldsymbol{x} \equiv [x_1, x_2, \ldots, x_n]^T$ is called a positive vector ($\boldsymbol{x} > \boldsymbol{0}$), or a nonnegative vector ($\boldsymbol{x} \ge \boldsymbol{0}$), respectively, where T denotes the transposition.*

Definition 4 *(Notation)*

$$\langle\langle \boldsymbol{A}, \boldsymbol{B}, N\rangle\rangle \equiv \left[\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \ldots, \boldsymbol{A}^{N-1}\boldsymbol{B}\right] \in \mathbb{R}^{n\times Nm} \tag{4}$$

$$\boldsymbol{U}[N] \equiv \left[{\boldsymbol{u}_1}^T, {\boldsymbol{u}_2}^T, \ldots, {\boldsymbol{u}_N}^T\right]^T \in \mathbb{R}^{Nm} \tag{5}$$

$$\boldsymbol{E}_n \equiv [1, 2, \ldots, n]^T \in \mathbb{R}^{n} \tag{6}$$

$$\|\boldsymbol{x}\| \equiv (\boldsymbol{x}^T\boldsymbol{x})^{1/2} \tag{7}$$

$$\|\boldsymbol{A}\| \equiv \max_{\boldsymbol{x}\ne\boldsymbol{0}} \|\boldsymbol{A}\boldsymbol{x}\| \,/\, \|\boldsymbol{x}\| \tag{8}$$

Furthermore, we use $\boldsymbol{I}_n$ as the $n\times n$ identity matrix.

Definition 5 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be transformed into*

$$\boldsymbol{A} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \boldsymbol{0} \\ \tilde{\boldsymbol{A}}_{21} & \tilde{\boldsymbol{A}}_{22} \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} \tilde{\boldsymbol{B}}_{1} \\ \tilde{\boldsymbol{B}}_{2} \end{bmatrix},$$
$$\tilde{\boldsymbol{A}}_{11} \in \mathbb{R}^{\tilde{n}(1)\times\tilde{n}(1)},\ \tilde{\boldsymbol{A}}_{21} \in \mathbb{R}^{\tilde{n}(2)\times\tilde{n}(1)},$$
$$\tilde{\boldsymbol{A}}_{22} \in \mathbb{R}^{\tilde{n}(2)\times\tilde{n}(2)},\ \tilde{\boldsymbol{B}}_{1} \in \mathbb{R}^{\tilde{n}(1)\times m},$$
$$\tilde{\boldsymbol{B}}_{2} \in \mathbb{R}^{\tilde{n}(2)\times m},\ n = \tilde{n}(1) + \tilde{n}(2) \tag{9}$$

by a nonsingular real transformation of the state variable. Then system $\tilde{S}$ described by

$$\tilde{S} : \tilde{\boldsymbol{x}}(k+1) = \tilde{\boldsymbol{A}}_{22}\tilde{\boldsymbol{x}}(k) + \tilde{\boldsymbol{B}}_{2}\tilde{\boldsymbol{u}}(k)\,;\ k = 0, 1, 2, \ldots \tag{10}$$

where

$$\tilde{\boldsymbol{x}}(k) \in \mathbb{R}^{\tilde{n}(2)}$$

is called a subsystem of S.

3 Positive Reachability

To discuss the necessary and sufficient condition for system S to be positive reachable, we first give the following two lemmas. These are almost evident from the definitions and the fact that the reachability is invariant under any nonsingular real transformation of the state variable.

Lemma 1 *System S is positive reachable, if and only if for any $n \times 1$ vector $\boldsymbol{x}$, there exist a vector $\boldsymbol{U}[N]$ such that*

$$\langle\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\rangle \boldsymbol{U}[N] = \boldsymbol{x}\,,\ \boldsymbol{U}[N] \geq \boldsymbol{0} \tag{11}$$

Lemma 2 *If system S is positive reachable, then the following two conditions hold:*

$$\text{①}\quad \text{rank}\langle\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\rangle = n \tag{12}$$

② *any subsystem of S is positive reachable.*

Next, we have Theorem 1.

Theorem 1 *System S is positive reachable, if and only if the following two conditions hold:*

$$\text{①}\quad \text{rank}\langle\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\rangle = n \tag{13}$$

② *there exist a vector $\boldsymbol{U}[N]$ such that*

$$\langle\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\rangle \boldsymbol{U}[N] = \boldsymbol{0}\,,\ \boldsymbol{U}[N] \geq \boldsymbol{0}\,,\ N \geq n \tag{14}$$

$$\boldsymbol{u}_i > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{15}$$

The proof is in Appendix A.

Remark 1 *As is evident from the proof of Theorem 1, if system S is positive reachable, then any final state $\boldsymbol{x}_f$ can be reached in at most N steps where N is independent on $\boldsymbol{x}_f$.*

Further, we have Lemma 3.

Lemma 3 *System S is positive reachable, if and only if the following two conditions hold:*

$$\text{①}\quad \text{rank}\langle\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\rangle = n \tag{16}$$

② *there exist a vector $\boldsymbol{U}[N]$ such that*

$$\langle\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\rangle \boldsymbol{U}[N] = \boldsymbol{\epsilon}\,,\ \boldsymbol{U}[N] \geq \boldsymbol{0}\,,\ N \geq n \tag{17}$$

$$\boldsymbol{u}_i > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{18}$$

where $\boldsymbol{\epsilon}$ is some $n \times 1$ vector, and $\|\boldsymbol{\epsilon}\|$ is sufficiently small.

The proof is in Appendix B.

Next we decompose system S into the following two subsystems:

$$S_t \ :\ \boldsymbol{x}_t(k+1) = \boldsymbol{A}_t \boldsymbol{x}_t(k) + \boldsymbol{B}_t \boldsymbol{u}(k) \tag{19}$$

$$S_q \ :\ \boldsymbol{x}_q(k+1) = \boldsymbol{A}_q \boldsymbol{x}_q(k) + \boldsymbol{B}_q \boldsymbol{u}(k) \tag{20}$$

where

$$\boldsymbol{A}_t \in \mathbb{R}^{n_t \times n_t}\,,\ \boldsymbol{B}_t \in \mathbb{R}^{n_t \times m}\,,\ \boldsymbol{x}_t(k) \in \mathbb{R}^{n_t}\,,$$
$$\lambda_i(\boldsymbol{A}_t) < 0 \text{ or } \text{Im}\{\lambda_i(\boldsymbol{A}_t)\} \neq 0\,;\ i = 1, 2, \ldots, n_t \tag{21}$$

$$\boldsymbol{A}_q \in \mathbb{R}^{n_q \times n_q}\,,\ \boldsymbol{B}_q \in \mathbb{R}^{n_q \times m}\,,\ \boldsymbol{x}_q(k) \in \mathbb{R}^{n_q}\,,$$
$$\lambda_i(\boldsymbol{A}_q) \geq 0\,;\ i = 1, 2, \ldots, n_q \tag{22}$$

$$n = n_t + n_q \tag{23}$$

and $\lambda_i(\boldsymbol{A})$ denotes the i-th eigenvalue of $\boldsymbol{A}$.

Then we have Lemma 4.

Lemma 4 *System S is positive reachable, if and only if the following two conditions hold:*

$$\text{①}\quad \text{rank}\langle\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\rangle = n \tag{24}$$

② *system S_q is positive reachable.*

The proof is given in Appendix C.

From Lemma 4, the reachability of S can basically be reduced to that of S_q.

Next if system S_q has more than two distinct eigenvalues, then we can decompose system S_q into the following two subsystems:

$$S_a \ :\ \boldsymbol{x}_a(k+1) = \boldsymbol{A}_a \boldsymbol{x}_a(k) + \boldsymbol{B}_a \boldsymbol{u}(k) \tag{25}$$

$$S_b \ :\ \boldsymbol{x}_b(k+1) = \boldsymbol{A}_b \boldsymbol{x}_b(k) + \boldsymbol{B}_b \boldsymbol{u}(k) \tag{26}$$

where

$$\boldsymbol{A}_a \in \mathbb{R}^{n_a \times n_a}\,,\ \boldsymbol{B}_a \in \mathbb{R}^{n_a \times m}\,,\ \boldsymbol{x}_a(k) \in \mathbb{R}^{n_a}\,,$$
$$\boldsymbol{A}_b \in \mathbb{R}^{n_b \times n_b}\,,\ \boldsymbol{B}_b \in \mathbb{R}^{n_b \times m}\,,\ \boldsymbol{x}_b(k) \in \mathbb{R}^{n_b}\,,$$
$$0 \leq \lambda_i(\boldsymbol{A}_a) < \lambda_j(\boldsymbol{A}_b)\,;$$
$$i = 1, 2, \ldots, n_a\,;\ j = 1, 2, \ldots, n_b \tag{27}$$

$$n_q = n_a + n_b \tag{28}$$

Then we establish Lemma 5.

Lemma 5 *System S is positive reachable, if and only if the following three conditions hold:*

$$\text{①}\quad \text{rank}\langle\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\rangle = n \tag{29}$$

② *system S_a is positive reachable.*

③ *system S_b is positive reachable.*

The proof is given in Appendix D.

From Lemma 5, the reachability of S can be reduced to those of its subsystems S_a and S_b.

Furthermore, if system S_q has Q distinct nonnegative eigenvalues, then we can transform $\boldsymbol{A}_q$ and $\boldsymbol{B}_q$ into the following Jordan canonical form by a nonsingular real transformation:

$$\boldsymbol{A}_q = \text{block diag}\,[\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_Q] \in \mathbb{R}^{n_q \times n_q}\,,$$
$$\boldsymbol{B}_q = \left[{\boldsymbol{B}_1}^T, {\boldsymbol{B}_2}^T, \ldots, {\boldsymbol{B}_Q}^T\right]^T \in \mathbb{R}^{n_q \times m} \tag{30}$$

$$n_q = \sum_{i=1}^{Q} n(i) \tag{31}$$

$$\boldsymbol{A}_i = \text{block diag}\left[\boldsymbol{A}_{i1}, \boldsymbol{A}_{i2}, \ldots, \boldsymbol{A}_{ir(i)}\right] \in \mathbb{R}^{n(i)\times n(i)}\,,$$

$$\boldsymbol{B}_i = \left[{\boldsymbol{B}_{i1}}^T, {\boldsymbol{B}_{i2}}^T, \ldots, {\boldsymbol{B}_{ir(i)}}^T\right]^T \in \mathbb{R}^{n(i)\times m}\,,$$

$$n(i) = \sum_{j=1}^{r(i)} n(i,j)\,;\ i = 1,2,\ldots,Q \tag{32}$$

$$\boldsymbol{A}_{ij} = \boldsymbol{J}\left[\lambda_i, n(i,j)\right] \in \mathbb{R}^{n(i,j)\times n(i,j)}\,,$$

$$\boldsymbol{B}_{ij} = \left[{\boldsymbol{b}_{1ij}}^T, {\boldsymbol{b}_{2ij}}^T, \ldots, {\boldsymbol{b}_{n(i,j)ij}}^T\right]^T \in \mathbb{R}^{n(i,j)\times m}\,;$$

$$i = 1,2,\ldots,Q\,;\ j = 1,2,\ldots,r(i) \tag{33}$$

$$\boldsymbol{b}_{ijk} \in \mathbb{R}^{1\times m}\,;\ i = 1,2,\ldots,Q\,;\ j = 1,2,\ldots,r(i)\,;$$

$$k = 1,2,\ldots,n(i,j) \tag{34}$$

where $\boldsymbol{J}[\lambda, n]$ denotes the lower Jordan block of order n with eigenvalue λ. Further we can assume without loss of generality that

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_Q \tag{35}$$

$$1 \le n(i,1) \le n(i,2) \le \cdots \le n(i, r(i)) \tag{36}$$

Then by using Lemma 5 repeatedly, we can directly obtain Lemma 6.

Lemma 6 *System S is positive reachable, if and only if the following two conditions hold:*

① $\text{rank}\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n\rangle\!\rangle = n$ (37)

② *for each $i = 1,2,\ldots,Q$, system S_i described by*

$$Si\ :\ \boldsymbol{x}_i(k+1) = \boldsymbol{A}_i\boldsymbol{x}_i(k) + \boldsymbol{B}_i\boldsymbol{u}(k) \tag{38}$$

is positive reachable.

From Lemma 6, the reachability of S can be reduced to those of its Q subsystems S_i $(i = 1,2,\ldots,Q)$ with Q distinct non-negative eigenvalues.

Now we introduce the following Q systems:

$$S_i^*\ :\ \boldsymbol{x}_i^*(k+1) = \boldsymbol{A}_i^*\boldsymbol{x}_i^*(k) + \boldsymbol{B}_i^*\boldsymbol{u}(k) \tag{39}$$

where

$$\boldsymbol{A}_i^* \equiv \lambda_i \boldsymbol{I}_{r(i)}\,,\ \lambda_i \ge 0 \tag{40}$$

$$\boldsymbol{B}_i^* \equiv \left[{\boldsymbol{b}_{i11}}^T, {\boldsymbol{b}_{i21}}^T, \ldots, {\boldsymbol{b}_{ir(i)1}}^T\right]^T \in \mathbb{R}^{r(i)\times m} \tag{41}$$

for each $i = 1,2,\ldots,Q$.

Then, we have Theorem 2.

Theorem 2 *System S_i^* is positive reachable, if and only if the following two conditions hold:*

① $\text{rank}\boldsymbol{B}_i^* = r(i)$ (42)

② *there exist a positive vector $\boldsymbol{U}_i$ such that*

$$\boldsymbol{B}_i^*\boldsymbol{U}_i = \boldsymbol{0}\,,\ \boldsymbol{U}_i > \boldsymbol{0} \tag{43}$$

(Proof) Applying system S for system S_i^* in Theorem 1, we can easily obtain Theorem 2. Q.E.D.

Remark 2 *In Theorem 2, $r(i) < m$ is necessary for system S_i^* to be positive reachable. Thus, a single input system $(m = 1)$ which contains any nonnegative eigenvalues is not positive reachable. This agrees with the former results [2].*

When $r(i)$ is small, it is not so difficult to find the positive vector $\boldsymbol{U}$ which satisfies (43). Thus the reachability of S_i^* can be checked easily by using Theorem 2.

Now we consider the following two systems for any positive integers r, m, and P:

$$S^*\ :\ \boldsymbol{x}^*(k+1) = \boldsymbol{A}^*\boldsymbol{x}^*(k) + \boldsymbol{B}^*\boldsymbol{u}(k) \tag{44}$$

$$S_P^+\ :\ \boldsymbol{x}^+(k+1) = \boldsymbol{\Phi}_P\boldsymbol{x}^+(k) + \boldsymbol{\Gamma}_P\boldsymbol{u}(k) \tag{45}$$

where

$$\boldsymbol{A}^* \equiv \lambda\boldsymbol{I}_r\,,\ \lambda \ge 0\,,\ \boldsymbol{B}^* \in \mathbb{R}^{r\times m} \tag{46}$$

$$\boldsymbol{\Phi}_P \equiv \begin{bmatrix} \boldsymbol{A}^* & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{I}_r & \boldsymbol{A}^* & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{A}^* \end{bmatrix} \in \mathbb{R}^{rP\times rP} \tag{47}$$

$$\boldsymbol{\Gamma}_P \equiv \begin{bmatrix} \boldsymbol{B}_1^+ \\ \boldsymbol{B}_2^+ \\ \cdots \\ \boldsymbol{B}_P^+ \end{bmatrix} \in \mathbb{R}^{rP\times m} \tag{48}$$

$$\boldsymbol{B}_1^+ \equiv \boldsymbol{B}^*\,,\ \boldsymbol{B}_i^+ \in \mathbb{R}^{r\times m}\,;\ i = 1,2,\ldots,P \tag{49}$$

Then we can show the following lemma by mathematical induction method.

Lemma 7 *System S^* is positive reachable, if and only if system S_P^+ is positive reachable.*

Finally Theorem 3 can be obtained.

Theorem 3 *System S is positive reachable, if and only if the following two conditions hold:*

① $\text{rank}\boldsymbol{B}_i^* = r(i)$ (50)

② *for each $i = 1,2,\ldots,Q$, system S_i^* is positive reachable.*

The proof is given in Appendix E.

From Theorem 3, the reachability of S can be reduced to those of its Q subsystems S_i^* $(i = 1,2,\ldots,Q)$ with Q distinct non-negative eigenvalues. Because the dimension of system S_i^* is much smaller than that of system S, the reachability test can be simplified considerably by using Theorem 2 and Theorem 3.

4 Example

Consider a system S represented by

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix},$$
$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 1 & -4 \\ 3 & 1 & 1 & -2 \\ 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 \end{bmatrix} \tag{51}$$

where

$$n = 6\,,\ m = 5 \tag{52}$$

Then we have

$$\boldsymbol{A}_t = -3\,,\ \boldsymbol{B}_t = \begin{bmatrix} 1 & 2 & 1 & -4 \end{bmatrix} \tag{53}$$
$$\boldsymbol{A}_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$
$$\boldsymbol{B}_q = \begin{bmatrix} 3 & 1 & 1 & -2 \\ 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 \end{bmatrix} \tag{54}$$
$$n_t = 1\,,\ n_q = 4\,,\ Q = 2\,,\ \lambda_1 = 0\,,\ \lambda_2 = 2 \tag{55}$$
$$\boldsymbol{A}_1 = 0\,,\ \boldsymbol{B}_1 = \begin{bmatrix} 3 & 1 & 1 & -2 \end{bmatrix} = \boldsymbol{b}_{111} \tag{56}$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix},$$
$$\boldsymbol{B}_2 = \begin{bmatrix} 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_{121} \\ \boldsymbol{b}_{122} \\ \boldsymbol{b}_{222} \end{bmatrix} \tag{57}$$
$$n(1) = 1\,,\ n(2) = 3\,,\ r(1) = 1\,,\ r(2) = 2\,,$$
$$n(1,1) = 1\,,\ n(2,1) = 1\,,\ n(2,2) = 2 \tag{58}$$

Thus we get

$$\boldsymbol{A}_1^* = 0\,,\ \boldsymbol{B}_1^* = \begin{bmatrix} 3 & 1 & 1 & -2 \end{bmatrix} = \boldsymbol{b}_{111} \tag{59}$$
$$\boldsymbol{A}_2^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$\boldsymbol{B}_2^* = \begin{bmatrix} 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_{121} \\ \boldsymbol{b}_{122} \end{bmatrix} \tag{60}$$
$$\mathrm{rank}\boldsymbol{B}_1^* = 1 = r(1)\,,\ \mathrm{rank}\boldsymbol{B}_2^* = 2 = r(2) \tag{61}$$

Choosing $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ as

$$\boldsymbol{U}_1 = \begin{bmatrix} 1 & 1 & 2 & 3 \end{bmatrix} > \boldsymbol{0}\,,$$
$$\boldsymbol{U}_2 = \begin{bmatrix} 3 & 1 & 1 & 1 \end{bmatrix} > \boldsymbol{0} \tag{62}$$

we have

$$\boldsymbol{B}_1^*\boldsymbol{U}_1 = \boldsymbol{0}\,,\ \boldsymbol{B}_2^*\boldsymbol{U}_2 = \boldsymbol{0} \tag{63}$$

Thus subsystems S_1^* and S_2^* are positive reachable from Theorem 2. On the other hand we have

$$\mathrm{rank}\left[\boldsymbol{B}, \boldsymbol{AB}, \ldots, \boldsymbol{A}^5 B\right] = 6 \tag{64}$$

Thus, system S is positive reachable from Theorem 3.

5 Conclusions

This paper presents a necessary and sufficient condition for a multiple input discrete-time linear system to be positive reachable based on the Jordan canonical form. It is pointed out that the reachability of a given system can be reduced to those of its subsystems with nonnegative eigenvalues. Because the dimension of the subsystem is much smaller than that of the given system, the reachability test can be simplified considerably.

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Appendix A: Proof of Theorem 1

(Proof) *Necessity*: If system S is positive reachable, then condition ① is necessary by Lemma 2. Next consider the following $n \times 1$ vector $\boldsymbol{x}_o$

$$\boldsymbol{x}_o \equiv -\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\!\rangle \boldsymbol{E}_{nm} \tag{A.1}$$

Then there exist a vector $\boldsymbol{V}[M]$ such that

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, M \rangle\!\rangle \boldsymbol{V}[M] = \boldsymbol{x}_o\,,\ \boldsymbol{V}[M] \geq \boldsymbol{0}\,,\ M \geq 1 \tag{A.2}$$

from Lemma 1. Next let

$$\boldsymbol{U}[N] \equiv \left[\boldsymbol{V}[M]^T, \boldsymbol{0}\right]^T + \boldsymbol{E}_{nm}\,,\ N \equiv n\,; \quad \text{if } M < N \tag{A.3}$$

$$\boldsymbol{U}[N] \equiv \boldsymbol{V}[M] + \left[{\boldsymbol{E}_{nm}}^T, \boldsymbol{0}\right]^T\,,\ N \equiv M\,; \quad \text{if } M \geq N \tag{A.4}$$

Then from (A.1)–(A.4), we obtain

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\!\rangle \boldsymbol{U}[N] = \boldsymbol{0}\,,\ \boldsymbol{U}[N] \geq \boldsymbol{0}\,,\ N \geq n \tag{A.5}$$

$$\boldsymbol{u}_i \equiv \boldsymbol{v}_i + \boldsymbol{E}_m > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{A.6}$$

Sufficiency: If condition ① holds, then $\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\!\rangle$ contains n linearly independent vectors. Thus for any final state $\boldsymbol{x}_f$, there exist an $nm \times 1$ vector $\boldsymbol{V}[n]$ such that

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\!\rangle \boldsymbol{V}[n] = \boldsymbol{x}_f \tag{A.7}$$

Next if condition ② holds, then we have

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\!\rangle \boldsymbol{W}[N] = \boldsymbol{0}\,,\ \boldsymbol{W}[N] \geq \boldsymbol{0}\,,\ N \geq n \tag{A.8}$$

$$\boldsymbol{w}_i > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{A.9}$$

Thus for a sufficiently large positive number M, let

$$\boldsymbol{u}(N-i) \equiv M\boldsymbol{w}_i + \boldsymbol{v}_i \geq \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{A.10}$$

$$\boldsymbol{u}(N-i) \equiv M\boldsymbol{w}_i \geq \boldsymbol{0}\,;\ i = n+1, n+2, \ldots, N \tag{A.11}$$

Then from (A.7)–(A.11), we have

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\!\rangle \left[\boldsymbol{u}(N-1)^T, \ldots, \boldsymbol{u}(1)^T, \boldsymbol{u}(0)^T\right]^T = \boldsymbol{x}_f \tag{A.12}$$

The last equation means that a positive control sequence $\{\boldsymbol{u}(0), \boldsymbol{u}(1), \ldots, \boldsymbol{u}(N-1)\}$ will transfer the origin to the final state $\boldsymbol{x}_f$. Therefore, system S is positive reachable. Q.E.D.

Appendix B: Proof of Lemma 3

(Proof) *Necessity*: If system S is positive reachable, then conditions ① and ② in Theorem 1 hold. Because $\boldsymbol{0}$ is sufficiently small, conditions ① and ② in Lemma 3 hold.

Sufficiency: Suppose that conditions ① and ② hold. Then $\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\!\rangle$ contains n linearly independent vectors. Thus there exist an $nm \times 1$ vector $\boldsymbol{V}[n]$ such that

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, n \rangle\!\rangle \boldsymbol{V}[n] = -\boldsymbol{\epsilon} \tag{B.1}$$

where $\|\boldsymbol{v}_i\|$ $(i = 1, 2, \ldots, n)$ is sufficiently small because $\|\boldsymbol{\epsilon}\|$ is sufficiently small. Next from condition ②, there exist a vector $\boldsymbol{U}[N]$ such that

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\!\rangle \boldsymbol{U}[N] = \boldsymbol{\epsilon}\,,\ \boldsymbol{U}[N] \geq \boldsymbol{0}\,,\ N \geq n \tag{B.2}$$

$$\boldsymbol{u}_i > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{B.3}$$

If we let

$$\boldsymbol{W}[N] \equiv \boldsymbol{U}[N] + \left[\boldsymbol{V}[n]^T, \boldsymbol{0}\right]^T \tag{B.4}$$

then from (B.1)–(B.4), we have

$$\langle\!\langle \boldsymbol{A}, \boldsymbol{B}, N \rangle\!\rangle \boldsymbol{W} = \boldsymbol{0}\,,\ \boldsymbol{W} \geq \boldsymbol{0}\,,\ N \geq n \tag{B.5}$$

$$\boldsymbol{w}_i \equiv \boldsymbol{u}_i + \boldsymbol{v}_i > \boldsymbol{0}\,;\ i = 1, 2, \ldots, n \tag{B.6}$$

Thus from Theorem 1, system S is positive reachable.

Q.E.D.

Appendix C: Proof of Lemma 4

(Proof) *Necessity*: System S_q is a subsystem of S. Thus if system S is positive reachable, then conditions ① and ② hold by Lemma 2.

Sufficiency: Suppose that conditions ① and ② hold. Then system S_q is positive reachable. Thus from Theorem 1, there exist a vector $\boldsymbol{V}[M]$ such that

$$\langle\!\langle \boldsymbol{A}_q, \boldsymbol{B}_q, M \rangle\!\rangle \boldsymbol{V}[M] = \boldsymbol{0}\,,\ \boldsymbol{V}[M] \geq \boldsymbol{0}\,,\ M \geq n_q \tag{C.1}$$

$$\boldsymbol{v}_i > \boldsymbol{0}\,,\ i = 1, 2, \ldots, n_q \tag{C.2}$$

Next by a nonsingular transformation, we have the following equation from (25) and (26).

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_t & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_q \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_t \\ \boldsymbol{B}_q \end{bmatrix} \tag{C.3}$$

Now by modifying the results of [4], it is easy to derive that there exists a polynomial $f(z)$ with positive coefficients, such that

$$f(z) \equiv f_L z^L + \cdots + f_1 z + f_0 \tag{C.4}$$

$$f_i > 0\,;\ i = 0, 1, \ldots, L \tag{C.5}$$

$$f(\boldsymbol{A}_t) = \boldsymbol{0} \tag{C.6}$$

where L can be designated arbitrarily as far as

$$L \gg n_t \geq 1 \tag{C.7}$$

Thus from (C.3)–(C.6), we obtain

$$f(\boldsymbol{A}) = \begin{bmatrix} f(\boldsymbol{A}_t) & \boldsymbol{0} \\ \boldsymbol{0} & f(\boldsymbol{A}_q) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & f(\boldsymbol{A}_q) \end{bmatrix} \tag{C.8}$$

Therefore we get

$$f(\boldsymbol{A})\langle\langle \boldsymbol{A},\boldsymbol{B},M\rangle\rangle \boldsymbol{V}[M] = \left[\begin{array}{c} \boldsymbol{0} \\ f(\boldsymbol{A}_q)\langle\langle \boldsymbol{A}_q,\boldsymbol{B}_q,M\rangle\rangle \boldsymbol{V}[M] \end{array}\right] = \boldsymbol{0} \quad \text{(C.9)}$$

considering (C.1). Here we can designate L as $L \gg n_t + M$ from (C.7). If we let

$$\boldsymbol{u}_i \equiv \sum_{j=1}^{M} f_{i-j}\boldsymbol{v}_j\,;\; i = 1,2,\ldots,L+M \quad \text{(C.10)}$$

where

$$f_i \equiv 0\,;\; i<0\,,\; i>L\,,$$
$$\boldsymbol{v}_i \equiv \boldsymbol{0}\,;\; i>M\,,$$
$$N \equiv L+M \geq n_t + n_q = n \quad \text{(C.11)}$$

then from (C.9)–(C.11) we obtain

$$\langle\langle \boldsymbol{A},\boldsymbol{B},N\rangle\rangle \boldsymbol{U}[N] = \boldsymbol{0}\,,\; \boldsymbol{U}[N]\geq \boldsymbol{0}\,,\; N\geq n \quad \text{(C.12)}$$
$$\boldsymbol{u}_i > \boldsymbol{0}\,;\; i=1,2,\ldots,n \quad \text{(C.13)}$$

Therefore, by Theorem 1 system S is positive reachable. Q.E.D.

Appendix D: Proof of Lemma 5

(Proof) *Necessity*: Systems S_a and S_b are subsystems of S. Thus if system S is positive reachable, then conditions ①–③ hold by Lemma 2.

Sufficiency: Suppose that conditions ①–③ hold. Then from Lemma 4, it is sufficient to show that system S_q is positive reachable.

Now by a nonsingular transformation, we have the following equation from (25) and (26).

$$\boldsymbol{A}_q = \left[\begin{array}{cc} \boldsymbol{A}_a & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_b \end{array}\right],\; \boldsymbol{B}_q = \left[\begin{array}{c} \boldsymbol{B}_a \\ \boldsymbol{B}_b \end{array}\right] \quad \text{(D.1)}$$

Next consider the following $n_a \times 1$ vector $\boldsymbol{x}_a$

$$\boldsymbol{x}_a \equiv -\langle\langle \boldsymbol{A}_a,\boldsymbol{B}_a,n_q\rangle\rangle \boldsymbol{E}_{mn_q} \quad \text{(D.2)}$$

Then there exist a vector $\boldsymbol{V}[M]$ such that

$$\langle\langle \boldsymbol{A}_a,\boldsymbol{B}_a,M\rangle\rangle \boldsymbol{V}[M] = \boldsymbol{x}_a\,,\; \boldsymbol{V}[M]\geq \boldsymbol{0}\,,\; M\geq 1 \quad \text{(D.3)}$$

from Lemma 1. Thus, by the similar way discussed in (A.1)–(A.6), we obtain

$$\langle\langle \boldsymbol{A}_a,\boldsymbol{B}_a,L\rangle\rangle \boldsymbol{W}[L] = \boldsymbol{0}\,,\; \boldsymbol{W}[L]\geq \boldsymbol{0}\,,\; L\geq n_q \quad \text{(D.4)}$$
$$\boldsymbol{w}_i > \boldsymbol{0}\,;\; i=1,2,\ldots,n_q \quad \text{(D.5)}$$

Further we consider the following $n_b \times 1$ vector $\boldsymbol{x}_b$

$$\boldsymbol{x}_b \equiv -(\lambda_b)^{N_c}(\boldsymbol{A}_b)^{-L-N_c}\langle\langle \boldsymbol{A}_b,\boldsymbol{B}_b,L\rangle\rangle \boldsymbol{W}[L] \quad \text{(D.6)}$$

where λ_b is any eigenvalues of $\boldsymbol{A}_b$ and N_c is a sufficiently large positive integer. Then by Lemma 1, there exist a vector $\boldsymbol{Y}[N_b]$ such that

$$\langle\langle \boldsymbol{A}_b,\boldsymbol{B}_b,N_b\rangle\rangle \boldsymbol{Y}[N_b] = \boldsymbol{x}_b\,,\; \boldsymbol{Y}[N_b]\geq \boldsymbol{0}\,,\; N_b\geq 1 \quad \text{(D.7)}$$
$$\boldsymbol{y}_i \geq \boldsymbol{0}\,;\; i=1,2,\ldots,N_b \quad \text{(D.8)}$$

Next let

$$N = L + N_c + N_b \quad \text{(D.9)}$$
$$\boldsymbol{U}[N] \equiv \left[\boldsymbol{W}[L]^T, \boldsymbol{0}, (\lambda_b)^{-N_c}\boldsymbol{Y}[N_b]^T\right]^T \quad \text{(D.10)}$$

Then from (D.1)–(D.10) we have

$$\langle\langle \boldsymbol{A}_q,\boldsymbol{B}_q,N\rangle\rangle \boldsymbol{U}[N] = \left[\begin{array}{c} \boldsymbol{\epsilon} \\ \boldsymbol{0} \end{array}\right] \quad \text{(D.11)}$$
$$\boldsymbol{U}[N] \geq 0\,; \quad \text{(D.12)}$$
$$\boldsymbol{u}_i > \boldsymbol{0}\,;\; i=1,2,\ldots,n_q \quad \text{(D.13)}$$
$$\boldsymbol{\epsilon} \equiv (\lambda_b^{-1}\boldsymbol{A}_a)^{N_c}(\boldsymbol{A}_a)^L\langle\langle \boldsymbol{A}_a,\boldsymbol{B}_a,N\rangle\rangle \boldsymbol{Y}[N_b] \quad \text{(D.14)}$$

Because all of the eigenvalues of the matrix ${\lambda_b}^{-1}\boldsymbol{A}_a$ are smaller than unity and N_c is sufficiently large, $\|\boldsymbol{\epsilon}\|$ is sufficiently small. Hence, by Lemma 3, system S is positive reachable. Q.E.D.

Appendix E: Proof of Theorem 3

(Proof) *Necessity*: For each $i = 1,2,\ldots,Q$, system S_i^* is a subsystem of S. Thus if system S is positive reachable, then conditions ① and ② hold by Lemma 2.

Sufficiency: Suppose that conditions ① and ② hold. Now, we consider the following system for each $i = 1,2,\ldots,Q$:

$$S_i^+ \;:\; \boldsymbol{x}_i^+(k+1) = \boldsymbol{A}_i^+\boldsymbol{x}_i^+(k) + \boldsymbol{B}_i^+\boldsymbol{u}(k) \quad \text{(E.1)}$$

$$\boldsymbol{A}_i^+ \equiv \left[\begin{array}{ccccc} \boldsymbol{A}_i^* & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{I}_{r(i)} & \boldsymbol{A}_i^* & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I}_{r(i)} & \boldsymbol{A}_i^* \end{array}\right] \in \boldsymbol{R}^{r(i)P(i)\times r(i)P(i)} \quad \text{(E.2)}$$

$$\boldsymbol{B}_i^+ \equiv \left[\begin{array}{c} \boldsymbol{B}_{i1}^+ \\ \boldsymbol{B}_{i2}^+ \\ \cdots \\ \boldsymbol{B}_{iP(i)}^+ \end{array}\right] \in \boldsymbol{R}^{r(i)P(i)\times m} \quad \text{(E.3)}$$

$$\boldsymbol{B}_{i1}^+ \equiv \boldsymbol{B}_i^* \quad \text{(E.4)}$$
$$P(i) \equiv n(i, r(i)) \quad \text{(E.5)}$$

Then by Lemma 7, system S_i^+ is positive reachable because system S_i^* is positive reachable. It is easy to show that $\boldsymbol{A}_i^+$ and $\boldsymbol{B}_i^+$ can be chosen such that system S_i is a subsystem of S_i^+. Thus by Lemma 2, system S_i is positive reachable. Therefore from Lemma 6, system S is positive reachable. Q.E.D.