FLATNESS BASED ASYMPTOTIC DISTURBANCE REJECTION FOR LINEAR AND NONLINEAR SYSTEMS

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Abstract

This contribution presents a flatness based approach for disturbance decoupling and asymptotic disturbance rejection for linear and nonlinear systems with measurable disturbances. Different from previous flatness based approaches the disturbance input is regarded as an additional fictitious input. This on the one hand enlarges the class of systems, where flatness based disturbance decoupling and asymptotic disturbance rejection is feasible, and on the other hand facilitates the control design.

1 Introduction

The flatness based approach to the analysis and control of nonlinear systems (see e.g. the introductory reference [4]) is an important design strategy for nonlinear control systems. In order to introduce the notion of flatness, consider the nth order nonlinear system

$$\dot{x} = f(x,u) \tag{1}$$

with a smooth vector function f and p smooth inputs u. The system (1) is called *(differentially) flat* if there exists an output

$$y_f = \Phi(x,u,\dot{u},\ldots,\overset{(\alpha)}{u}) \tag{2}$$

with $\dim(y_f) = \dim(u)$, such that the system variables x und u can be expressed by the output (2) and a finite number of its time derivatives according to

$$x = \psi_x(y_f,\dot{y}_f,\ldots,\overset{(\beta)}{y_f}) \tag{3}$$

$$u = \psi_u(y_f,\dot{y}_f,\ldots,\overset{(\beta+1)}{y_f}) \tag{4}$$

If these conditions at least hold locally, then the output y_f is called a *flat output* and the system is a *flat system*. An inherent advantage of flat systems is, that the design of tracking controllers is simplified due to the parametrization of the system variables by the flat output y_f. Since flat systems are feedback linearizable by so-called endogenous feedback, a linear tracking error dynamics is always attainable for the closed loop system (see [4]). However, in practical applications disturbances z affect the system resulting in a deviation between the closed loop system trajectory and the reference trajectory. If a tracking observer (see [5]) is employed to reconstruct the states needed for the error feedback, the closed loop system can become unstable, since the tracking observer is designed on the basis of a linearization about the reference trajectory. However, if the disturbances are measurable or observable, a disturbance feedforward controller can completely or at least asymptotically reject the tracking errors caused by disturbances. So far disturbances were treated as time-varying parameters within the flatness based system parametrization (see [2] and [9]). As a consequence the flatness based approach is only possible if the system is flat with respect to the inputs u. In order to enlarge the class of nonlinear systems, where a flatness based disturbance decoupling or asymptotic disturbance rejection is possible, this contribution regards the disturbance input z as an additional fictitious input, which also simplifies the parameterization of the system variables by the flat output. The basic idea for solving the tracking problem in the presence of disturbances is to use the flatness based parameterization of the input u and the disturbance input z to obtain two differential equations, which are driven by the reference trajectory of the output to be controlled and the disturbance input. The solution of these differential equations yields a reference trajectory for the flat output, which takes the disturbances into account, such that the disturbance does not affect the output to be controlled (disturbance decoupling) or is at least rejected asymptotically (asymptotic disturbance rejection).

The next section presents a state space and a polynomial approach for computing a flat output for disturbed linear systems. The disturbance decoupling and the asymptotic disturbance rejection problem are solved using the proposed flatness based parameterization. In Section 3 the results for linear systems are extended to nonlinear systems. A simple examples demonstrates the proposed asymptotic disturbance rejection for a nonlinear system.

2 Disturbance rejection for linear systems

2.1 Flatness of disturbed linear systems

Consider a time invariant linear system of nth order with one input u and one disturbance input z, described by the state equation

$$\dot{x} = Ax + B\begin{bmatrix} u \\ z \end{bmatrix}, \tag{5}$$

Assume that $\operatorname{rank} B = 2 \leq n$ and that the system is completely controllable with respect to u and z, that is

$$\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n \tag{6}$$

holds. Then there exists a linear state transformation

$$\zeta = \begin{bmatrix} \zeta_1^1 & \dots & \zeta_{\kappa_1}^1 & \zeta_1^2 & \dots & \zeta_{\kappa_2}^2 \end{bmatrix}^T = Tx, \quad \kappa_1 + \kappa_2 = n \tag{7}$$

which transforms system (5) into *controller form* (see e.g. [7])

$$\begin{aligned} \dot{\zeta}_k^1 &= \zeta_{k+1}^1, \quad k = 1(1)\kappa_1 - 1 & (8)\\ \dot{\zeta}_{\kappa_1}^1 &= -\sum_{k=1}^{\kappa_1} a_{1k}^1 \zeta_k^1 - \sum_{k=1}^{\kappa_2} a_{2k}^1 \zeta_k^2 + b_1^1 u + b_2^1 z & (9)\\ \dot{\zeta}_k^2 &= \zeta_{k+1}^2, \quad k = 1(1)\kappa_2 - 1 & (10)\\ \dot{\zeta}_{\kappa_2}^2 &= -\sum_{k=1}^{\kappa_1} a_{1k}^2 \zeta_k^1 - \sum_{k=1}^{\kappa_2} a_{2k}^2 \zeta_k^2 + b_1^2 u + b_2^2 z & (11) \end{aligned}$$

In (8)-(11) $\zeta_1^i, \dots, \zeta_{\kappa_i}^i$, $i = 1(1)2$, denote the states of the ith subsystem of order κ_i, where the quantities κ_i are the *controllability indices* of the system (5). Next it is shown, that

$$y_f = \begin{bmatrix} y_{f1} & y_{f2} \end{bmatrix}^T = \begin{bmatrix} \zeta_1^1 & \zeta_1^2 \end{bmatrix}^T \tag{12}$$

is a flat output for the system (5) according to the definition given in the introduction (see also [1]). To this end insert (12) in (9) and (11) and use (8) and (10) giving

$$\begin{aligned} \overset{(\kappa_1)}{y}_{f1} &= -\sum_{k=1}^{\kappa_1} a_{1k}^1 \overset{(k-1)}{y}_{f1} - \sum_{k=1}^{\kappa_2} a_{2k}^1 \overset{(k-1)}{y}_{f2} + b_1^1 u + b_2^1 z & (13)\\ \overset{(\kappa_2)}{y}_{f2} &= -\sum_{k=1}^{\kappa_1} a_{1k}^2 \overset{(k-1)}{y}_{f1} - \sum_{k=1}^{\kappa_2} a_{2k}^2 \overset{(k-1)}{y}_{f2} + b_1^2 u + b_2^2 z & (14) \end{aligned}$$

Using the differential operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and the operator polynomials

$$a_j^i(D) = \sum_{k=1}^{\kappa_j} a_{jk}^i D^{k-1}, \quad j = 1(1)2, \quad i = 1(1)2 \tag{15}$$

equations (13) and (14) can be rewritten as

$$\begin{bmatrix} D^{\kappa_1} + a_1^1(D) & a_2^1(D) \\ a_1^2(D) & D^{\kappa_2} + a_2^2(D) \end{bmatrix} y_f = \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix} \tag{16}$$

which can be represented more compactly by

$$P(D)y_f = \bar{B}\begin{bmatrix} u \\ z \end{bmatrix} \tag{17}$$

with the (2,2) polynomial matrix $P(D)$ and the constant (2,2) matrix $\bar{B}$. Since $\det \bar{B} \neq 0$ always holds for the controller form (8)-(11), one can solve (17) for $\begin{bmatrix} u & z \end{bmatrix}^T$ yielding

$$\begin{bmatrix} u \\ z \end{bmatrix} = \bar{B}^{-1} P(D) y_f = N(D) y_f \tag{18}$$

with the (2,2) polynomial matrix $N(D)$. Thus the system inputs u and z can be represented by y_f and its time derivatives according to (4). Also the system states x of (5) can be expressed by y_f and its time derivatives by using (7)

$$\begin{aligned} x &= T^{-1}\zeta = T^{-1}\begin{bmatrix} \zeta_1^1 & \dots & \zeta_{\kappa_1}^1 & \zeta_1^2 & \dots & \zeta_{\kappa_2}^2 \end{bmatrix}^T \\ &= T^{-1}\begin{bmatrix} y_{f1} & \dots & \overset{(\kappa_1-1)}{y}_{f1} & y_{f2} & \dots & \overset{(\kappa_2-1)}{y}_{f2} \end{bmatrix}^T \end{aligned} \tag{19}$$

which results in

$$x = Z_x(D) y_f \tag{20}$$

with the (n,2) polynomial matrix

$$Z_x(D) = T^{-1} \operatorname{diag}(\begin{bmatrix} 1 & \dots & D^{\kappa_1 - 1} \end{bmatrix}^T, \begin{bmatrix} 1 & \dots & D^{\kappa_2 - 1} \end{bmatrix}^T) \tag{21}$$

when written in matrix form. In view of (7) and (12) the flat output y_f is a function of x only, such that (2) is satisfied. This shows, that the controllability of the system (5) with respect to its inputs u and z is sufficient for flatness. This the more one can show that this condition is also necessary (see [4]).

2.2 Disturbance decoupling for linear systems

The problem of *disturbance decoupling with disturbance measurement* for the system (5) amounts to finding a control input u, such that the output to be controlled

$$y = c^T x \tag{22}$$

is completely unaffected by the measurable disturbance z. This problem can readily be solved on the basis of the flatness property of system (5) by using a feedforward controller. This controller decouples the disturbance z from the output y and assures exact tracking of a given trajectory for appropriate initial conditions $x(0)$ of system (5). In order to compute this feedforward controller the output y is expressed by the flat ouput y_f using (20)

$$y = c^T Z_x(D) y_f = Z(D) y_f \tag{23}$$

with the (1,2) polynomial matrix $Z(D)$. Introduce the (1,2) polynomial matrices $N_u(D)$ and $N_z(D)$ by

$$\begin{bmatrix} u \\ z \end{bmatrix} = N(D) y_f = \begin{bmatrix} N_u(D) \\ N_z(D) \end{bmatrix} y_f \tag{24}$$

in view of (18) then the feedforward controller

$$u_d = N_u(D) y_{f,d} \tag{25}$$

which achieves exact tracking of a desired trajectory $y_{f,d}$ for the flat output y_f, directly follows from (24). In order to solve the disturbance decoupling problem one has to compute the trajectory $y_{f,d}$ from the desired trajectory y_d for y and taking into account the relation $z = N_z(D)y_f$ (see (24)). In view of (23) the flat output has to satisfy

$$Z(D)y_{f,d} \overset{!}{=} y_d \tag{26}$$

which gives with (24) the differential equation

$$\begin{bmatrix} Z(D) \\ N_z(D) \end{bmatrix} y_{f,d} = \begin{bmatrix} y_d \\ z \end{bmatrix} \tag{27}$$

for the trajectory $y_{f,d}$ with the given trajectory y_d and the measured disturbance z as inputs. A prerequisite for exact tracking using the feedforward controller (25) is, that the initial conditions $x(0)$ of system (5) fulfill $x = Z_x(D)y_{f,d}$ (see (20)) at $t = 0$. Note that the differential equation (27) is stable if all poles being solutions of

$$\det \begin{bmatrix} Z(s) \\ N_z(s) \end{bmatrix} = 0 \tag{28}$$

are located in the open left half plane (see [7]). If differential equation (27) is unstable, one must perform a suitable trajectory planning for y_d, such that all systems states and inputs remain within given bounds.

2.3 Asymptotic disturbance rejection for linear systems

If the initial conditions $x(0)$ of the system do not satisfy $x = Z_x(D)y_{f,d}$ (see (20)) at time instant $t = 0$, then only *asymptotic disturbance rejection with disturbance measurement* is possible. Here one seeks a control input u for system (5), such that the tracking error $e = y - y_d$ satisfies

$$\lim_{t\to\infty} e(t) = 0 \tag{29}$$

in the presence of a measurable disturbance z. In the following this control problem is solved by using a flatness based tracking controller, which achieves asymptotic tracking of a given trajectory for y. In order to derive this tracking controller substitute (15) in (13) and (14)

$$\overset{(\kappa_1)}{y}_{f1} = -a_1^1(D)y_{f1} - a_2^1(D)y_{f2} + b_1^1 u + b_2^1 z \tag{30}$$

$$\overset{(\kappa_2)}{y}_{f2} = -a_1^2(D)y_{f1} - a_2^2(D)y_{f2} + b_1^2 u + b_2^2 z \tag{31}$$

Let $b_1^1 \neq 0$, then by inspection of (30) one can introduce a new input $\bar{u}$ by

$$\bar{u} = -a_1^1(D)y_{f1} - a_2^1(D)y_{f2} + b_1^1 u + b_2^1 z \tag{32}$$

yielding a regular feedback. Note that in view of $\det \bar{B} \neq 0$, it is always possible to introduce a new input $\bar{u}$ (i.e. if $b_1^1 = 0$ then $b_1^2 \neq 0$). By solving (32) for u one obtains the control law

$$u = \frac{1}{b_1^1}\left(\bar{u} + a_1^1(D)y_{f1} + a_2^1(D)y_{f2} - b_2^1 z\right) \tag{33}$$

which achieves

$$\overset{(\kappa_1)}{y}_{f1} = \bar{u} \tag{34}$$

for the closed loop system. A desired dynamics

$$\overset{(\kappa_1)}{e}_{f1} + \tilde{a}_{\kappa_1-1} \overset{(\kappa_1-1)}{e}_{f1} + \ldots + \tilde{a}_0 e_{f1} = 0 \tag{35}$$

for the tracking error $e_{f1} = y_{f1} - y_{f1,d}$ is assigned to the closed loop system by the error feedback

$$\bar{u} = \overset{(\kappa_1)}{y}_{f1,d} - \left(\tilde{a}_{\kappa_1-1} \overset{(\kappa_1-1)}{e}_{f1} + \ldots + \tilde{a}_0 e_{f1}\right) \tag{36}$$

Since one is interested to track y_d, the trajectory $y_{f1,d}$ for the first element y_{f1} of the flat output y_f has to be computed from the trajectory y_d of the output y to be controlled by solving the differential equation

$$Z_1(D)y_{f1,d} = y_d - Z_2(D)y_{f2} \tag{37}$$

This differential equation follows from (23) by writing the (1,2) polynomial matrix $Z(D)$ in the form

$$Z(D) = \begin{bmatrix} Z_1(D) & Z_2(D) \end{bmatrix} \tag{38}$$

The second element y_{f2} of the flat output, which is an input to (37), has to be computed by solving the differential equation (31) for y_{f2} with feedback (33) and the disturbance z as inputs.

The feedback (33) decouples the y_{f2}-subsystem (31) from the controlled y_{f1}-subsystem (30). Hence internal stability of the closed loop system depends on the stability of the y_{f2}-subsystem. To see this, introduce new state variables

$$\xi = \begin{bmatrix} \xi_1 & \ldots & \xi_{\kappa_1} \end{bmatrix}^T = \begin{bmatrix} y_{f1} & \ldots & \overset{(\kappa_1-1)}{y}_{f1} \end{bmatrix}^T \tag{39}$$

$$\eta = \begin{bmatrix} \eta_1 & \ldots & \eta_{\kappa_2} \end{bmatrix}^T = \begin{bmatrix} y_{f2} & \ldots & \overset{(\kappa_2-1)}{y}_{f2} \end{bmatrix}^T \tag{40}$$

and consider (30) and (31) with feedback (33). Then the new state equations for (39) and (40) read

$$\dot{\xi} = J_{\kappa_1}\xi + e_{\kappa_1}\bar{u} \tag{41}$$

$$\dot{\eta} = (J_{\kappa_2} + e_{\kappa_2}q^T)\eta + e_{\kappa_2}\left(p^T\xi + g_1\bar{u} + g_2 z\right) \tag{42}$$

$$y_{f1} = \xi_1 \tag{43}$$

where e_i is the ith unit vector, J_m is a (m,m) matrix with zero elements except for ones on the first upper secondary diagonal. In (42) the quantities q^T, p^T are vectors of appropriate dimensions and g_1 and g_2 are scalars. The resulting state equations (41)-(43) are in Byrnes Isidori normal form (see [6]), where ξ are the states of the output dynamics of y_{f1} and η are the states of the zero dynamics for the system with error feedback (36). Thus internal stability of the closed loop system only depends on the stability of the zero dynamics (42) driven by the disturbance z and the reference inputs. If the zero dynamics (42) is unstable and $b_1^2 \neq 0$, then one can try to get an internally stable closed loop system by introducing the new input

$$\bar{u} = -a_1^2(D)y_{f1} - a_2^2(D)y_{f2} + b_1^2 u + b_2^2 z \tag{44}$$

related to (31). A stable zero dynamics then possibly results from (30).

2.4 Polynomial approach to disturbance rejection

The flatness based parameterizations (18) and (20) of the system variables can directly be obtained from the state equation (5) by coprime matrix fraction conversion. To this end consider the coprime matrix fraction conversion related to the transfer behaviour

$$x(s) = (sI-A)^{-1}B\begin{bmatrix}u(s)\\z(s)\end{bmatrix} = Z_x(s)N^{-1}(s)\begin{bmatrix}u(s)\\z(s)\end{bmatrix} \quad (45)$$

of system (5), where $Z_x(s)$ and $N(s)$ are right coprime matrices with $N(s)$ column reduced (see e.g. [7]). Since the transfer matrix in (45) has no direct feedthrough these polynomial matrices satisfy $\delta_{ci}[Z_x(s)] < \delta_{ci}[N(s)]$, $i = 1(1)2$, where $\delta_{ci}[\cdot]$ is the ith column degree of a polynomial matrix (i.e. the highest degree of the polynomials in the corresponding column). Then the parameterization of u and z can be obtained by setting

$$y_f(s) = N^{-1}(s)\begin{bmatrix}u(s)\\z(s)\end{bmatrix} \quad (46)$$

yielding

$$\begin{aligned}\begin{bmatrix}u\\z\end{bmatrix} &= N(D)y_f = \begin{bmatrix}\psi_u(y_f,\dot{y}_f,\ldots,\overset{(\beta+1)}{y_f})\\ \psi_z(y_f,\dot{y}_f,\ldots,\overset{(\beta+1)}{y_f})\end{bmatrix} &(47)\\ x &= Z_x(D)y_f = \psi_x(y_f,\dot{y}_f,\ldots,\overset{(\beta)}{y_f}) &(48)\end{aligned}$$

in the time domain. The relations (47) and (48) are valid flatness based parameterizations if y_f in (46) satisfies (2). This property directly follows from the Bezout identity by controllability of system (5) (see [7]). Note that (47) and (48) are linear versions of (3) and (4). By setting $\delta_{ci}[N(D)] = \kappa_i$, $i = 1(1)2$, (47) can be written as

$$\begin{aligned}\begin{bmatrix}u\\z\end{bmatrix} &= \Gamma_c[N(D)]\begin{bmatrix}D^{\delta_{c1}[N(D)]} & 0\\ 0 & D^{\delta_{c2}[N(D)]}\end{bmatrix}y_f + N_R(D)y_f\\ &= \begin{bmatrix}\gamma_{11}\overset{(\kappa_1)}{y_{f1}} & \gamma_{12}\overset{(\kappa_2)}{y_{f2}}\\ \gamma_{21}\overset{(\kappa_1)}{y_{f1}} & \gamma_{22}\overset{(\kappa_2)}{y_{f2}}\end{bmatrix}y_f + N_R(D)y_f &(49)\end{aligned}$$

with $\Gamma_c[\cdot] = [\gamma_{ij}]$ the highest column degree coefficient matrix and $N_R(D)$ is a polynomial matrix with polynomials of lower degrees. By comparing (49) with (16) one obtains

$$\begin{aligned}\bar{B} &= \Gamma_c^{-1}[N(D)] &(50)\\ \begin{bmatrix}a_1^1(D) & a_2^1(D)\\ a_1^2(D) & a_2^2(D)\end{bmatrix} &= \Gamma_c^{-1}[N(D)]N_R(D) &(51)\end{aligned}$$

provided $\Gamma_c[\cdot]$ is nonsingular. This property follows from the column reducedness of $N(s)$. In order to introduce a new input $\bar{u}$ (see (32)) on the basis of representation (49) solve (49) for the highest time derivatives of y_f, which leads to

$$\begin{bmatrix}\overset{(\kappa_1)}{y_{f1}}\\ \overset{(\kappa_2)}{y_{f2}}\end{bmatrix} = \frac{1}{\det\Gamma_c[N(D)]}\begin{bmatrix}\gamma_{22} & -\gamma_{12}\\ -\gamma_{21} & \gamma_{11}\end{bmatrix}\left(\begin{bmatrix}u\\z\end{bmatrix} - N_R(D)y_f\right) \quad (52)$$

By comparing (49) with (52) it follows that the new input $\overset{(\kappa_1)}{y_{f1}}= \bar{u}$ can be introduced if $\gamma_{22} \neq 0$, i.e. $\overset{(\kappa_2)}{y_{f2}}$ has to appear in the flatness based parameterization of z (see (49)). The same holds true for introducing $\overset{(\kappa_2)}{y_{f2}}= \bar{u}$, where $\overset{(\kappa_1)}{y_{f1}}$ has to appear in the expression for z.

3 Disturbance rejection for nonlinear systems

3.1 Flatness of disturbed nonlinear systems

In this section the flatness based disturbance decoupling approach for linear systems is extended to the nonlinear case. The nth order nonlinear system under consideration is given by the state equation

$$\dot{x} = f(x,u,z) \quad (53)$$

with one input u and one disturbance input z. In the following it is assumed that rank $\frac{\partial f(x,u,z)}{\partial(u,z)} = 2 \leq n$ and that the nonlinear system (53) is flat with respect to u and z. Thus in light of the definition of flatness in the introduction there exists a flat output $y_f = \begin{bmatrix}y_{f1} & y_{f2}\end{bmatrix}^T$

$$y_f = \Phi(x,u,z,\dot{u},\dot{z},\ldots,\overset{(\alpha)}{u},\overset{(\alpha)}{z}) \quad (54)$$

such that

$$\begin{aligned}x &= \psi_x(y_f,\dot{y}_f,\ldots,\overset{(\beta)}{y_f}) &(55)\\ u &= \psi_u(y_f,\dot{y}_f,\ldots,\overset{(\beta+1)}{y_f}) &(56)\\ z &= \psi_z(y_f,\dot{y}_f,\ldots,\overset{(\beta+1)}{y_f}) &(57)\end{aligned}$$

In contrast to linear systems there is no necessary and sufficient condition to be known for flatness of nonlinear systems, such that there exists no systematic method for constructing flat outputs. However, in many cases it is possible to use physical insight into the design problem to determine a flat output.

3.2 Disturbance decoupling for nonlinear systems

The problem of *disturbance decoupling with disturbance measurement* for the system (53) amounts to finding a control input u, such that the output to be controlled

$$y = h(x) \quad (58)$$

is completely uneffected by the measurable disturbance z. A flatness based representation of the output y in (58) is obtained by inserting (55) in (58) giving

$$y = \psi_y(y_f,\dot{y}_f,\ldots,\overset{(\beta)}{y_f}) \quad (59)$$

As in the linear case the disturbance decoupling problem can be solved by computing a flatness based feedforward controller. This controller is obtained from (56)

$$u_d = \psi_u(y_{f,d},\dot{y}_{f,d},\ldots,\overset{(\beta+1)}{y_{f,d}}) \quad (60)$$

and assures exact tracking of the trajectory $y_{f,d}$ for appropriate initial conditions $x(0)$ of system (53). In order to solve the disturbance decoupling problem the trajectory $y_{f,d}$ for the flat output y_f has to satisfy

$$\psi_y(y_{f,d}, \dot{y}_{f,d}, \ldots, \overset{(\beta)}{y}_{f,d}) \overset{!}{=} y_d \tag{61}$$

such that with (57) the trajectory $y_{f,d}$ follows from solving the implicit differential equations

$$\begin{aligned} \psi_y(y_{f,d}, \dot{y}_{f,d}, \ldots, \overset{(\beta)}{y}_{f,d}) &= y_d &(62)\\ \psi_z(y_{f,d}, \dot{y}_{f,d}, \ldots, \overset{(\beta+1)}{y}_{f,d}) &= z &(63) \end{aligned}$$

where the trajectory y_d for y and the measured disturbance z are inputs. Exact tracking and thus disturbance decoupling is only feasible if

$$x(0) = \psi_x(y_{f,d}(0), \dot{y}_{f,d}(0), \ldots, \overset{(\beta)}{y}_{f,d}(0)) \tag{64}$$

holds. For an unbounded solution $y_{f,d}$ of (62) and (63) one must perform a suitable trajectory planning for y_d in order to ensure boundedness of x and u_d. The flatness based solution of the multivariable version of the disturbance decoupling problem considered in this section can be found in [8].

3.3 Asymptotic disturbance rejection for nonlinear systems

If condition (64) does not hold one can achieve at least asymptotic disturbance rejection as defined in the beginning of Section 2.3. For computing a flatness based tracking controller which solves the asymptotic disturbance rejection problem, one has to introduce a new input $\bar{u}$ (see Section 2.3). To this end consider (57) in the form

$$z = \psi_z(y_f, \dot{y}_f, \ldots, \overset{(\beta_1)}{y}_{f1}, \overset{(\beta_2)}{y}_{f2}) \tag{65}$$

Now let κ_i, $i = 1(1)2$, be the orders of the highest time derivatives of y_{fi} appearing in (56) and in (65). Then by the discussion in Section 2.4 it is reasonable that one can introduce the new input $\overset{(\kappa_1)}{y}_{f1} = \bar{u}$ if $\beta_2 = \kappa_2$ holds or the new input $\overset{(\kappa_2)}{y}_{f2} = \bar{u}$ if $\beta_1 = \kappa_1$ respectively. If both $\overset{(\kappa_1)}{y}_{f1}$ and $\overset{(\kappa_2)}{y}_{f2}$ appear in (65) one can choose the new input $\bar{u}$ arbitrarily provided the corresponding highest time derivative appears in (56). In order to justify this procedure assume that $\beta_2 = \kappa_2$ and $\beta_1 < \kappa_1$, then according to (65) $\overset{(\kappa_2)}{y}_{f2}$ is completely determined by system variables and cannot be chosen as a new input. Since $\overset{(\kappa_1)}{y}_{f1}$ does not appear in (65) it has to appear in (56) and can be introduced as new input. The condition for introducing $\overset{(\kappa_2)}{y}_{f2}$ can be justified in the same way. If both highest time derivatives of the flat output appear in (65) it is assumed that at least one of them appears in (56), which can be chosen as a new input.

In the following assume that $\overset{(\kappa_1)}{y}_{f1} = \bar{u}$ has been chosen as new input. Then in order to stabilize the tracking of the trajectory $y_{f1,d}$ one uses the error feedback (36) to assign the error dynamics (35). The resulting tracking controller reads

$$u = \psi_u(y_f, \dot{y}_f, \ldots, \overset{(\kappa_1-1)}{y}_{f1}, \bar{u}, \overset{(\kappa_2)}{y}_{f2}) \tag{66}$$

in view of (56). By observing (61) and (65) the trajectory $y_{f1,d}$ solving the asymptotic disturbance rejection problem is computed from the trajectory y_d by the implicit differential equations

$$\begin{aligned} \psi_y(y_{f1,d}, \dot{y}_{f1,d}, \ldots, \overset{(\kappa_1-1)}{y}_{f1,d}, y_{f2}, \dot{y}_{f2}, \ldots, \overset{(\kappa_2-1)}{y}_{f2}) &= y_d &(67)\\ \psi_z(y_{f1}, \dot{y}_{f1}, \ldots, \overset{(\kappa_1-1)}{y}_{f1}, \bar{u}, y_{f2}, \dot{y}_{f2}, \ldots, \overset{(\kappa_2)}{y}_{f2}) &= z &(68) \end{aligned}$$

The control law (66) decouples the y_{f2}-subsystem from the controlled y_{f1}-subsystem $\overset{(\kappa_1)}{y}_{f1} = \bar{u}$ resulting in a zero dynamics for the closed loop system. This is better seen by introducing new states according to (39) and (40) to obtain the Byrnes Isidori normal form

$$\begin{aligned} \dot{\xi} &= J_{\kappa_1}\xi + e_{\kappa_1}\bar{u} &(69)\\ \dot{\eta} &= q(\eta, \xi, \bar{u}, z) &(70)\\ y_{f1} &= \xi_1 &(71) \end{aligned}$$

where the nonlinear $(\kappa_2,1)$ vector function $q(\cdot)$ results from solving (56) and (57) for the highest time derivatives of y_f.

It should be noted that if $\kappa_1 + \kappa_2 > n$ holds, the tracking controller (66) is a dynamic state feedback controller. However, introducing dynamic elements in the controller can be circumvented by using *quasi-static state feedback* (see [9] for details).

4 Example

Consider the following nonlinear system of order 3

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_3 + x_2x_3 &(72)\\ \dot{x}_2 &= -3x_2 + x_1^2 + z &(73)\\ \dot{x}_3 &= -x_2 - x_3 + u &(74)\\ y &= x_1 + x_3 &(75) \end{aligned}$$

with one input u and one disturbance input z. The output to be controlled is denoted by y. A flat output y_f for the system (72)-(74) is given by

$$y_f = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{76}$$

since x_3, z and u can be expressed as

$$\begin{aligned} x_3 &= \frac{\dot{y}_{f1} + 2y_{f1}}{1 + y_{f2}} &(77)\\ z &= \dot{y}_{f2} + 3y_{f2} - y_{f1}^2\\ &= \psi_z(y_{f1}, y_{f2}, \dot{y}_{f2}) &(78)\\ u &= \frac{1}{1+y_{f2}}\left(\ddot{y}_{f1} + 3\dot{y}_{f1} + 2y_{f1} - \frac{\dot{y}_{f2}(\dot{y}_{f1} + 2y_{f1})}{1+y_{f2}}\right) + y_{f2}\\ &= \psi_u(y_{f1}, \dot{y}_{f1}, \ddot{y}_{f1}, y_{f2}, \dot{y}_{f2}) &(79) \end{aligned}$$

In order to solve the asymptotic disturbance decoupling problem (Section 3.3) one has to introduce a new input $\bar{u}$. In view of (78) the only possible choice is

$$\ddot{y}_{f1} = \bar{u} \tag{80}$$

since $\dot{y}_{f2}$ in (78) cannot be chosen independently from the system variables. With the error feedback

$$\bar{u} = \ddot{y}_{f1,d} - \tilde{a}_1(\dot{y}_{f1} - \dot{y}_{f1,d}) - \tilde{a}_0(y_{f1} - y_{f1,d}) \tag{81}$$

one assigns the dynamics

$$\ddot{e}_{f1} + \tilde{a}_1\dot{e}_{f1} + \tilde{a}_0 e_{f1} = 0 \tag{82}$$

to the tracking error $e_{f1} = y_{f1} - y_{f1,d}$. The tracking controller following from (79) reads

$$u = \psi_u(y_{f1}, \dot{y}_{f1}, \bar{u}, y_{f2}, \dot{y}_{f2}) \tag{83}$$

This controller decouples a subsystem from the output behaviour related to y_{f1}. By introducing

$$\xi_1 = y_{f1}, \quad \xi_2 = \dot{y}_{f1}, \quad \eta = y_{f2} \tag{84}$$

this can be verified by looking at the corresponding Byrnes Isidori normal form

$$\dot{\xi}_1 = \xi_2 \tag{85}$$
$$\dot{\xi}_2 = \bar{u} \tag{86}$$
$$\dot{\eta} = -3\eta + \xi_1^2 + z \tag{87}$$
$$y_{f1} = \xi_1 \tag{88}$$

resulting from (78) and (80). Thus one has a linear and stable (undriven) zero dynamics (87) driven by the disturbance input z. The trajectory $y_{f1,d}$ solving the asymptotic disturbance decoupling problem is obtained from the trajectory $y_d \equiv 0$ by

$$y = \frac{3y_{f1} + y_{f1}y_{f2} + \dot{y}_{f1}}{1 + y_{f2}} \overset{!}{\equiv} 0 \tag{89}$$

where the flatness based parametrization of y follows from (75), (76) and (77). Relation (89) simplifies to

$$3y_{f1} + y_{f1}y_{f2} + \dot{y}_{f1} = 0 \tag{90}$$

for $y_{f2} \neq -1$. Since the tracking controller needs the second time derivative of $y_{f1,d}$ (see (81)) one differentiates (90) with respect to time giving with (87) the differential equations

$$\ddot{y}_{f1,d} + (3+\eta)\dot{y}_{f1,d} + (\xi_1^2 + z - 3\eta)y_{f1,d} = 0 \tag{91}$$
$$\dot{\eta} + 3\eta - \xi_1^2 - z = 0 \tag{92}$$

to be solved for $y_{f1,d}$. A valid choice for the initial values of (91) and (92) satisfying (90) is $\dot{y}_{f1,d}(0) = y_{f1,d}(0) = 0$.

5 Conclusions

In this contribution the disturbance decoupling and the asymptotic disturbance rejection for linear and nonlinear flat systems with measurable disturbances was considered. By introducing the disturbance input as an additional fictitious input a larger class of linear and nonlinear systems can be treated within the flatness based approach. In some cases the controller also needs time derivatives of the measured disturbances. Then estimates for the time derivatives can be obtained by employing standard filtering techniques (see e.g. [3]).

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