PID-TYPE CONTROLLER SYNTHESIS VIA π-SHARING THEORY

I-Kong Fong*, **Jong-Kuen Horng**, **Chih-Chin Hsu**

Department of Electrical Engineering
National Taiwan University
Taipei, Taiwan 10617
Republic of China
*E-mail: ikfong@cc.ee.ntu.edu.tw

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Abstract

The π-sharing theory is an extension of the passivity theory to simultaneously accommodate the state and input-output stability. In this paper, the π-sharing theory is used to develop a procedure for synthesizing PID-type controllers that stabilize multivariable linear time-invariant systems. The proposed method is based on the linear matrix inequality formulation and is easy to apply. Numerical examples are provided to show the effectiveness of the method.

1 Introduction

The π-sharing theory [11], which simultaneously accommodates state and input-output stability, offers a less known, but alternative way of analyzing system stability. As an extension of the concepts of passivity [7] and dissipativity [21], the π-sharing theory is particularly useful when applied to feedback systems, because it uses the so-called π-coefficients to describe the "energy storage and dissipation" of the subsystems. Hence, for a feedback system consisting of two subsystems to be stable, a subsystem is allowed to be non-passive, as long as the other subsystem is "passive enough". In [11], the theory is presented in the context of discrete-time single variable systems, and in [10] the corresponding theory for continuous-time multivariable systems is developed. Moreover, the originally difficult task of finding usable π-coefficients is overcome for linear time-invariant (LTI) systems in [10], where the problem is translated into the solution of a set of linear matrix inequalities (LMI's) [3]. Thus for any finite dimensional LTI systems, time-invariant π-coefficients may be obtained conveniently.

In contrast to the less known π-sharing theory, the PID controller is probably the most popular idea in the field of control engineering. Determination of the controller parameters of a PID controller has many different ways, ranging from empirical tuning methods [1, 5, 12] such as the celebrated Ziegler-Nichols rule, to sophisticated methods based on mathematical theory [2, 6, 8, 13, 14, 15, 16, 18, 20] and neuro-fuzzy theory [9, 17, 19]. However, for multivariable systems described by the state-space model how to obtain stabilizing PID controllers is still a challenging problem. In this paper, the multivariable π-sharing theory is utilized to establish a set of LMI conditions for a given multivariable LTI system. If there are feasible solutions to these LMI conditions, then stabilizing PID-type controllers may be found systematically for the system in discussion. Examples will be given to show the effectiveness of the proposed method.

Before we start, some notations adopted in this paper are introduced first. We use $\mathbf{X} \geq \mathbf{0}$ to denote that the matrix $\mathbf{X}$ is symmetric and positive semi-definite, and $\mathbf{X} \geq \mathbf{Y}$ to denote $\mathbf{X} - \mathbf{Y} \geq \mathbf{0}$. Similar definitions apply to symmetric positive/negative definite matrices. If $\mathbf{X} > \mathbf{0}$, then $\mathbf{X}^{1/2}$ denotes the positive definite matrix such that $\mathbf{X}^{1/2}\mathbf{X}^{1/2} = \mathbf{X}$. Let $\mathbf{x}(t)$ and $\mathbf{\Phi}(t)$, respectively, be any real vector and symmetric matrix functions of time t, then $(\mathbf{\Phi})|\mathbf{x}(t)|^2 = \mathbf{x}^{\mathrm{T}}(t)\mathbf{\Phi}(t)\mathbf{x}(t)$ and $(\mathbf{\Phi})\|\mathbf{x}\|_T^2 = \int_0^T \mathbf{x}^{\mathrm{T}}(t)\mathbf{\Phi}(t)\mathbf{x}(t)dt$, where $T \geq 0$ is a constant. If $\mathbf{\Phi} = \mathbf{I}$, the identity matrix, then it is omitted from the notations. Finally, we let $\|\mathbf{X}\|$ represent the induced two-norm of the matrix $\mathbf{X}$.

2 Multivariable π-sharing theory

For easy reference, the continuous-time multivariable π-sharing theory developed in [10] is briefly reviewed in this Section, but restricted to the LTI case. Consider the system S in state-space form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathcal{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathcal{R}^m$ is the input vector, $\mathbf{y}(t) \in \mathcal{R}^m$ is the output vector, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are constant matrices of appropriate dimensions. The system S is said [10, 11] to be π-*sharing* with respect to π-coefficients $\{\mathbf{\Gamma}, \mathbf{Q}, \mathbf{P}, \mathbf{R}\}$, if for all $T \geq 0$

$$\begin{aligned} \int_0^T \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t)dt \;\geq\; & (\mathbf{\Gamma})|\mathbf{x}(T)|^2 - (\mathbf{\Gamma})|\mathbf{x}(0)|^2 + (\mathbf{Q})\|\mathbf{x}\|_T^2 \\ & + (\mathbf{P})\|\mathbf{y}\|_T^2 + (\mathbf{R})\|\mathbf{u}\|_T^2, \end{aligned} \tag{2}$$

where $\mathbf{\Gamma}, \mathbf{Q} \in \mathcal{R}^{n\times n}$ are positive semi-definite symmetric matrices, and $\mathbf{P}, \mathbf{R} \in \mathcal{R}^{m\times m}$ are symmetric matrices. Basically (2) describes energy dissipativity [21] of the system S with a quadratic type energy supply rate. In [10, 11], it is pointed out that the left hand side of (2) can be interpreted as the energy supplied to the system from outside sources, and is positive if S sinks energy. In the right hand side of (2), $(\mathbf{\Gamma})|\mathbf{x}(T)|^2 - (\mathbf{\Gamma})|\mathbf{x}(0)|^2$ and $(\mathbf{Q})\|\mathbf{x}\|_T$ respectively parameterize energy stored and dissipated in state trajectory, and $(\mathbf{P})\|\mathbf{y}\|_T^2 + (\mathbf{R})\|\mathbf{u}\|_T^2$ characterizes energy exchange of S with systems connected to it. Define the *dissipativity matrix* of the system S as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12}^{\mathrm{T}} & \mathbf{M}_{22} \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} \mathbf{M}_{11} &= \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma} + \mathbf{\Gamma}\mathbf{A} + \mathbf{Q} + \mathbf{C}^{\mathrm{T}}\mathbf{P}\mathbf{C}, \\ \mathbf{M}_{12} &= \mathbf{\Gamma}\mathbf{B} - \frac{1}{2}\mathbf{C}^{\mathrm{T}} + \mathbf{C}^{\mathrm{T}}\mathbf{P}\mathbf{D}, \\ \mathbf{M}_{22} &= \mathbf{D}^{\mathrm{T}}\mathbf{P}\mathbf{D} - \frac{1}{2}(\mathbf{D} + \mathbf{D}^{\mathrm{T}}) + \mathbf{R}. \end{aligned}$$

The next lemma qualifies a set of π-coefficients in terms of the negative semi-definiteness of the dissipativity matrix.

Lemma 1 [10, 11] *If $\mathbf{M} \le \mathbf{0}$, $\mathbf{\Gamma} \ge \mathbf{0}$, and $\mathbf{Q} \ge \mathbf{0}$, then system S in (1) is π-sharing with respect to $\{\mathbf{\Gamma}, \mathbf{Q}, \mathbf{P}, \mathbf{R}\}$.*

Note that the conditions in the above lemma are LMIs with respect to the variables $\{\mathbf{\Gamma}, \mathbf{Q}, \mathbf{P}, \mathbf{R}\}$ for a given S.

In the π-sharing theory, the π-stability is defined to include state and input-output stability at the same time. Below is the definition of the π-stability.

Definition 1 [10, 11] *The system (1) is π-stable, if there exist $\gamma_1, \dots, \gamma_4 \in \mathcal{R}$ such that*

$$\begin{aligned} &\|\mathbf{y}\|_T \le \gamma_1\|\mathbf{u}\|_T + \gamma_2|\mathbf{x}(0)| \\ \textit{and}\quad & \\ &\sup_{0\le t\le T}|\mathbf{x}(t)| \le \gamma_3\|\mathbf{u}\|_T + \gamma_4|\mathbf{x}(0)| \end{aligned}$$

for all $\mathbf{u}(t) \in \mathcal{R}^m$, $\mathbf{x}(0) \in \mathcal{R}^n$, and $T \ge 0$.

Note that the first condition in Definition 1 is the condition for $\mathcal{L}_2$ stability, and the second condition implies stability in the sense of Lyapunov when the external input $\mathbf{u} \equiv \mathbf{0}$. The next lemma, adapted from [10, 11] for the LTI case, gives a sufficient condition of the π-stability in terms of the π-coefficients.

Lemma 2 *If $\mathbf{R} \ge r_0\mathbf{I}$, $\mathbf{P} \ge p_0\mathbf{I} > \mathbf{0}$ and $\mathbf{\Gamma} \ge \gamma\mathbf{I} > \mathbf{0}$, then the system S in (1) is π-stable with $\gamma_1 = (1 + \sqrt{p_0\delta})/p_0$, $\gamma_2 = \sqrt{\gamma_0/p_0}$, and $\gamma_3 = \gamma_4 = \sqrt{\xi/\gamma}$, where $\delta = |\min\{0, r_0\}|$, $\gamma_0 =$ the maximum eigenvalue of $\mathbf{\Gamma}$, and $\xi = \max\{\gamma_0,\ \delta + (1 + \sqrt{p_0\delta})/p_0,\ \sqrt{\gamma_0/p_0}\}$.*

An important advantage of the π-sharing theory is that π-coefficients of the feedback system shown in Fig. 1 can be expressed in a composite form of those of individual subsystems, as the following lemma shows.

Lemma 3 [10, 11] *In Fig. 1, let the system S be the feedback connection of subsystems S_1 and S_2, which are π-sharing with respect to $\{\mathbf{\Gamma}_1, \mathbf{Q}_1, \mathbf{P}_1, \mathbf{R}_1\}$ and $\{\mathbf{\Gamma}_2, \mathbf{Q}_2, \mathbf{P}_2, \mathbf{R}_2\}$, respectively. If $\mathbf{R}_1 + \mathbf{P}_2 > \mathbf{0}$, then S is π-sharing with respect to the composite π-coefficients*

$$\begin{aligned} \{\mathbf{\Gamma}, \mathbf{Q}, \mathbf{P}, \mathbf{R}\} = \Big\{ &\begin{bmatrix} \mathbf{\Gamma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}, \\ &\mathbf{P}_1 + \mathbf{R}_2, \mathbf{R}_1[\mathbf{R}_1 + \mathbf{P}_2]^{-1}\mathbf{P}_2 \Big\}. \end{aligned} \tag{4}$$

By the above lemma, a sufficient condition to ensure that S is π-stable is that $\mathbf{\Gamma}_1 > \mathbf{0}$, $\mathbf{\Gamma}_2 > \mathbf{0}$, $\mathbf{P}_1 + \mathbf{R}_2 > \mathbf{0}$, and $\mathbf{R}_1 + \mathbf{P}_2 > \mathbf{0}$. Note that the condition is a set of LMIs.

3 PID-type controller synthesis

The feedback system in Fig. 2 has a square LTI plant in the forward path, a PI-controller in the feedback path, a disturbance input $\mathbf{w}$, and a reference input $\mathbf{r}$. In this paper the effect of the disturbance is not considered, so it is assumed that $\mathbf{w} = \mathbf{0}$. Notice that though Fig. 2 looks like a feedback configuration, the PI-controller is actually in the forward path with respect to the input-output pair $(\mathbf{r}, \mathbf{y}_1)$. Let the plant be described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_p\mathbf{x}(t) + \mathbf{B}_p\mathbf{u}_1(t) \\ \mathbf{y}_1(t) &= \mathbf{C}_p\mathbf{x}(t) + \mathbf{D}_p\mathbf{u}_1(t), \end{aligned} \tag{5}$$

where $\mathbf{x}(t) \in \mathcal{R}^n$, $\mathbf{u}_1(t)$, $\mathbf{y}_1(t) \in \mathcal{R}^m$, and $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p, \mathbf{D}_p$ are given constant matrices with appropriate dimensions. The PI-controller is described by

$$\begin{aligned} \dot{\mathbf{x}}_c(t) &= \mathbf{B}_c\mathbf{u}_2(t) \\ \mathbf{y}_2(t) &= \mathbf{C}_c\mathbf{x}_c(t) + \mathbf{D}_c\mathbf{u}_2(t), \end{aligned} \tag{6}$$

where $\mathbf{x}_c(t)$, $\mathbf{u}_2(t)$, $\mathbf{y}_2(t) \in \mathcal{R}^m$, and $\mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c$ are constant matrices with appropriate dimensions. It is assumed that $\mathbf{B}_c$ is given, or set to $\mathbf{I}$ by default. The problem is to find the I-gain $\mathbf{C}_c$ and P-gain $\mathbf{D}_c$ so that the feedback system is π-stable.

Based on the multivariable π-sharing theory stated in Section 2, it is easy to see that if there exist $\mathbf{C}_c$, $\mathbf{D}_c$, $\{\mathbf{\Gamma}_1, \mathbf{Q}_1, \mathbf{P}_1, \mathbf{R}_1\}$, and $\{\mathbf{\Gamma}_2, \mathbf{Q}_2, \mathbf{P}_2, \mathbf{R}_2\}$ such that

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{A}_p^{\mathrm{T}}\mathbf{\Gamma}_1 + \mathbf{\Gamma}_1\mathbf{A}_p + \mathbf{Q}_1 + \mathbf{C}_p^{\mathrm{T}}\mathbf{P}_1\mathbf{C}_p & \mathbf{\Gamma}_1\mathbf{B}_p - \frac{1}{2}\mathbf{C}_p^{\mathrm{T}} + \mathbf{C}_p^{\mathrm{T}}\mathbf{P}_1\mathbf{D}_p \\ \mathbf{B}_p^{\mathrm{T}}\mathbf{\Gamma}_1 - \frac{1}{2}\mathbf{C}_p + \mathbf{D}_p^{\mathrm{T}}\mathbf{P}_1\mathbf{C}_p & \mathbf{D}_p^{\mathrm{T}}\mathbf{P}_1\mathbf{D}_p - \frac{1}{2}(\mathbf{D}_p + \mathbf{D}_p^{\mathrm{T}}) + \mathbf{R}_1 \end{bmatrix} \le \mathbf{0}, \tag{7}$$

$$\mathbf{\Gamma}_1 > \mathbf{0},\ \mathbf{Q}_1 \ge \mathbf{0}, \tag{8}$$

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{Q}_2 + \mathbf{C}_c^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_c & \mathbf{\Gamma}_2\mathbf{B}_c - \frac{1}{2}\mathbf{C}_c^{\mathrm{T}} + \mathbf{C}_c^{\mathrm{T}}\mathbf{P}_2\mathbf{D}_c \\ \mathbf{B}_c^{\mathrm{T}}\mathbf{\Gamma}_2 - \frac{1}{2}\mathbf{C}_c + \mathbf{D}_c^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_c & \mathbf{D}_c^{\mathrm{T}}\mathbf{P}_2\mathbf{D}_c - \frac{1}{2}(\mathbf{D}_c + \mathbf{D}_c^{\mathrm{T}}) + \mathbf{R}_2 \end{bmatrix} \le \mathbf{0}, \tag{9}$$

$$\mathbf{\Gamma}_2 > \mathbf{0},\ \mathbf{Q}_2 \ge \mathbf{0}, \tag{10}$$

$$\mathbf{P}_1 + \mathbf{R}_2 > \mathbf{0},\ \mathbf{R}_1 + \mathbf{P}_2 > \mathbf{0}, \tag{11}$$

then our goal is reached. To find feasible solutions of the above matrix inequalities, a procedure is suggested here. First ignore the inequalities in (9) and (10). Then add an extra matrix inequality $\mathbf{P}_2 < \mathbf{0}$, which together with (7), (8), and (11) form a set of LMIs with respect to the variables $\mathbf{\Gamma}_1$, $\mathbf{Q}_1$, $\mathbf{P}_1$, $\mathbf{R}_1$, $\mathbf{P}_2$, and $\mathbf{R}_2$. To continue, it is assumed that there exists a set of feasible solution for the LMIs with $\mathbf{P}_2 < \mathbf{0}$, which may be found by using suitable computing softwares [4]. Now, the matrix inequality (9) is re-written as

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{Q}_2 & \mathbf{\Gamma}_2\mathbf{B}_c \\ \mathbf{B}_c^{\mathrm{T}}\mathbf{\Gamma}_2 & \mathbf{R}_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{C}_c^{\mathrm{T}} \\ \mathbf{0} & \mathbf{D}_c^{\mathrm{T}} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}_c & \mathbf{D}_c \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{C}_c^{\mathrm{T}} \\ \mathbf{0} & \mathbf{D}_c^{\mathrm{T}} \end{bmatrix}\mathbf{P}_2\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}_c & \mathbf{D}_c \end{bmatrix} \le \mathbf{0},$$

or equivalently,

$$\begin{bmatrix} \mathbf{Q}_2 & \mathbf{\Gamma}_2\mathbf{B}_c \\ \mathbf{B}_c^{\mathrm{T}}\mathbf{\Gamma}_2 & \mathbf{R}_2 + \frac{1}{4}\hat{\mathbf{P}}_2^{-1} \end{bmatrix} \le \begin{bmatrix} \mathbf{0} & \mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{P}}_2^{1/2} \\ \mathbf{0} & \mathbf{D}_c^{\mathrm{T}}\hat{\mathbf{P}}_2^{1/2}+\frac{1}{2}\hat{\mathbf{P}}_2^{-1/2} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{P}}_2^{1/2}\mathbf{C}_c & \hat{\mathbf{P}}_2^{1/2}\mathbf{D}_c+\frac{1}{2}\hat{\mathbf{P}}_2^{-1/2} \end{bmatrix},$$

where $\hat{\mathbf{P}}_2 = -\mathbf{P}_2 > \mathbf{0}$. The above matrix inequality holds if

$$\mathbf{R}_2+\frac{1}{4}\hat{\mathbf{P}}_2^{-1} \le (\mathbf{D}_c^{\mathrm{T}}\hat{\mathbf{P}}_2^{1/2}+\frac{1}{2}\hat{\mathbf{P}}_2^{-1/2})(\hat{\mathbf{P}}_2^{1/2}\mathbf{D}_c+\frac{1}{2}\hat{\mathbf{P}}_2^{-1/2}), \quad (12)$$

$$\mathbf{\Gamma}_2\mathbf{B}_c = (\mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{P}}_2^{1/2})(\hat{\mathbf{P}}_2^{1/2}\mathbf{D}_c + \frac{1}{2}\hat{\mathbf{P}}_2^{-1/2}), \quad (13)$$

$$\mathbf{Q}_2 \le \mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{P}}_2\mathbf{C}_c. \quad (14)$$

The matrix inequality (12) is satisfied by setting

$$\mathbf{D}_c = \hat{\mathbf{P}}_2^{-1/2}(\mathbf{R}_2 + \frac{1}{4}\hat{\mathbf{P}}_2^{-1} + \mathbf{Z})^{1/2} - \frac{1}{2}\hat{\mathbf{P}}_2^{-1}, \quad (15)$$

where $\mathbf{Z}$ may be any symmetric matrix making $\mathbf{R}_2 + \frac{1}{4}\hat{\mathbf{P}}_2^{-1} + \mathbf{Z} > \mathbf{0}$. Also, in (15) $\mathbf{D}_c$ makes $\mathbf{D}_c^{\mathrm{T}}\hat{\mathbf{P}}_2 + \frac{1}{2}\mathbf{I} = (\mathbf{R}_2 + \frac{1}{4}\hat{\mathbf{P}}_2^{-1} + \mathbf{Z})^{1/2}\hat{\mathbf{P}}_2^{1/2}$, a nonsingular matrix. Hence the matrix equation (13) may be satisfied by setting

$$\mathbf{C}_c = (\mathbf{D}_c^{\mathrm{T}}\hat{\mathbf{P}}_2 + \frac{1}{2}\mathbf{I})^{-1}\mathbf{B}_c^{\mathrm{T}}\mathbf{\Gamma}_2, \quad (16)$$

where $\mathbf{\Gamma}_2$ is at designer's choice, but it needs to be positive definite to satisfy (10). Finally, any $\mathbf{Q}_2 \ge \mathbf{0}$ which is less than or equal to $\mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{P}}_2\mathbf{C}_c \ge \mathbf{0}$ may be selected to satisfy the matrix inequality (14) and the condition about $\mathbf{Q}_2$ in (10). However, the actual determination of $\mathbf{Q}_2$ is not necessary as it does not affect the selections of controller gains $\mathbf{C}_c$ and $\mathbf{D}_c$.

The proposed PI-controller synthesis procedure contains three matrices $\mathbf{B}_c$, $\mathbf{Z}$, and $\mathbf{\Gamma}_2$ that may be tuned by the designer. A simple default choice is $\mathbf{B}_c = \mathbf{I}$, $\mathbf{Z} = \mathbf{0}$, and $\mathbf{\Gamma}_2 = g\mathbf{I}$, where $g > 0$ is the gain adjustment factor for the I-gain $\mathbf{C}_c$. While the second half part of the procedure is straightforward, at this point it is appropriate to consider more closely the LMIs that must be solved in the first half part of the procedure. The second LMI $\mathbf{R}_1 + \mathbf{P}_2 > \mathbf{0}$ in (11) and the extra LMI $\mathbf{P}_2 < \mathbf{0}$ clearly require $\mathbf{R}_1 > \mathbf{0}$, but then the $(2,2)$-block of the main LMI in (7) dictates that $\mathbf{D}_p$ be at least nonsingular. Thus for any plant with the singular direct transmission gain matrix, the proposed procedure can not be applied. However, there is a remedy. For the plant described by (5) with a singular $\mathbf{D}_p$, consider the control system configuration displayed in Fig. 3, where a feedforward compensation with the transfer function matrix $\frac{s}{s+\alpha}\mathbf{D}_f$ is utilized to produce an augmented plant with input $\mathbf{u}_1$ and output $\hat{\mathbf{y}}_1$. Clearly the direct transmission gain matrix of the augmented plant is $\mathbf{D}_f + \mathbf{D}_p$, which can always be arranged to be nonsingular. The objective of the high-pass filter $\frac{s}{s+\alpha}$ with $\alpha \gg 1$ is to ensure that inside the bandwidth of the control system, responses of the true plant output $\mathbf{y}_1$ will be close to those of the augmented plant output $\hat{\mathbf{y}}_1$. In particular, when a PI-controller is called for, it is usually desired that $\mathbf{y}_1$ has no steady-state errors in response to the reference input $\mathbf{r}$ consisting of the step signals. This is guaranteed by using the high-pass filter, provided the augmented system is stabilized by the PI-controller.

After a PI-controller (6) is designed for the augmented plant in Fig. 3, it is seen that

$$\begin{aligned} \mathbf{u}_1(s) &= -\mathbf{F}(s)[\hat{\mathbf{y}}_1(s) - \mathbf{r}(s)] \\ &= \mathbf{F}(s)[\mathbf{r}(s) - \mathbf{y}_1(s)] - \frac{s}{s+\alpha}\mathbf{F}(s)\mathbf{D}_f\mathbf{u}_1(s), \end{aligned}$$

where $\mathbf{F}(s) = \frac{1}{s}\mathbf{C}_c\mathbf{B}_c + \mathbf{D}_c$. Thus

$$\mathbf{u}_1(s) = \frac{s+\alpha}{s}[s(\mathbf{I} + \mathbf{D}_c\mathbf{D}_f) + \mathbf{C}_c\mathbf{B}_c\mathbf{D}_f + \alpha\mathbf{I}]^{-1} (s\mathbf{D}_c + \mathbf{C}_c\mathbf{B}_c)[\mathbf{r}(s) - \mathbf{y}_1(s)]. \quad (17)$$

This expression enables one to return to the basic control system configuration in Fig. 2, but the PI-controller therein is replaced by a controller with the transfer function matrix in (17). Note that the modified controller is basically a PID-type, as can be seen in the special case of single-input-single-output systems, where (17) reduces to

$$u_1(s) = \frac{(s+\alpha)(sD_c + C_cB_c)}{s[s(1+D_cD_f) + C_cB_cD_f + \alpha]}[r(s) - y_1(s)],$$

similar in the form to the standard realizable PID controller

$$u_1(s) = \frac{c_Ds^2 + c_Ps + c_I}{s(s/\beta+1)}[r(s) - y_1(s)]$$

with $\beta \gg 1$.

4 Examples

Example 1: For the plant (5) with $n = 3$, $m = 2$,

$$\mathbf{A}_p = \begin{bmatrix} -5 & 44 & -60 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_p = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{D}_p = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

the proposed PI-controller synthesis procedure can be applied directly. However, our numerical experiences show that if the LMIs $\mathbf{P}_2 < \mathbf{0}$, (7), (8), and (11) are solved blindly for any feasible solutions without setting any preferences, then often in the solution $\|\mathbf{R}_2\|$ is large and $\|\mathbf{P}_2\| = \|\hat{\mathbf{P}}_2\|$ is small, resulting in a large $\|\mathbf{D}_c\|$. Hence an objective function $\|\mathbf{R}_2\| - \|\mathbf{P}_2\|$ is set to be minimized. Based on $\mathbf{B}_c = \mathbf{I}$, $\mathbf{Z} = \mathbf{0}$, and $\mathbf{\Gamma}_2 = 1000\mathbf{I}$, the optimal solution leads to

$$\mathbf{C}_c = \begin{bmatrix} 1999.9997 & 0.0011 \\ 0.0011 & 1999.9924 \end{bmatrix},$$
$$\mathbf{D}_c = \begin{bmatrix} 101.8133 & 0.0027 \\ 0.0003 & 101.8135 \end{bmatrix}.$$

The feedback system response to $\mathbf{r}(t) = [\ r_1(t) \quad r_2(t)\ ]^{\mathrm{T}}$, where $r_1(t)$ is the unit-step signal and $r_2(t) = 0$, is shown in Fig. 4a. It is seen that for $\mathbf{y}_1(t)$ there are no steady-state errors as desired, and the decoupling effect is pretty good as $\mathbf{y}_2(t)$ is kept small. Also, in Fig. 4b the two control inputs of $\mathbf{u}_1(t)$ have moderate magnitudes. When $r_1(t) = 0$ and $r_2(t)$ is the unit-step signal, the response characteristics are about the same, except that the magnitude of $\mathbf{u}_1(t)$ is larger.

Example 2: For the plant (5) with $n = 2$, $m = 1$,

$$\mathbf{A}_p = \begin{bmatrix} 1.5 & -0.5 \\ 1 & -1 \end{bmatrix},\ \mathbf{B}_p = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\ \mathbf{C}_p = [\ 1 \quad 0.5\ ],$$

and $\mathbf{D}_p = 0$, a feedforward compensation $\frac{0.8s}{s+200}$ is adopted to form the augmented plant. Then corresponding to $B_c = 1$, $Z = 0$, and $\Gamma_2 = 1$, the gains $C_c = 2.0000$ and $D_c = 4.8028$ are obtained from applying the optimization procedure introduced in *Example 1*. The simulation results in Fig. 5 shows the response of the feedback system to the step reference signal $r(t)$, where it can be seen that, except during a short transient period, $y_1(t)$ and $\hat{y}_1(t)$ are very close.

5 Conclusion

In this paper the multivariable π-sharing theory is utilized to develop a procedure for synthesizing PID-type controllers. An LMI-based sufficient condition is established to let the users of the procedure easily determine if a PI-controller can be found. Notice that the π-sharing theory is originally devised to handle nonlinear systems. Therefore if for any nonlinear system a set of suitable π-coefficients exists, then the procedure proposed in this paper has the potential to be applicable as well. This part is currently under investigations.

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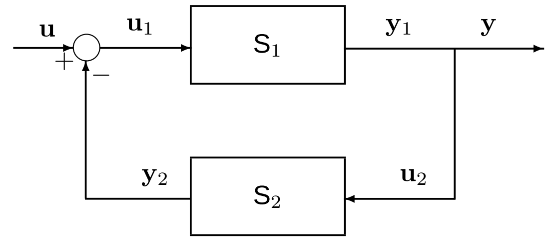


Figure 1: A feedback system.

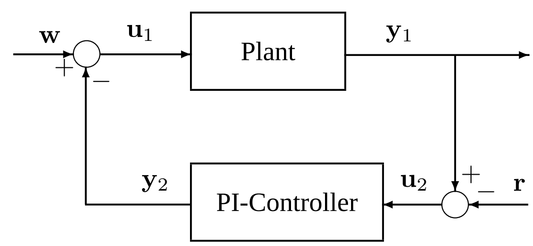


Figure 2: A feedback system with a PI-controller.

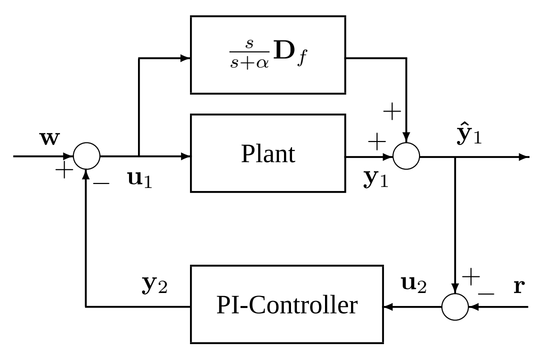
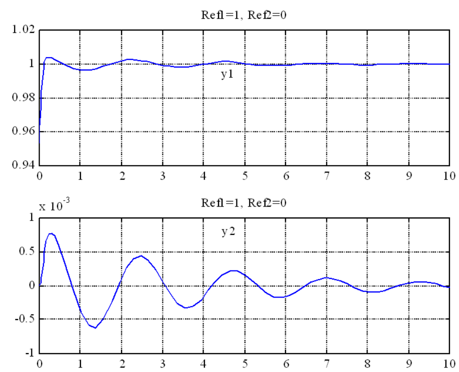
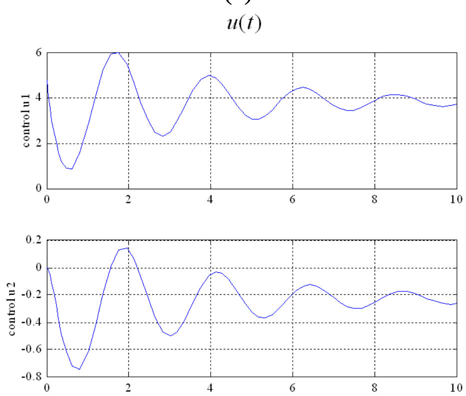


Figure 3: A plant controlled by a feedforward controller and a PI-controller.



(a)



(b)

Figure 4: (a) Output responses of the system in Example 1 due to r_1 = unit step, and $r_2 = 0$. (b) Control inputs of the system in Example 1 due to r_1 = unit step, and $r_2 = 0$.

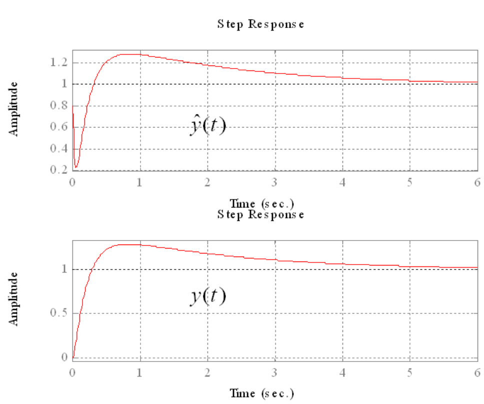


Figure 5: Step responses of the system in Example 2.