UNIFORM ASYMPTOTIC STABILITY OF NON-AUTONOMOUS PARAMETERIZED DISCRETE-TIME CASCADES: A CASE STUDY[1]

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Abstract
Recently, a framework for controller design of sampled-data nonlinear systems via their approximate discrete-time models has been proposed in the literature. This framework naturally leads to investigation of families of parameterized discrete-time systems. In this paper we present a case-study in cascades-based control. That is, we show the utility of a theorem for stability of parameterized discrete-time cascades in the tracking control of the unicycle benchmark. This application is fairly illustrative of the technical differences and obstacles encountered in the analysis of discrete-time parameterized systems and therefore motivates a formal study of parameterized discrete-time systems.

1 Introduction

The prevalence of digitally controlled systems and the fact that the nonlinearities in the plant model can often not be neglected, strongly motivate the area of nonlinear sampled-data systems. An important method for controller design for sampled-data nonlinear systems consists of obtaining the exact (respectively, approximate) discrete-time model of the plant and then designing the controller based on the discrete-time plant model. We refer to this method as exact (respectively approximate) discrete-time design (DTD). The main pitfall of the approximate DTD method is that if one is not careful with the choice of the approximate model and the design of the controller, it is possible that a controller asymptotically stabilizes the approximate plant model but *not* the exact model. It is noteworthy that this fact concerns even linear systems [13].

Motivated by this fact, a framework for nonlinear sampled-data controller design via approximate discrete-time models has been proposed in [13, 14, 15]. These results are very similar in spirit to results from the numerical analysis literature (see e.g. [18]) that apply to continuous-time control systems. In [13] checkable conditions on the continuous-time plant model, the controller and the approximate discrete-time model are presented which guarantee that if the controller stabilizes the approximate model, it would also stabilize the exact discrete-time model. Furthermore, in [14] it was shown that stability of the exact discrete-time model under mild conditions guarantees also stability of the sampled-data system.

Very recently, in [12, 9] we established new results on stability of parameterized discrete-time cascaded systems which complement the above-mentioned framework. These results contribute to what we may call *cascades-based* control. Roughly speaking, this approach aims at designing controllers in cases when the closed loop system has a cascaded structure. Moreover, the closed loop dynamics shall verify certain structural conditions imposed either on the functions that define the closed loop dynamics or indirectly, in terms of properties of Lyapunov-like functions. See [1, 7, 8, 17] and references therein for a large number of results and applications in this area, in the continuous-time context. In the context of discrete-time systems cascade based results are scarce with a notable exception of [4] where results based on the property of input-to-state stability (ISS) property are presented.

This paper focuses on a case-study in cascades-based control. In particular, we revisit the well studied problem of tracking control of the unicycle benchmark but in the case when the controller is to be implemented digitally. Since we design the controller using the approximate DTD method this naturally leads to investigation of families of parameterized discrete-time systems. The approach that we follow here parallels that of [16] however, the result presented here is not a simple translation of its counterpart in continuous-time. Indeed, the proofs we establish here are notably different and the sufficient conditions we impose are tailored specifically for discrete-time parameterized discrete-time systems. Moreover, we illustrate how one can redesign control algorithms originally designed for continuous-time models, in order to improve the per-

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formance of the sampled-data system.

Notation. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for all $t > 0$, $\beta(\cdot, t) \in \mathcal{K}$, for all $s > 0$, $\beta(s, \cdot)$ is decreasing to zero. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{N}$ if $\gamma(\cdot)$ is continuous and nondecreasing. We denote by $|\cdot|$ the Euclidean norm of vectors. We denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of the real and natural numbers respectively. For an arbitrary $r \in \mathbb{R}$ we use the notation $\lfloor r \rfloor := \max_{z \in \mathbb{Z}, z \leq r} z$. Given strictly positive real numbers L, T we use the following notation $\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor$. For a parameterized discrete-time system $x(k+1) = F_T(k, x(k))$ we denote its solution with initial conditions $k_\circ \geq 0$ and $x_\circ \in \mathbb{R}^n$, by $\phi^x_T(k, k_\circ, x_\circ)$.

2 Parameterized discrete-time systems

We consider the class of systems

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^m$ are respectively the state and control input. We assume that for any given $x_\circ$ and $u(\cdot)$ the differential equation in (1) has a unique solution defined on its maximal interval of existence $[0, t_{\max})$. The control input is taken to be a piecewise constant signal $u(t) = u(kT) =: u(k)$, $\forall t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period.

In the approximate DTD method that we take, the goal is to design a controller based on an approximate discrete-time model:

$$x(k+1) = F^a_T(k, x(k), u(k)) \ . \tag{2}$$

For instance, if f is locally Lipschitz in t and x, the Euler approximate model can be defined as $x(k+1) = x(k) + Tf(kT, x(k), u(k))$ and it can be shown to be an $O(T^2)$ approximation of the exact discrete-time model. On the other hand, if f is measurable in t, then a modified "Euler" model that is $O(T^2)$ approximation of the exact model is given by $x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(\tau, x(k), u(k)) d\tau$.

In our work, the sampling period T is assumed to be a design parameter which can be *arbitrarily assigned*. Since we are dealing with a family of approximate discrete-time models F^a_T, *parameterized* by T, in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by T. Thus, we consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(k, x(k), z(k)) \\ u(k) &= u_T(k, x(k), z(k)) \ , \end{aligned} \tag{3}$$

where $z \in \mathbb{R}^{n_z}$. We emphasize again that if the controller (3) stabilizes the approximate model (2) for all small T, this does not guarantee that the same controller would approximately stabilize the exact model

$$x(k+1) = F^e_T(k, x(k), u(k)) \tag{4}$$

where $F^e_T(k, x, u) := x + \int_{kT}^{(k+1)T} f(\tau, x(\tau), u) d\tau$ for all small T. In [13, Theorem 1] sufficient conditions for this to hold were given for autonomous systems. The following result generalizes [13, Theorem 1] to time-varying systems and it gives sufficient conditions to guarantee that any controller that stabilizes an approximate model will also stabilize the exact model for sufficiently fast sampling[1]. Let $\widetilde{F}^a_T(k, \tilde{x})$ and $\widetilde{F}^e_T(k, \tilde{x})$, where $\tilde{x} = (x^T \ z^T)^T$, denote the right hand sides of the closed-loop systems (2), (3) and (4), (3) respectively.

Theorem 1 *Suppose that there exists $\beta \in \mathcal{KL}$ such that for any strictly positive numbers $(L, \eta, \Delta, \delta)$ there exist $K, T^* > 0$ and $\rho \in \mathcal{K}_\infty$ such that:*
(i) **SP-UAS of approximate:** *For all $k_\circ \geq 0$, $|\tilde{x}(k_\circ)| \leq \Delta$ and $T \in (0, T^*)$ the solutions of (2), (3) satisfy for all $k \geq k_\circ \geq 0$,*

$$|\phi^a_T(k, k_\circ, \tilde{x}(k_\circ))| \leq \beta(|\tilde{x}(k_\circ)|, T(k - k_\circ)) + \delta.$$

(ii) **strong multiple-step consistency between $\widetilde{F}^a_T$ and $\widetilde{F}^e_T$:** *For all $\tilde{x}_1, \tilde{x}_2$ with $\max\{|\tilde{x}_1|, |\tilde{x}_2|\} \leq \Delta$, $k \geq 0$ and $T \in (0, T^*)$ we have*

$$\begin{aligned} \left| \widetilde{F}^a_T(k, \tilde{x}_1) - \widetilde{F}^a_T(k, \tilde{x}_2) \right| &\leq (1 + KT) |\tilde{x}_1 - \tilde{x}_2| \\ \left| \widetilde{F}^e_T(k, \tilde{x}) - \widetilde{F}^a_T(k, \tilde{x}) \right| &\leq T\rho(T) \ . \end{aligned}$$

Then, for any strictly positive real numbers $(\widetilde{\Delta}, \widetilde{\delta})$ there exists $\widetilde{T} > 0$ such that for all $k_\circ \geq 0$, $|\tilde{x}(k_\circ)| \leq \widetilde{\Delta}$ and $T \in (0, \widetilde{T})$ the solutions of (4), (3) satisfy:
(iii) **SP-UAS of exact:** *For all $k \geq k_\circ \geq 0$,*

$$|\phi^e_T(k, k_\circ, \tilde{x}(k_\circ))| \leq \beta(|\tilde{x}(k_\circ)|, T(k - k_\circ)) + \widetilde{\delta}.$$

We stress that the consistency condition in Theorem 1 is checkable although F^e_T is not known in general. Indeed, it is ensured by choosing an appropriate consistent approximate model F^a_T for controller design. A range of such approximate models can be found in standard books on numerical analysis [18] (if f is independent of t). This result motivates the following definition.

Definition 1 (SP-UAS) *The family of the parameterized time-varying systems*

$$y(k+1) = F_T(k, y(k)) \tag{5}$$

[1] The proof is omitted since it follows from that of [13, Theorem 1] with straightforward changes.

is semiglobally practically uniformly asymptotically stable SP-UAS (resp. uniformly globally asymptotically stable UGAS) if there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers (Δ, ν) there exists $T^* > 0$ (resp. there exists $T^* > 0$) such that for all $k_\circ \geq 0$, $y(k_\circ) = y_\circ$ with $|y_\circ| \leq \Delta$, $T \in (0, T^*)$ (resp. $y(k_\circ) = y_\circ$ with $y_\circ \in \mathbb{R}^n$, $T \in (0, T^*)$) and for all $k \geq k_\circ$ the following holds:

$$\begin{aligned} &|\phi_T^y(k, k_\circ, y_\circ)| \leq \max\{\beta(|y_\circ|, (k-k_\circ)T), \nu\} \\ &(\textit{resp. } |\phi_T^y(k, k_\circ, y_\circ)| \leq \beta(|y_\circ|, (k-k_\circ)T)\ ). \end{aligned} \tag{6}$$

2.1 Results on UGAS of cascades

Our control approach relies on the ability of designing a controller so that $\widetilde{F}_T^a(k, \tilde{x})$ has the following cascaded structure:

$$x(k+1) = f_T(k, x(k), z(k)) \tag{7}$$
$$z(k+1) = g_T(k, z(k)), \tag{8}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$ and T is the sampling period. We present here the technical tools that will allow us to conclude UGAS for the case-study of the unicycle. To this end, we introduce the following two technical definitions.

Definition 2 *The system (5) is uniformly globally bounded (UGB), if there exist $\kappa \in \mathcal{K}_\infty$, c and $T^* > 0$ such that for all $k \geq k_\circ \geq 0$, $y(k_\circ) = y_\circ$, $y_\circ \in \mathbb{R}^n$ and $T \in (0, T^*)$ it holds that $|\phi_T^y(k, k_\circ, y_\circ)| \leq \kappa(|y_\circ|) + c$.*

Definition 3 *The family of the parameterized time-varying systems (5) is Lyapunov UGAS if there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$, $L \in \mathcal{N}$, $T^* > 0$ and for each $T \in (0, T^*)$ a continuous function $V_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $y \in \mathbb{R}^n$, all $k \geq k_\circ \geq 0$ and all $T \in (0, T^*)$ we have that*

$$\alpha_1(|y|) \leq V_T(k, y) \leq \alpha_2(|y|) \tag{9}$$
$$V_T(k+1, F_T(k, y)) - V_T(k, y) \leq -T\alpha_3(|y|) \tag{10}$$

and for all $x, r, s \in \mathbb{R}^n$,

$$|\, V_T(k, r) - V_T(k, s)\, | \leq L\left(\max\{|r|, |s|\}\right) |r - s|\ . \tag{11}$$

Remark 1 *We note that the properties in Definitions 1 and 3 are very related. In particular, it was shown in [13] that (9), (10) are equivalent to UGAS in Definition 1 (see (6)). However, the converse Lyapunov theorem in [13] does not produce a Lyapunov function satisfying the condition (11) and we believe that constructing such converse Lyapunov functions is an open problem in the literature.*

Assumption 1 *There exist $\gamma_2 \in \mathcal{N}$, $\gamma_1, \gamma_3 \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $\xi \in \mathbb{R}^n$, $k \geq 0$ and $T \in (0, T^*)$ we have $|f_T(k, x, z)| \leq \gamma_1(|\xi|)$ and $|f_T(k, x, z) - f_T(k, x, 0)| \leq T\gamma_2(|x|)\gamma_3(|z|)$, where $\xi^\top = [x^\top\, y^\top]$.*

Theorem 2 *Suppose that f_T of the system (7) satisfies Assumption 1. Then, the system (7), (8) is UGAS if the following conditions hold: (i) The system $x(k+1) = f_T(k, x(k), 0)$ is Lyapunov UGAS; (ii) The system (8) is UGAS; (iii) The system (7), (8) satisfies the property UGB.*

We stress that UGB is in general difficult to check. In [9] we present several sufficient conditions for this property to hold and which are inspired by [1, 17]. For the sake of completeness, we close this section with a result for UGB which we will use later on in the case study. To this end, we introduce the following technical hypothesis.

Assumption 2 *Suppose that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2, \varphi \in \mathcal{K}_\infty$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2 \in \mathcal{N}$, $c, T^* > 0$ and for each $T \in (0, T^*)$ there exists $V_T : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}$, $k \geq 0$ and $T \in (0, T^*)$ we have that :*

$$\tilde{\alpha}_1(|x|) \leq V_T(k, x) \leq \tilde{\alpha}_2(|x|) + c \tag{12}$$
$$\begin{aligned} &V_T(k+1, f_T(k, x, z)) - V_T(k+1, f_T(k, x, 0)) \\ &\qquad \leq T\tilde{\gamma}_1(|z|)\varphi(V_T(k, x)) + T\tilde{\gamma}_2(|z|) \end{aligned} \tag{13}$$
$$V(k+1, f_T(k, x, 0)) - V_T(k, x) \leq 0 \tag{14}$$
$$\int_1^\infty \frac{ds}{\varphi(s)} = \infty\ . \tag{15}$$

Proposition 1 *[10] Consider the system (7) with input z and under Assumption 2. If furthermore the solutions of the system (8) satisfy the summability condition*

$$T\sum_{k=k_\circ}^{\infty} \mu\left(|\phi_T^z(k, k_\circ, z_\circ)|\right) \leq \rho(|x_\circ|)\,, \tag{16}$$

with some $\rho \in \mathcal{K}_\infty$ and $\mu(s) := \tilde{\gamma}_1(s) + \frac{\tilde{\gamma}_2(s)}{\varphi(1)}$ then, the system (7) is UGB.

3 Tracking control of the unicycle

We revisit now the problem of tracking control of a mobile robot of the unicycle type. This problem has been thoroughly studied in the continuous-time context via many different approaches (see [6] for a survey; for a more recent text with an updated list of references see [7]). To illustrate the utility of our results we will revisit the cascades approach used in [16] for a 3 degrees of freedom cart. The results may be extended to higher dimension systems, following for instance [7].

While the problem setting is the same as considered in the continuous-time context, we will see that the proof techniques employed in the discrete-time case are quite different. For instance, since we deal with approximate discrete-time models, some important structural characteristics are lost. Hence, we believe that the proofs of this section are interesting in their own right.

According to [5] the context of the problem can be set as follows. We have a mobile robot with two directional wheels and two "fixed" wheels and whose motion is described by

$$\dot{x} = v\cos\theta;\ \dot{y} = v\sin\theta;\ \dot{\theta} = \omega\ , \tag{17}$$

where x, y are the Cartesian coordinates of the center of the axis joining the directional wheels and θ is the orientation angle of the directional wheels. The robot is required to follow a trajectory generated by an exosystem, i.e., a fictitious "reference robot" with kinematics

$$\dot{x}_r = v_r(t)\cos\theta_r;\quad \dot{y}_r = v_r(t)\sin\theta_r;\quad \dot{\theta}_r = \omega_r(t) \tag{18}$$

where $v_r(t)$ and $\omega_r(t)$ are *given* reference velocities. Then, the tracking errors satisfy the set of equations (see [5, Lemma 1])

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r(t)\cos\theta_e &&(19a)\\ \dot{y}_e &= -\omega x_e + v_r(t)\sin\theta_e &&(19b)\\ \dot{\theta}_e &= \omega_r(t) - \omega v\ , &&(19c)\end{aligned}$$

where $(\cdot)_e := (\cdot)_r - (\cdot)$. The system is velocity-controlled, i.e., the control problem reduces to finding control inputs ω and v (which also correspond to the actual angular and linear velocities of the cart) such that the origin of (19) is UGAS.

There are numerous solutions to this problem in the context of continuous-time (e.g. [3, 11] and [7] for a recent literature review). Here, we will revisit the cascaded approach proposed in [16] whose main feature is that the control laws are *linear*. To illustrate and motivate our results we will proceed to solve the same problem with a *linear* time-varying discrete-time controller which we will redesign based on the Euler-discretization of the error dynamics,

$$\begin{aligned} x_e(k+1) &= x_e(k) + T[\,\omega y_e(k) - v + v_r(k)\cos\theta_e(k)\,]\\ y_e(k+1) &= y_e(k) + T[\,-\omega x_e(k) + v_r(k)\sin\theta_e(k)\,]\\ \theta_e(k+1) &= \theta_e(k) + T[\,\omega_r(k) - \omega\,]\,. \end{aligned} \tag{20}$$

Thus, our control problem consists of designing v and ω such that (20) is UGAS.

We will solve this control problem following a similar approach to that of [16] where it was shown using results for continuous-time cascaded systems, that the system (19) in closed loop with $v = v_r(t) + a_2 x_e$ and $\omega = \omega_r(t) + a_1\theta_e$ is UGAS for appropriately chosen a_1 and a_2.

The controller structure that we use

$$\omega = \omega_r(k) + a_1\theta_e(k);\ v = v_r(k) + a_2 x_e(k) + T\vartheta \tag{21}$$

where a_1, a_2, $\omega_r(k)$ and $v_r(k)$ come from the continuous-time control law proposed in [16] and ϑ is an extra control input which depends on k, x_e and y_e and that we will design with the aim of improving the system's performance. More specifically, the motivation for the control laws above is that as in the continuous-time context, the closed loop system

$$\begin{aligned} \begin{matrix} x_e(k+1) \\ y_e(k+1)\end{matrix} &\begin{matrix} = \\ = \end{matrix} \underbrace{\begin{matrix}(1-Ta_2)x_e(k) + T\omega_r(k)y_e(k) - T^2\vartheta \\ y_e(k) - T\omega_r(k)x_e(k)\end{matrix}}_{F_{1T}(k,x(k))} + \\ &\underbrace{\begin{matrix} T[\,a_1\theta_e(k)y_e(k) - v_r(k) + v_r(k)\cos\theta_e(k)\,] \\ T[\,-a_1\theta_e(k)x_e(k) + v_r(k)\sin\theta_e(k)\,]\end{matrix}}_{G_T(k,x(k),z(k))} &&(22)\\ \theta_e(k+1) &= (1-Ta_1)\theta_e(k) =:\ F_{2T}(k,x(k))\,, &&(23)\end{aligned}$$

where $z := \theta$ and $x :=$ col$[x_e, y_e]$, has a cascaded structure.

Hence, the control laws (21) are designed with two main ideas in mind: 1) to have as simple as possible controllers; 2) that the closed loop system verifies the conditions of our main results for cascades. More specifically, notice that the bottom subsystem (23) is independent of x_e and y_e and is UGES[2] for values of a_1 sufficiently small ($T^*a_1 < 1$). Also, the interconnection term $G_T(k,x,z)$ is linear in x_e and y_e. Hence, our results suggest that we only need to design ϑ as a function of x_e and y_e only so that the zero-input (i.e., with $G_T \equiv 0$) subsystem in (22) be UGAS (or possibly, UGES).

Notice that in the particular case that $\vartheta \equiv 0$ we obtain the emulated (discretized) continuous-time control law. However, as we will illustrate below, when carefully defined this extra degree of freedom in the control design allows to improve performance and, on occasions, to enlarge the domain of attraction with respect to that of the emulated continuous-time control law. Thus, our control scheme can be regarded as a *redesign* of the cascaded-based continuous-time controller of [16]. Simulation results at the end of this section will illustrate this.

[2] That is, there exist c, λ and $T^* > 0$ such that $\left|\phi_T^{\theta_e}(k,k_\circ,\theta_e(k_\circ))\right| \le c\,|\theta_e(k_\circ)|\,e^{-\lambda T(k-k_\circ)}$ for all $k \ge k_\circ \ge 0$, all $\theta_e(k_\circ)\in\mathbb{R}$ and all $T\in(0,T^*)$.

Here, we will establish UGES of the zero-input system based on a property of persistency of excitation. However, we will need a specific reformulation of this property[3] within the framework of discrete-time parameterized systems. This is introduced next. To compact the notation, in the sequel we will use $\omega_{r_k} := \omega_r(k)$.

Definition 4 (PE) *Let $\omega_r : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ be a function produced by sampling a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ at rate T. The function ω_r is said to be persistently exciting (PE) if there exist positive numbers μ, L and T^* such that for all $T \in (0, T^*)$ and all $j \geq 0$ we have $T \sum_{k=j}^{j+\ell_{L,T}} \omega_{r_k}^2 \geq \mu$.*

Proposition 2 (Main result) *Consider the system (20) in closed loop with (21). Assume that the signal ω_{r_k} is PE and there exists $\hat{T}, w_M > 0$ such that for all $k \geq 0$ and $T \in (0, \hat{T})$*

$$\max\left\{ |v_{r_k}|, |\omega_{r_k}|, \frac{|\omega_{r_k} - \omega_{r_{k-1}}|}{T} \right\} \leq w_M \ . \qquad (24)$$

Then, there exists $a_2 > 0$ such that for all $K, a_1 > 0$ and $\vartheta(k,x)$ with $|\vartheta(k,x)| \leq K|x|$, the system is UGAS.

Proof. It follows by invoking Theorem 2. Firstly, we see that Assumption 1 holds trivially in view of item 1 of the proposition. To see more clearly, notice that (24) implies that $f_T(\cdot,\cdot,\cdot)$ is continuous and uniformly bounded in the first argument. Item 1 of the proposition also implies that there exists $c > 0$ independent of T, such that for all $T \in (0, T^*)$, we have that $|G_T(k,x,z)| \leq Tc|z|(|x|+1)$. Secondly, it is evident that the origin of (23) is uniformly globally exponentially stable for any a_1 and any $T \in (0, T^*)$ where $T^* > 0$ is such that $1 > a_1 T^* > 0$ and therefore the trajectories $\phi_T^z(\cdot,\cdot,\cdot)$ are uniformly summable, that is, it satisfies (16) with $\mu(s) = s$ and $\rho(s) \propto s$.

<u>*Proof of UGES of $x(k+1) = F_{1T}(k, x(k))$*</u>: Consider the function $V_T(k,x) := |x|^2 - \varepsilon \omega_{r_{k-1}} x_e y_e$ with $\varepsilon := \alpha_y + T$ and $\alpha_y > 0$. Observe that this function is positive definite and radially unbounded for sufficiently small α_y, T^* and w_M; indeed, we have that

$$c_1 |x|^2 \leq V_T(k,x) \leq c_2 |x|^2 \qquad (25)$$

with $c_1 := (1 - 0.5(\alpha_y + T^*)w_M)$ and $c_2 := (1 + 0.5(\alpha_y + T^*)w_M)$ which are clearly independent of T.

One can also show that there exists $K_1 > 0$ such that for all $T \in (0, T^*)$, $x \in \mathbb{R}^2$ and $k \geq 0$,

$$\frac{\Delta V_T}{T} \leq -(\alpha_x x_e^2 + \alpha_y \omega_{r_k}^2 y_e^2) + TK_1 |x|^2 \ . \qquad (26)$$

[3]With respect to the original one from [2].

Let us introduce the following auxiliary function:

$$W_T(k,x) := -T \sum_{i=k}^{\infty} e^{(k-i)T} \omega_{r_i} y_e^2$$

for which we claim the following (for the proof, see the Appendix).

Claim 1 *Suppose that the signal ω_{r_k} is PE. Suppose also that there exists $w_M > 0$ such that for all $i \geq 0$ we have $|\omega_{r_i}| \leq w_M$. Then, there exist strictly positive numbers $T^*, c_3, c_4, K_2, \tilde{\alpha}_y$ such that for all $T \in (0, T^*)$, $k \geq 0$ and $z_e \in \mathbb{R}^2$ we have*

$$-c_3 y_e^2 \ \leq \ W_T(k,x) \ \leq \ -c_4 y_e^2 \qquad (27)$$

$$\frac{\Delta W_T}{T} \ \leq \ \omega_{r_k}^2 y_e^2 - \tilde{\alpha}_y y_e^2 + K_2 x_e^2 \ . \qquad (28)$$

Let now T^* be generated by the claim above and such that (25) and (26) hold. Then, we can complete the proof by showing that there exists $\epsilon > 0$ and $\tilde{T} > 0$ such that $U_T(k,x) := V_T(k,x) + \epsilon W_T(k,x)$ is a Lyapunov function that proves UGES. Indeed, let $\epsilon := \min\left\{\frac{c_1}{2c_3}, \frac{\alpha_x}{2K_2}, \alpha_y\right\}$ and $\tilde{T} := \min\left\{T^*, \frac{1}{2K_1}\min\left\{\frac{\alpha_x}{2}, \epsilon\tilde{\alpha}_y\right\}\right\}$ where $\tilde{\alpha}_y$ comes from (28). Then it is easy to show that U_T satisfies

$$\frac{c_1}{2}|x|^2 \ \leq \ U_T(k,x) \ \leq c_2 |x|^2 \qquad (29)$$

$$\frac{\Delta U_T}{T} \ \leq \ -\tilde{c}_3 |x|^2 \ , \qquad (30)$$

where $\tilde{c}_3 = \frac{1}{2}\min\{\frac{\alpha_x}{2}, \epsilon\tilde{\alpha}_y\}$. This completes the proof invoking standard Lyapunov arguments.

<u>*Proof of UGB*</u>: We invoke Proposition 1. It is worth recalling to avoid confusion in the notation, that the state x in Proposition 1 corresponds here to with $x = \mathrm{col}[x_e, y_e]$ and the input z in Proposition 1 corresponds here to $z := \theta_e$, . Hence, we proceed to verify the conditions of the proposition with $f_T(k,x,z) := F_{1T}(k,x) + G_T(k,x,z)$ as defined in (22) and $V_T(k,x) = U_T(k,x)$. The bounds (12) and (14) hold from (29) and (30). The conditions (13) and (15) hold with $\varphi(s) = s$, $\tilde{\gamma}_2 = d_2 s$, and $\tilde{\gamma}_1(s) := d_3 s$, $d_2, d_3 > 0$. This is because $U_T(k,\cdot)$ is quadratic and $G_T(k,x,z)$ contains terms of linear growth in x for each fixed k and z and trigonometric functions, which can be over-bounded by a linear function of x. Also, $G_T(k,x,\cdot)$ can be over-bounded by a linear function for each fixed k and x. Finally, (16) holds with a linear function $\rho(s) := \rho\, s$ since $\mu(s)$ is in this case a non decreasing function of linear growth and $\phi_T^{\theta_e}(k)$ decays uniformly exponentially to zero.

Now we illustrate how we can use Proposition 2 to improve the performance of the system with the redesigned controller. Since the correction ϑ can be chosen arbitrarily, we can choose it so that negativity of

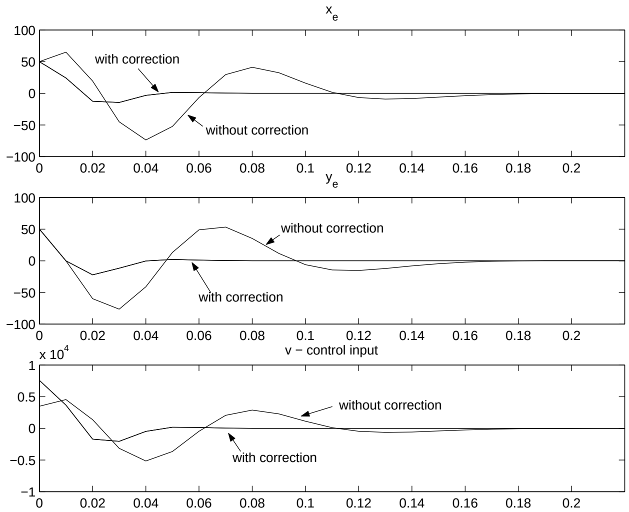

Figure 1: Tracking errors for x and y and the control input v given by (21), (31).

ΔV is enhanced, that is,

$$\vartheta(k,x) := \frac{(a_2^2 + \omega_{r_k}^2 - \varepsilon a_2)x_e - (2a_2\omega_{r_k} - \varepsilon\omega_{r_k}^3)y_e}{2(1-a_2T) + \varepsilon\omega_{r_k}^2 T} \tag{31}$$

with $\varepsilon = \alpha_y + T$. The simulations in Figure 1 show that the performance is considerably improved. We have simulated the system above in SIMULINK™ of MATLAB™ with $a_2 = 70$, $a_1 = 10$, $w_r(k) = 20\sin(kT)$, $T = 0.01$ and $\alpha_y = 2 - T$. We show only the responses for the states x_e and y_e as well as v since these are the only variables affected by the additional input $T\vartheta$. We show simulations for the system's response with $\vartheta = 0$ and with $\vartheta = 0.5[\,(a_2^2 + \omega_{r_k}^2 - \varepsilon a_2)x_e - (2a_2\omega_{r_k} - \varepsilon\omega_{r_k}^3)y_e\,]$. The best apparent performance is for the latter. It is also clear from the plots, that even though the correction ϑ is linear in the state and actually ($\vartheta \approx \mathcal{O}(1)\,|x|$), this correction is not comparable to "adding gain" to the control input. Notice that in this case the resulting control effort is actually smaller than in the case of the continuous-time based controller (i.e., when $\vartheta = 0$).

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