ON THE EQUIVALENCE TO FEEDFORWARD FORMS

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Keywords: Feedforward systems, invariant distributions, controlled invariant distributions, equivalence.

Abstract

The problem of (local) coordinates and feedback equivalence of single-input affine nonlinear systems to feedforward forms is studied and solved. The general theory is applied to linear systems, and is illustrated through a simple four dimensional example describing the dynamics of a food-chain.

1 Introduction

Following the seminal work [16] of Teel on stabilization of nonlinear systems, a new class of systems, denominated *feedforward systems* has attracted the attention of the nonlinear control community.

Feedforward systems are in general not feedback linearizable and occur naturally in the model of simple physical systems, *e.g.* the cart and pendulum system, the ball and beam (with friction). Therefore they have been regarded as an interesting class of truly nonlinear systems, for which methods such as feedback linearization or backstepping cannot be applied. Most of the attention of the researches has been devoted to the stabilization problem and several semiglobal and global stabilization results have been derived, either via full state feedback and via measurement feedback, see *e.g.* [5, 9, 16, 10, 8, 1, 6] and [14, 4] and the references therein.

Most of the aforementioned works start from the assumption that the system to be controlled is already in feedforward form. As a result little attention has been devoted to the problem of the intrinsic characterization of feedforward systems, *i.e.* to the problem of deciding when a given nonlinear system can be (locally or globally) transformed, via a coordinates or a feedback transformation, into a feedforward system.

Notable exceptions are the results in [16, Appendix 1] and [15, 13]. In [16, Appendix 1] the problem of feedback equivalence of a perturbed chain of integrators to a feedforward system is studied and some sufficient conditions are proposed. However, it must be noted that the (sufficient) conditions in [16, Appendix 1] rely on a special structure of the system to be transformed and on a special form for the transformed system. In particular it is required that the transformed system is controllable in the first approximation.

In this paper, we focus our attention on control affine systems, as the more general case of non-affine systems can be dealt with using the idea of dynamic extension, which has already been used in the framework of feedback linearization of non-affine systems, see *e.g.* [11, Theorem 6.12]. Also, as the definition of feedforward systems is not unique [5, 9, 16], we consider only the simplest possible description of feedforward systems. Moreover, we do not take into consideration controllability and detectability issues.

The present paper is organized as follows. In Section 2 we give the definition of the *feedforward forms* we are interested in and we state precisely the problems under investigation. Section 3 contains the main results of the paper, namely necessary and sufficient conditions to transform a single-input affine nonlinear control system into a feedforward system either via coordinates or via feedback transformations. Note that these necessary and sufficient conditions are only conceptual, *i.e.* they rely on the existence of solutions of certain partial differential equations, hence in practice they may not be easy to use. Our method complements the approach of [15], where the problem of *approximate* equivalence to feedforward forms has been addressed using the series expansions tools developed in [7]. In Section 4 we discuss the special case of linear systems. Finally, Sections 5 and 6 contain an illustrative example, concluding remarks and a discussion on open problems and future research directions.

2 Definitions, basic facts and problem formulation

As the notions of invariant distribution and of controlled invariant distribution will play a relevant role in the following developments we recall here their definition [3].

Definition 1 *A distribution Δ is said to be* invariant *under the vector field f if*

$$\tau \in \Delta \Rightarrow [f, \tau] \in \Delta,$$

i.e. *the Lie bracket of f with every vector field τ in Δ is again a vector field in Δ.*

Definition 2 *Consider a single-input system described by equations of the form*

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with $x \in U \subset I\!\!R^n$.
A distribution Δ is said to be controlled invariant *on U if there exists a feedback control law described by equations of the form $u = \alpha(x) + \beta(x)v$ such that*

$$[f + g\alpha, \Delta] \subset \Delta, \quad [g\beta, \Delta] \subset \Delta$$

for all x in U. A distribution Δ is said to be locally controlled invariant *if for each $x \in U$ there exists a neighborhood U^0 of x with the property that Δ is controlled invariant on U^0.*

The notion of local controlled invariance can be easily tested in geometric terms, as expressed in the following statement (see [3, Lemma 6.2.1]).

Lemma 1 *Let Δ be an involutive distribution. Suppose Δ and $\Delta + \mathrm{span}\{g(x)\}$ are non-singular on U. Then Δ is locally controlled invariant if and only if*

$$[f,\Delta] \subset \Delta + \mathrm{span}\{g(x)\},\ [g,\Delta] \subset \Delta + \mathrm{span}\{g(x)\}.$$

We now define two special classes of systems, referred to as *feedforward forms* and *strict feedforward forms*, respectively.

Definition 3 *The single-input affine nonlinear system (1), with state $x \in U \subset I\!R^n$, is said to be in* feedforward form *if the vector fields $f(x)$ and $g(x)$ are described by equations of the form*

$$f(x)= \begin{bmatrix} f_1(x_1,\cdots,x_n) \\ f_2(x_2,\cdots,x_n) \\ \vdots \\ f_n(x_n) \end{bmatrix} \quad g(x)= \begin{bmatrix} g_1(x_1,\cdots,x_n) \\ g_2(x_2,\cdots,x_n) \\ \vdots \\ g_n(x_n) \end{bmatrix}.$$

Definition 4 *The single-input affine nonlinear system (1), with state $x \in U \subset I\!R^n$, is said to be in* strict feedforward form *if the vector fields $f(x)$ and $g(x)$ are described by equations of the form*

$$f(x)= \begin{bmatrix} f_1(x_2,\cdots,x_n) \\ f_2(x_3,\cdots,x_n) \\ \vdots \\ 0 \end{bmatrix} \quad g(x)= \begin{bmatrix} g_1(x_2,\cdots,x_n) \\ g_2(x_3,\cdots,x_n) \\ \vdots \\ c \end{bmatrix},$$

with $c \in I\!R$. Note that, without loss of generality, we can assume $c = 1$.

We are now ready to give a precise formulation of the problem addressed in the paper.

Problem 1 (Coordinates equivalence problem) *Given a single-input nonlinear system described by equations of the form (1), with $x \in U \subset I\!R^n$, and a point $x^0 \in U$ find (if possible) a neighborhood U^0 of x^0 and a coordinates transformation $z = \Phi(x)$, defined on U^0, such that, in the new coordinates z, the system is in (strict) feedforward form.*

Problem 2 (Feedback equivalence problem) *Given a single-input nonlinear system described by equations of the form (1), with $x \in U \subset I\!R^n$, and a point $x^0 \in U$ find (if possible) a neighborhood U^0 of x^0, a coordinates transformation $z = \Phi(x)$, defined on U^0, and a state feedback control law*

$$u = \alpha(x) + \beta(x)v$$

such that, in the new coordinates z, the closed loop system

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v$$

is in (strict) feedforward form.

3 Main results

3.1 Coordinates equivalence

Proposition 1 *The system (1) is locally coordinates equivalent to a system in feedforward form if and only if there exists a nested sequence of distributions*

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{n-1}, \tag{2}$$

with Δ_i i-dimensional and involutive, such that

$$[f,\Delta_i] \subset \Delta_i \tag{3}$$

and

$$[g,\Delta_i] \subset \Delta_i, \tag{4}$$

for all $i = 1,\cdots,n-1$.

Remark 1 The above statement can be found, in possibly different forms, in [3, 11]. It is here presented to highlight the difference between the problems of coordinates and feedback equivalence to feedforward forms and of that to strict feedforward forms. Moreover, our proof (see also [6, Chapter 9]) is different from the proof provided therein.

Proof.(Only if) The proof is trivial, hence omitted.

(If) The sufficiency can be proven using iteratively the result in [11, Theorem 3.49] (see also the results in [3, Lemma 1.6.3, Proposition 1.7.2]). It can be shown that the original n-dimensional problem can be reduced to a $(n-1)$-dimensional problem using a coordinates transformation. Then, the $(n-1)$-dimensional problem has the same properties of the original one, hence we show that the reduction procedure can be applied $(n-1)$ times to obtain a system in feedforward form. ◁

The conditions in Proposition 1 can be strengthened to obtain a characterization of strict feedforward systems, as illustrated in the following statement.

Proposition 2 *The system (1) is locally coordinates equivalent to a system in strict feedforward form if and only if there exist a nested sequence of distributions as in expression (2), with Δ_i i-dimensional and involutive and n real functions $\lambda_1(x),\lambda_2(x),\cdots,\lambda_n(x)$, with $\lambda_i(x^0) = 0$ and $d\lambda_i(x^0) \neq 0$[1], such that*

$$\begin{array}{rcccl} \Delta_1^\perp & \not\ni & d\lambda_1 & \in & \mathbb{R}^n \\ \Delta_2^\perp & \not\ni & d\lambda_2 & \in & \Delta_1^\perp \\ & & \vdots & & \\ \Delta_{n-1}^\perp & \not\ni & d\lambda_{n-1} & \in & \Delta_{n-2}^\perp \\ 0 & \neq & d\lambda_n & \in & \Delta_{n-1}^\perp \end{array} \tag{5}$$

and

$$\begin{array}{rclcrcl} dL_f\lambda_1 & \in & \Delta_1^\perp & & dL_g\lambda_1 & \in & \Delta_1^\perp \\ \vdots & & & & \vdots & & \\ dL_f\lambda_{n-1} & \in & \Delta_{n-1}^\perp & & dL_g\lambda_{n-1} & \in & \Delta_{n-1}^\perp \\ dL_f\lambda_n & = & 0 & & dL_g\lambda_n & = & 0. \end{array} \tag{6}$$

Proof. (Only If) Assume there exists a local coordinates transformation $y = \Phi(x)$ such that the transformed system with state y is in strict feedforward form. Then, simple computations show that the distributions $\Delta_i = \mathrm{span}\{\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_i}\}$ are i-dimensional and involutive. Moreover, the functions $\lambda_1(y) = y_1,\ \lambda_2(y) = y_2,\ \cdots,\lambda_n(y) = y_n$, fulfil conditions (5) and (6) and $d\lambda_i(0) \neq 0$.

(If) The existence of a series of nested distributions that are involutive and non-singular implies the existence of a set of coordinates

$$z = \Phi(x), \tag{7}$$

in which the distributions are described by the expressions

$$\Delta_i(z) = \mathrm{span}\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\ \cdots, \frac{\partial}{\partial z_i}\right\}. \tag{8}$$

Next, consider n functions $\lambda_1(z),\lambda_2(z),\cdots,\lambda_n(z)$ such that conditions (5) and (6) holds. Because of the expression (8), condition (5) implies that each function $\lambda_i(z)$ will in fact be a function of $(z_i,z_{i+1},\cdots,z_n)$ with $\frac{\partial\lambda_i}{\partial z_i}$ not identically equal to zero at zero, for all i. For that, and because $d\lambda_i(0) \neq 0$,

[1] Without loss of generality, it will be assumed for the rest of the paper that $x^0 = 0$.

the functions $\lambda_1(z), \lambda_2(z), \cdots, \lambda_n(z)$ define a local diffeomorphism

$$y = \Lambda(z) = \begin{bmatrix} \lambda_1(z_1, z_2, \cdots, z_{n-1}, z_n) \\ \lambda_2(z_2, \cdots, z_{n-1}, z_n) \\ \vdots \\ \lambda_n(z_n) \end{bmatrix}. \tag{9}$$

We obtain

$$\dot{y} = \begin{bmatrix} L_{\tilde{f}}\lambda_1(z) \\ \vdots \\ L_{\tilde{f}}\lambda_n(z) \end{bmatrix}_{z=\Lambda^{-1}(y)} + \begin{bmatrix} L_{\tilde{g}}\lambda_1(z) \\ \vdots \\ L_{\tilde{g}}\lambda_n(z) \end{bmatrix}_{z=\Lambda^{-1}(y)} u.$$

Conditions (6), written in the z-coordinates, are

$$\begin{array}{ll} dL_{\tilde{f}}\lambda_i \in \left[\text{span}\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_i}\}\right]^{\perp} & \\ & i = 1, \cdots, n-1 \\ dL_{\tilde{g}}\lambda_i \in \left[\text{span}\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_i}\}\right]^{\perp} & \\ dL_{\tilde{f}}\lambda_n = 0, \ dL_{\tilde{g}}\lambda_n = 0, & \end{array}$$

The claim then follows from the interpretation of the relations above and from the triangular structure of the transformation (9). ◁

Remark 2 For a system to be coordinates equivalent to a strict feedforward form it, necessarily, has to be coordinates equivalent to a feedforward form. Indeed conditions (3) and (4), *i.e.* the invariance of the distributions under the vector fields f and g, are implied by the existence of the functions $\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x)$ with the properties (5) and (6). This can be easily verified considering the change of coordinates (9) and conditions (6), and keeping in mind that

$$\Delta_i(\omega) = \text{span}\left\{\frac{\partial}{\partial \omega_1}, \frac{\partial}{\partial \omega_2} \cdots, \frac{\partial}{\partial \omega_i}\right\} \text{ for } \omega = \{z, y\}.$$

Remark 3 To establish coordinates equivalence to a strict feedforward form, one might be tempted to consider the conditions

$$[f, \Delta_i] \subset \Delta_{i-1}, \quad [g, \Delta_i] \subset \Delta_{i-1}. \tag{10}$$

However, for a vector field satisfying condition (10) the following implications are true (as in the rest of the paper, we denote with $\tilde{g}$ the vector field expressed in the coordinates in which the set of distributions Δ_i are expressed by (8)). For all $i = 1, \cdots, n$ and all scalar functions $\mu(z)$ the vector field $\mu(z)\frac{\partial}{\partial z_i}$ is in Δ_i, whereas all vectors in Δ_{i-1} have zero entries in the positions $i, i+1, \cdots, n$. Then, by (10)

$$\frac{\partial \mu}{\partial z_i}\tilde{g} - \mu(z)\frac{\partial \tilde{g}}{\partial z_i} \subset \Delta_{i-1} \Rightarrow \frac{\partial \mu}{\partial z_i}\tilde{g}_i - \mu(z)\frac{\partial \tilde{g}_i}{\partial z_i} = 0.$$

But the only way that this is true for all functions $\mu(z)$ is if $\tilde{g}_i = 0$ for all $i = 1, \cdots, n$, *i.e.* $\tilde{g}(z) = g(x) = 0$. The same for the vector field f. Hence conditions (10) are not correct.

Remark 4 The solution of the coordinates equivalence problem relies upon the solution of a set of partial differential equations, namely the partial differential equations defining the distributions Δ_i. For example, the distribution Δ_1 can be computed solving the system of (linear) partial differential equations

$$\begin{array}{l} \frac{\partial f}{\partial x}\,\delta(x) - \frac{\partial \delta}{\partial x}\,f(x) = \alpha(x)\delta(x) \\ \frac{\partial g}{\partial x}\,\delta(x) - \frac{\partial \delta}{\partial x}\,g(x) = \beta(x)\delta(x) \end{array} \tag{11}$$

in the unknown $\alpha(x)$, $\beta(x)$ and $\delta(x) = \text{col}\{\delta_1(x), \delta_2(x) \cdots \delta_n(x)\}$. It is worth noting that the existence of a solution of the above system of partial differential equations strongly depends upon the choice of the functions $\alpha(x)$ and $\beta(x)$.

3.2 A necessary condition for strict feedforward forms

As explained in Remark 4, the applicability of the results in Propositions 1 and 2 depends on the solution of some nontrivial partial differential equations. On the other hand, it is known that a necessary and sufficient condition for a linear system to be coordinates equivalent to a system in strict feedforward form is that its eigenvalues are equal to zero. Derivation of similar, easy-to-check necessary conditions for nonlinear systems would prove very useful. In this section we present a necessary condition for the coordinates equivalence of a nonlinear system (1) to a system in strict feedforward form. For simplicity, we restrict ourselves to the first step of the procedure *i.e.* we are looking for equivalence to the form

$$\dot{y} = \begin{bmatrix} \hat{f}_1(y_2, \cdots, y_n) \\ \hat{f}_2(y_2, \cdots, y_n) \\ \vdots \\ \hat{f}_n(y_2, \cdots, y_n) \end{bmatrix} + \begin{bmatrix} \hat{g}_1(y_2, \cdots, y_n) \\ \hat{g}_2(y_2, \cdots, y_n) \\ \vdots \\ \hat{g}_n(y_2, \cdots, y_n) \end{bmatrix} u. \tag{12}$$

Consider the one-dimensional involutive distribution $\Delta_1 = \text{span}\{\tau_1\}$. According to Proposition 2 (see also the explanation in [6, Remark 9.1]), for a system that is coordinates equivalent to a *at-the-first-step* strict feedforward form, *i.e.* to a system of the form (12), there exist a one-dimensional distribution Δ_1 and n real functions $\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x)$ such that the following hold.

1. Conditions (5): $\Delta_1^{\perp} \not\ni d\lambda_1 \in \mathbb{R}^n$ and $d\lambda_i \in \Delta_1^{\perp}$, for $i = 2, \cdots, n$, *i.e.*
$$< d\lambda_1, \tau_1 > \neq 0 \text{ and } < d\lambda_i, \tau_1 >= 0. \tag{13}$$
2. Condition (6): $dL_f\lambda_i \in \Delta_1^{\perp}$ for $i = 2, \cdots, n$, *i.e.* (see [11]),
$$\begin{array}{c} < d\left(< d\lambda_i, f >\right), \tau_1 >= 0 \iff \\ d\lambda_i \frac{\partial f}{\partial x}\tau_1 + f'\frac{\partial^2 \lambda_i}{\partial x^2}\tau_1 = 0. \end{array} \tag{14}$$

As mentioned in Remark 2, the distribution Δ_1 is invariant under the vector field f (and the same holds for g), *i.e.* there exists a real function $k(x)$ such that

$$[f, \tau_1] = \frac{\partial \tau_1}{\partial x}f - \frac{\partial f}{\partial x}\tau_1 = k(x)\tau_1. \tag{15}$$

Multiplying both sides of equation (15) by the nonsingular matrix

$$M = \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \\ \vdots \\ d\lambda_n \end{bmatrix} \tag{16}$$

and using (13) and (14) we obtain a matrix equation, the first row of which rewrites as

$$d\lambda_1\frac{\partial \tau_1}{\partial x}f + \tau_1'\frac{\partial^2 \lambda_1}{\partial x^2}f = k(x)d\lambda_1\tau_1,$$

i.e.

$$\frac{\partial}{\partial x}(L_{\tau_1}\lambda_1)f = k(x)L_{\tau_1}\lambda_1 \tag{17}$$

while the rest $n-1$ rows are automatically satisfied.
Moreover, by equation (13) $L_{\tau_1}\lambda_1 \neq 0$, hence $k(0) = 0$. A necessary condition can be obtained by the conditions (15) and (17).

Corollary 1 *If system (1) is coordinates equivalent to a system which is at-the-first-step in a strict feedforward form, then there exist a vector field $\tau_1 \neq 0$ and functions $\lambda_1(x)$ and $k(x)$ with $d\lambda_1 \neq 0$ and $k(0) = 0$ such that equations (15) and (17) hold.*

3.3 Feedback equivalence

In this section we show how the conditions expressed in Propositions 1 and 2 have to be modified in order to solve the feedback equivalence problem, *i.e.* Problem 2.

Proposition 3 *Consider system (1). Let*

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{n-1}, \tag{18}$$

be a nested sequence of distributions with Δ_i i-dimensional and involutive, and such that the distributions $\Delta_i + \mathrm{span}\{g(x)\}$ are non-singular.
Then the system (1) is locally feedback equivalent to a system in feedforward form if and only if

$$[f, \Delta_i] \subset \mathrm{span}\{g\} + \Delta_i \textit{ and } [g, \Delta_i] \subset \mathrm{span}\{g\} + \Delta_i, \tag{19}$$

for all $i = 1, \cdots, n-1$.

Proof. (Only if) The proof of the necessity is straightforward, hence it is omitted.
(If) We prove the sufficiency in a constructive, iterative way, as for Proposition 1. By equations (19) the distribution Δ_1, is a controlled invariant distribution for system (1). Hence, by [3, Lemma 6.2.1], there exists a local change of coordinates $z = \Phi(x)$ and a feedback control law $u = \alpha(x) + \beta(x)v$ such that, in the new coordinates, the closed loop system is described by equations of the form $\dot{z} = \tilde{f}(z) + \tilde{g}(z)v$, with

$$\tilde{f}(z) = \begin{bmatrix} \tilde{f}_1(z_1, \cdots, z_n) \\ \tilde{f}_2(z_2, \cdots, z_n) \\ \vdots \\ \tilde{f}_n(z_2, \cdots, z_n) \end{bmatrix} \quad \tilde{g}(z) = \begin{bmatrix} \tilde{g}_1(z_1, \cdots, z_n) \\ \tilde{g}_2(z_2, \cdots, z_n) \\ \vdots \\ \tilde{g}_n(z_2, \cdots, z_n) \end{bmatrix}.$$

Moreover, in the new coordinates one has $\Delta_1 = \mathrm{span}\{\frac{\partial}{\partial z_1}\}$. Repeating the same arguments used in the proof of Proposition 1 we conclude that the original n-dimensional problem of feedback equivalence is now reduced to a $(n-1)$-dimensional problem. This reduced problem has the same properties of the original one, hence further reduction can be performed. It is worth noting that the feedback transformation used at step i does not depend upon the first $(i-1)$ coordinates, hence the change of feedback used at step i does not change the structure of the first $i-1$ equations of the system. ◁

Remark 5 The conditions for feedback equivalence are obviously weaker then the conditions for coordinates equivalence. In particular, the partial differential equations defining Δ_1 are (in the case of feedback equivalence)

$$\begin{aligned} \frac{\partial f}{\partial x}\,\delta(x) - \frac{\partial \delta}{\partial x}\,f(x) = \alpha(x)\delta(x) + \gamma(x)g(x) \\ \frac{\partial g}{\partial x}\,\delta(x) - \frac{\partial \delta}{\partial x}\,g(x) = \beta(x)\delta(x) + \eta(x)g(x), \end{aligned} \tag{20}$$

with the unknown $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\eta(x)$ and $\delta(x) = \mathrm{col}\{\delta_1(x), \delta_2(x) \cdots \delta_n(x)\}$. This partial differential equations are (in principle) easier to solve then the corresponding equations (11) because of the presence of two more free *parameters*, namely $\gamma(x)$ and $\eta(x)$.

Remark 6 It is worth noting that the results concerning the feedback equivalence are not as elegant as the ones on coordinates equivalence. This is because in the proof of the sufficiency of the feedback equivalence we need the technical assumption that the distributions $\Delta_i + \mathrm{span}\{g(x)\}$ be non-singular. This assumption is required to construct the feedback laws inducing the invariance of the distributions Δ_i (in the closed loop system), but it is not necessary. As a matter of fact the regularity of the above distributions is not needed in the proof of the necessity. This fact should not be surprising and is a direct consequence of the small gap existing in the characterization of locally controlled invariant distributions, as observed in [3, Remark 6.2.1].

To address the problem of feedback equivalence to strict feedforward forms, we present first the following preliminary result.

Proposition 4 *The system (1) is locally feedback equivalent to a system in strict feedforward form if and only if there exists a coordinates transformation $z = \Phi(x)$ such that system (1) written in the z-coordinates is described by equations of the form*

$$\dot{z} = b(z)(c(z) + e(z)u) + h(z), \tag{21}$$

where $c(z)$ and $e(z)$ are scalar functions, with $e(0) \neq 0$, and $b(z)$ and $h(z)$ are vector fields in strict feedforward form, i.e.,

$$b(z) = \begin{bmatrix} b_1(z_2, \cdots, z_n) \\ \vdots \\ b_{n-1}(z_n) \\ c_d \end{bmatrix}, \quad h(z) = \begin{bmatrix} h_1(z_2, \cdots, z_n) \\ \vdots \\ h_{n-1}(z_n) \\ c_h \end{bmatrix}.$$

Proof. (Only if) Suppose that system (1) is locally feedback equivalent to a system in strict feedforward form. Then there exists a feedback $u(x) = \alpha(x) + \beta(x)v$, with $\beta(x) \neq 0$ and a coordinates transformation $z = \Phi(x)$ such that the system $\dot{z} = \tilde{f}_f(z) + \tilde{g}_f(z)v$ is in strict feedforward form, with $f_f(x) = f(x) + g(x)\alpha(x)$, $g_f(x) = g(x)\beta(x)$, $\tilde{f}_f(z) = \frac{\partial \Phi}{\partial x} f_f(x)\big|_{x=\Phi^{-1}(z)}$ and $\tilde{g}_f(z) = \frac{\partial \Phi}{\partial x} g_f(x)\big|_{x=\Phi^{-1}(z)}$.
Applying the transformation $z = \Phi(x)$ to the original system (1), one gets

$$\dot{z} = \tilde{f}(z) + \tilde{g}(z)u \tag{22}$$

which is necessarily transformed into a strict feedforward form by the feedback

$$u(z) = u(x)|_{x=\Phi^{-1}(z)} = \tilde{\alpha}(z) + \tilde{\beta}(z)v.$$

This implies that for the vector field $\tilde{g}_f$ one has

$$\begin{aligned} \tilde{g_f}(z) &= \tilde{g}(z)\tilde{\beta}(z) \\ &\Updownarrow \\ \tilde{g_f}_1(z_2, \cdots, z_n) &= \tilde{g}_1(z)\tilde{\beta}(z) \\ &\vdots \\ \tilde{g_f}_{n-1}(z_n) &= \tilde{g}_{n-1}(z)\tilde{\beta}(z) \\ c_g &= \tilde{g}_n(z)\tilde{\beta}(z), \end{aligned} \tag{23}$$

with $c_g \neq 0$. Similar computations can be drawn for the vector field $\tilde{f}_f$ and thus, it becomes clear that $\tilde{\alpha}(z)$ and $\tilde{\beta}(z)$ are such that

$$\begin{aligned} \tilde{\beta}(z) &= \frac{c_g}{\tilde{g}_n(z)} = \frac{\tilde{g_f}_{n-1}(z_n)}{\tilde{g}_{n-1}(z)} = \cdots = \frac{\tilde{g_f}_1(z_2, \cdots, z_n)}{\tilde{g}_1(z)} \\ \tilde{\alpha}(z) &= \frac{c_f - \tilde{f}_n(z)}{\tilde{g}_n(z)} = \cdots = \frac{\tilde{f_f}_1(z_2, \cdots, z_n) - \tilde{f}_1(z)}{\tilde{g}_1(z)}. \end{aligned}$$

From this, with tedious but straighforward computations (see [6]), it follows that the vector fields $\tilde{g}(z)$ and $\tilde{f}(z)$ must possess a special structure, namely,

$$\tilde{g}(z) = \psi_g(z)\tilde{g}_n(z) \text{ and } \tilde{f}(z) = \psi_f(z) + \psi_g(z)\tilde{f}_n(z), \quad (24)$$

where $\psi_g(z)$ and $\psi_f(z)$ are vector fields in strict feedforward form.

From equations (24), we conclude that system (22) can be written in the form of equation (21) with $c(z) = \tilde{f}_n(z)$, $e(z) = \tilde{g}_n(z)$, $b(z) = \psi_g(z)$ and $h(z) = \psi_f(z)$, *i.e.* the vector fields $\tilde{f}(z)$ and $\tilde{g}(z)$ are described by

$$\tilde{f}(z) = h(z) + b(z)\tilde{f}_n(z),\ \tilde{g}(z) = b(z)\tilde{g}_n(z),$$

which proves the claim.

(If) Consider a system described by (21). Applying the feedback

$$u(z,v) = -\frac{c(z)}{e(z)} + \frac{1}{e(z)}v$$

we get

$$\dot{z} = h(z) + b(z)v$$

which is in strict feedforward form and feedback equivalent to system (1). $\triangleleft$

Proposition 4 gives necessary and sufficient conditions for feedback equivalence to strict feedforward forms, however, these conditions are not phrased in the geometric framework that has been used for the results presented so far. We overcome this shortcoming with the next result.

Proposition 5 *The system (1) is locally feedback equivalent to a system in a strict feedforward form, if and only if the following hold.*

1. *There exists a nested sequence of distributions (18) with Δ_i i-dimensional and involutive, and such that the distributions*
$$\Delta_i + \mathrm{span}\{g(x)\}$$
are non-singular.

2. *There exist n real functions $\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x)$ with $d\lambda_i(0) \neq 0$, such that (5) holds, $L_g\lambda_n \neq 0$ and moreover*

$$\begin{array}{c} dL_f\lambda_1 - \frac{L_g\lambda_1}{L_g\lambda_n} dL_f\lambda_n \in \Delta_1^\perp \\ dL_f\lambda_2 - \frac{L_g\lambda_2}{L_g\lambda_n} dL_f\lambda_n \in \Delta_2^\perp \\ \vdots \\ dL_f\lambda_{n-1} - \frac{L_g\lambda_{n-1}}{L_g\lambda_n} dL_f\lambda_n \in \Delta_{n-1}^\perp \\ dL_g\lambda_1 - \frac{L_g\lambda_1}{L_g\lambda_n} dL_g\lambda_n \in \Delta_1^\perp \\ dL_g\lambda_2 - \frac{L_g\lambda_2}{L_g\lambda_n} dL_g\lambda_n \in \Delta_2^\perp \\ \vdots \\ dL_g\lambda_{n-1} - \frac{L_g\lambda_{n-1}}{L_g\lambda_n} dL_g\lambda_n \in \Delta_{n-1}^\perp . \end{array} \quad (25)$$

Proof. (Only if) This is trivially proven following the steps in the proof of Proposition 4.

(If) Consider system (1) and the change of coordinates

$$z = \Lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \\ \vdots \\ \lambda_n(x) \end{bmatrix},$$

where the functions λ_i are such that condition (5) holds, with the nested sequence of distributions of expression (18). Writing the system equations in the z-coordinates and applying the feedback

$$u = -\frac{L_f\lambda_n}{L_g\lambda_n} + \frac{1}{L_g\lambda_n}v. \quad (26)$$

we obtain

$$\dot{z} = \begin{bmatrix} L_f\lambda_1 - \frac{L_f\lambda_n}{L_g\lambda_n}L_g\lambda_1 \\ L_f\lambda_2 - \frac{L_f\lambda_n}{L_g\lambda_n}L_g\lambda_2 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{L_g\lambda_1}{L_g\lambda_n} \\ \frac{L_g\lambda_2}{L_g\lambda_n} \\ \vdots \\ 1 \end{bmatrix} v \stackrel{\triangle}{=} f^\star + g^\star v.$$

It can be easily checked that the vector fields $f^\star$ and $g^\star$ are coordinates equivalent to strict feedforward form. $\triangleleft$

4 Linear systems

Consider a linear single-input system described by equations of the form

$$\dot{x} = Ax + bu. \quad (27)$$

For such a system Problems 1 and 2 can be easily solved, as shown in the following (simple) statements.

Corollary 2 *The system (27) is coordinates equivalent to a feedforward form if and only if all the eigenvalues of the matrix A are real.*

Proof. The claim can be easily proved using standard linear algebra. However, it is interesting to observe that, under the stated assumptions the partial differential equations used to define the distributions Δ_i can be trivially solved. For example, the partial differential equations for Δ_1 reduces to

$$A\,\delta(x) - \frac{\partial\delta(x)}{\partial x}\,Ax = \alpha(x)\delta(x)$$

$$-\frac{\partial\delta(x)}{\partial x}\,B = \beta(x)\delta(x),$$

with $\delta(x) = \mathrm{col}(\delta_1(x), \delta_2(x), \cdots, \delta_n(x))$. These equations admit the obvious solution $\alpha(x) = \lambda_i$, $\beta(x) = 0$ and $\delta(x) = v_i$, with λ_i and v_i such that $Av_i = \lambda_i v_i$. $\triangleleft$

Corollary 3 *The system (27) is coordinates equivalent to a strict feedforward form if and only if all the eigenvalues of the matrix A are equal to 0.*

Corollary 4 *Any linear controllable system is feedback equivalent to a strict feedforward form.*

5 Examples

The theory presented in Section 3 is illustrated through an example. Consider a four dimensional model of a food-chain system [2, 12], *i.e.*

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_1x_2 \\ \dot{x}_2 & = & -x_1x_2 - x_2 + x_2x_3 \\ \dot{x}_3 & = & -x_2x_3 - x_3 + x_3x_4 \\ \dot{x}_4 & = & -x_3x_4 - x_4 + u. \end{array} \quad (28)$$

This system describes the behavior of a (normalized) four species ecologies, in which the species described by x_2 and x_3 act as preys and predators, x_1 acts as predator and x_4 acts as prey. The species described by x_4 is *fed* by the environment through the input signal u. Obviously, the system is defined in the (open) positive orthant $\mathbb{R}^4_+$, which is a positive invariant set for all trajectories as long as $u > 0$.

The qualitative behavior of system (28) has been extensively studied in the biological and game theory communities, whereas a few control problems have been discussed in [12]. System (28) is not in feedforward form, and there is no obvious change of feedback and/or change of coordinates that transforms the system into a feedforward form. Nevertheless, the system can be (locally) transformed into a feedforward form, as detailed in the following statement.

Proposition 6 *System (28) is locally feedback equivalent to a feedforward form around any point* $x_0 \in I\!R^4_+$.

Proof. Simple but tedious computations show that the distribution

$$\Delta_1 = \mathrm{span}\left\{ \begin{bmatrix} x_3 \\ 0 \\ x_3 \\ 2x_2 - x_4 \end{bmatrix} \right\}$$

is a controlled invariant distributions for system (28). Hence, following the construction in [3, Lemma 1.6.1] we define (locally) new coordinates $z_1 = \phi_1(x)$, $z_2 = \phi_2(x)$, $z_3 = \phi_3(x)$ and $z_4 = \phi_4(x)$ such that

$$\mathrm{span}\{d\phi_2, d\phi_3, d\phi_4\} = \Delta_1^\perp.$$

A possible choice for such new coordinates is

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 - x_2 - x_1 \\ x_3(x_4 - 2x_2) \end{bmatrix}.$$

The above coordinates transformation is a diffeomorphism on $I\!R^4_+$. Written in the z-coordinates, the system is brought to a feedforward form with a re-definition of the control u. ◁

6 Conclusions and outlook

The problems of local coordinates and feedback equivalence of a single-input nonlinear system to a class of feedforward systems have been studied and a geometric characterization of the problems has been proposed. The investigation has been restricted to a special subclass, however, similar conditions can be given for more general classes of systems and in particular for the so-called block feedforward systems.
The solution of the global problem requires existence of global (controlled) invariant distributions and completeness of the vector fields in the distributions. On the other hand, no major differences are to be expected in dealing with multi-input systems.

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