FORCED OSCILLATIONS IN FIRST ORDER SYSTEMS

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Abstract

Forced oscillations is a phenomenon in which a nonlinear closed loop system is forced to operate at the same frequency as the external forcing sinusoid. This paper derives the necessary and sufficient conditions for which forced oscillations can be achieved when a first order linear system is placed in a closed loop relay feedback configuration. It is shown that a fundamental requirement for forced oscillations is the existence of a pair of switch points which are separated by exactly half the period of the external forcing signal. The conditions of forced oscillations for this class of plants are derived by ensuring that this basic requirement is satisfied. The results completely characterise the forced oscillations conditions for all first order plants with dead times. Simulation studies are given to illustrate these results.

1 Introduction

Relays have become common tools for the auto-tuning of PI or PID controllers since the initial work of Astrom in [1]. They are mainly used to identify points on the process Nyquist curve, from which essential information for tuning such controllers can be extracted. While the use of relays in auto-tuning have been shown to be successful for single loop systems, there are still many unanswered questions in multi-loop systems.

One of these pertains to the existence of different limit cycling patterns that are possible in a multi-loop system where the interactions between sub-systems are significant. In cases where the oscillation patterns are simple, extensions from the single to multi-loop systems are straightforward. In other complex cases, more investigations have to be carried out in order to arrive at a feasible design. In [2], multi-input multi-output (MIMO) systems were classified according to the different modes of oscillations that they exhibit when placed on closed loop relay control where all relay amplitudes are the same. Some design procedures were also given to handle the different types of MIMO systems classified according to this approach.

In many MIMO auto-tuning designs (see [3]), the existence of different modes is either ignored or some adaptive means is provided to find a suitable point where tuning can proceed, [4]. The adaptive tuning approach is novel, interesting but time consuming. A non-adaptive procedure is given in [5] to set the relay amplitudes in such a way that all signals in the loops have the same frequency (also referred to as Mode 1 behaviour). This result was based on the observation that Mode 1 behaviours can be viewed as forced osillations occuring in one or more loops. By understanding the conditions for forced oscillations, relay amplitudes which lead to Mode 1 behaviours can be prescribed, without any adaptation.

In this paper, an extension of [5], with more complete results for the existence of forced oscillations in single loop relay feedback systems, is given and analysed. In particular, necessary and sufficient conditions for such an existence are given when the linear dynamic model in the relay feedback system is a first order plus deadtime (FOPDT) model. This class of plants is of special interest because many industrial processes can be approximated by the FOPDT model.

This paper is organised as follows. Section 2 gives the general switching equations for a FOPDT plant in a relay feedback system driven by an external sinusoidal forcing signal at the output. In Section 3, the fundamental requirement to achieve forced oscillations is proposed. This is followed by Section 4 where the minimum amplitude of the forcing signal required to enforce forced oscillations is solved. Simulation studies are given in Section 5. Finally, we conclude the paper in Section 6.

2 General Switching Equations

In the configuration of Figure 1, the nonlinear element is an ideal relay and $g(s)$ is a FOPDT model. For a given ω_f and $R > 0$ of the external forcing signal,

$$f(t) = R \sin \omega_f t, \tag{1}$$

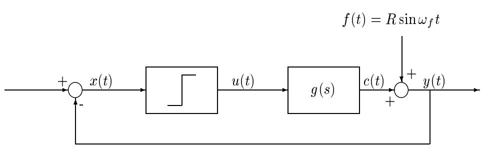

Figure 1: Forced oscillation configuration for SISO system

the steady state signals of $c(t)$, and hence $y(t)$, are periodic with frequency not necessarily equal to ω_f. If $c(t)$ indeed has a frequency ω_f, then forced oscillation is said to have taken place.

In any case, $u(t)$ will be a signal, switching between $+d$ and $-d$ depending on $y(t)$. $c(t)$ is then the response of $g(s)$ to $u(t)$. The switching conditions satisfy

$$y(t_i) = c(t_i) + f(t_i) = 0 \tag{2}$$

at any switching instant, $t = t_i$, assumed unknown. t_i is defined to be a switching instant if $u(t_i)$ is discontinuous at time $t = t_i$.

Under the most general conditions of the forcing signal, $f(t)$ and $x(0^+) < 0$, switching is assumed to take place and, for $d = 1$, $u(t)$ is a sequence of unit rectangular pulses that changes when $x(t)$ changes sign. Hence

$$u(t) = -[u_s(t) + 2\sum_{k=1}^{\infty}(-1)^k u_s(t - t_k)] \tag{3}$$

where $u_s(t - t_k)$ denotes the delayed unit step function. Assuming zero initial conditions, $c(t)$ in response to $u(t)$ in (3) can be written as :

$$c(t) = -[h(t) + 2\sum_{k=1}^{\infty}(-1)^k h(t - t_k)] \tag{4}$$

where $h(t)$ is the unit step response of $g(s)$. In expanded form (4) may be written as

$$c(t) = \begin{cases} -h(t) & t \in [0, t_1) \\ -h(t) + 2h(t - t_1) & t \in [t_1, t_2) \\ \vdots & \vdots \end{cases} \tag{5}$$

(5) determines both the transient and steady state processes in the configuration of Figure 1 for a general linear system, $g(s)$, in the relay feedback loop. From here on, $g(s)$ is assumed to have the following transfer function

$$g(s) = \frac{Ke^{-Ls}}{Ts + 1} \tag{6}$$

where K is the static gain, T is the time constant and L is the time delay. These equations will be used in the following sections to show several results on forced oscillations. We show that

(a) If $t_{i_0+1} = t_{i_0} + T_f/2$ for some $i_0 >> 0$ and $R > 0$, then steady state switching or limit cycling occurs with a frequency of $\omega_f = 2\pi/T_f$ and periodic oscillations with frequency ω_f are established after $t = t_{i_0}$.

(b) The necessary and sufficient conditions for forced oscillations to occur in the class of plants in (6) are presented.

(c) For forced oscillations to occur at a frequency of ω_f, we require $R \geq |c(t_{i_0})|$ where $c(t_{i_0})$ is the value of $c(t)$ at the switching instant, t_{i_0}, during steady state. For some plants, depending on the L/T ratio, this minimum R is not sufficient to enforce forced oscillations.

3 Existence of Forced Oscillations

Forced oscillation due to an external forcing signal, $f(t)$, is a well known phenomenon which happens when R is sufficiently large for a given ω_f. However, if R does not exceed a certain threshold for a given ω_f, then forced oscillation cannot take place but instead, self-oscillations occurs as if $f(t) = 0$. This is well documented in [6] and [7]. The following proposition gives an ideal condition for which forced oscillations will take place when $L < T_f/2$. The analysis for $L > T_f/2$ is similar except that $c(t)$ is slightly different due to the relationship between L and switching instants t_i.

Proposition 1. *In the configuration of Figure 1, for a plant in (6) with $f(t) = R\sin\omega_f t$, where R is sufficiently large, forced oscillations with frequency $\omega_f < \pi/L$ exists with steady state periodic switching of the relay at frequency ω_f if there exists two switch points, t_{i_0+1} and t_{i_0} which satisfies*

$$t_{i_0+1} = t_{i_0} + T_f/2$$

for some $i_0 >> 1$. Furthermore, $c(t) = c(t + T_f)$ for all $t > t_{i_0}$.

Proof. $h(t)$ of a FOPDT system is given by

$$h(t) = K(1 - e^{-\frac{(t-L)}{T}})$$

assuming zero initial conditions. For $L < T_f/2$, $c(t)$ in (5) becomes

$$c(t) = \begin{cases} 0 & t \in [0, L) \\ -Z_t^0 & t \in [L, t_1 + L) \\ \vdots & \vdots \\ -[Z_t^0 + 2Z_t^{i-1}] & t \in [t_{i-1} + L, t_i + L) \end{cases} \tag{7}$$

$$Z_t^0 \triangleq h(t)$$

$$Z_t^{i-1} \triangleq \sum_{k=1}^{i-1}(-1)^k h(t - t_k) \tag{8}$$

Suppose two pairs of switches occur such that

$$t_{i_0+1} = t_{i_0} + \frac{T_f}{2} \tag{9}$$

$$t_{i_0+2} = t_{i_0+1} + \frac{T_u}{2} \tag{10}$$

where T_u is unknown. Then the switching condition, $y(t_{i_0}) = y(t_{i_0+1}) = y(t_{i_0+2}) = 0$ implies that

$$c(t_{i_0}) + R\sin(\omega_f t_{i_0}) = 0. \tag{11}$$

$$c(t_{i_0+1}) - R\sin(\omega_f t_{i_0}) = 0 \tag{12}$$

$$c(t_{i_0+2}) - R\sin(\omega_f(t_{i_0} + \frac{T_u}{2})) = 0 \tag{13}$$

and $c(t_{i_0+1}) = -c(t_{i_0})$ which, for $i_0 >> 1$, leads to

$$Z_{t_{i_0+1}}^{i_0-1} + Z_{t_{i_0}}^{i_0-1} - h(T_f/2) + C_0 = 0$$

where $Z_{t_{i_0+1}}^{i_0-1} = e^{-T_f/2T} Z_{t_{i_0}}^{i_0-1}$ and C_0 is the steady state value of $h(t)$. For i_0 odd, it follows that

$$Z_{t_{i_0}}^{i_0-1} = -\frac{Ke^{\frac{L}{T}}}{1+e^{\frac{T_f}{2T}}} \tag{14}$$

$$Z_{t_{i_0+1}}^{i_0-1} = -\frac{Ke^{\frac{L}{T}}e^{-\frac{T_f}{2T}}}{1+e^{\frac{T_f}{2T}}} \tag{15}$$

$$Z_{t_{i_0+2}}^{i_0-1} = -\frac{Ke^{\frac{L}{T}}e^{-\frac{T_f+T_u}{2T}}}{1+e^{\frac{T_f}{2T}}}. \tag{16}$$

With (14)-(16), $c(t_{i_0})$, $c(t_{i_0+1})$ and $c(t_{i_0+2})$ become

$$c(t_{i_0}) = -K + \frac{2Ke^{L/T}}{1+e^{T_f/2T}} \tag{17}$$

$$c(t_{i_0+1}) = K - \frac{2Ke^{L/T}}{1+e^{T_f/2T}} \tag{18}$$

$$c(t_{i_0+2}) = -K + \frac{2Ke^{L/T}e^{-T_u/2T}e^{T_f/2T}}{1+e^{T_f/2T}} \tag{19}$$

since $C_0 = K$ for the plant in (6). Hence, $c(t_{i_0})$, $c(t_{i_0+1})$ $c(t_{i_0+2})$ are dependent on the plant parameters as well as the switching intervals, T_f and T_u. Substituting (14) and (16) into (13), we arrive at

$$\begin{aligned} K \quad - \quad & 2Ke^{T_f/2T}e^{-T_u/2T}\frac{e^{L/T}}{1+e^{T_f/2T}} + \\ & \cos(\omega_f T_u/2)\left(K - 2K\frac{e^{L/T}}{1+e^{T_f/2T}}\right) + \\ & R\sin(\omega_f T_u/2)\cos\omega_f t_{i_0} = 0. \end{aligned} \tag{20}$$

Since (20) is satisfied for any arbitrarily large R and t_{i_0}, we now show that $T_u = T_f$. Suppose (20) is also satisfied for $R' > R$ with the corresponding switching instants occuring at $t_{i_0'}$ instead of t_{i_0}. Substituting these into (20), and subtracting we arrive at

$$\sin\omega_f T_u/2(R\cos\omega_f t_{i_0} - R'\cos\omega_f t_{i_0'}) = 0. \tag{21}$$

From (11), we have

$$\sin\omega_f t_{i_0} = -\frac{c(t_{i_0})}{R}, \quad \sin\omega_f t_{i_0'} = -\frac{c(t_{i_0'})}{R'}.$$

Thus, rewriting $c(t_{i_0}) = C$,

$$\omega_f t_{i_0} = \begin{cases} m\pi + \sin^{-1}|C|/R \text{ or } k\pi - \sin^{-1}|C|/R & C < 0 \\ m\pi - \sin^{-1}C/R \text{ or } k\pi + \sin^{-1}C/R & C > 0 \end{cases}$$

where m is even and k is odd. The same applies to the (.') quantities. As $c(t_{i_0})$ is only dependent on the plant parameters, $c(t_{i_0}) = c(t_{i_0'})$. It follows that, for any $c(t_{i_0})$,

$$\begin{aligned} R\cos\omega_f t_{i_0} &= \pm\sqrt{R^2 - c^2(t_{i_0})} \\ R'\cos\omega_f t_{i_0'} &= \pm\sqrt{R'^2 - c^2(t_{i_0})} \end{aligned}$$

Substituting into (21), we get

$$\sin\omega_f T_u/2\left[\pm\sqrt{R^2 - c^2(t_{i_0})} \mp \sqrt{R'^2 - c^2(t_{i_0})}\right] = 0$$

Since $R' > R$, the term is square bracket is non zero, and it follows that $\sin\omega_f T_u/2 = 0$ which implies that $T_u = \nu T_f$ where ν is any integer. We next show that only $\nu = 1$ is possible. Substituting $T_u = \nu T_f$ into (20), we have

$$\begin{aligned} & K - 2Ke^{T_f/2T}e^{-\nu T_f/2T}\frac{e^{L/T}}{1+e^{T_f/2T}} + \\ & \cos(\omega_f \nu T_f/2)\left(K - 2K\frac{e^{L/T}}{1+e^{T_f/2T}}\right) = 0. \end{aligned} \tag{22}$$

Since (22) only holds for $\nu = 1$, we conclude that $T_u = T_f$.

This result proves that if R is sufficiently large and two switching instants satisfying (9) exist, then subsequent switchings also satisfy (9). This establishes the steady state switching at a frequency of ω_f.

We next show that $c(t)$ is indeed periodic by proving that $c(t) = c(t + T_f)$ for all $t > t_{i_0}$, $i_0 >> 1$. It follows from (7) that

$$c(t_{i_0} + \triangle t) = \begin{cases} -(Z_{t_{i_0}+\triangle t}^0 + 2Z_{t_{i_0}+\triangle t}^{i_0-1}) & \triangle t \in [0, L] \\ -(Z_{t_{i_0}+\triangle t}^0 + 2Z_{t_{i_0}+\triangle t}^{i_0}) & \triangle t \in [L, T_f/2] \end{cases}$$

Since $Z_{t_{i_0}+\triangle t}^{i_0} = Z_{t_{i_0}+\triangle t}^{i_0-1} + (-1)^{i_0}h(t - t_{i_0} - \triangle t)$, we have

$$\begin{aligned} c(t_{i_0} \quad + \quad & \triangle t) = c(t_{i_0+2} + \triangle t) \\ = \quad & \begin{cases} -K + 2K\frac{e^{L/T}}{1+e^{T_f/2T}}e^{-\triangle t/T} & \triangle t \in [0, L] \\ K - 2K\frac{e^{L/T}e^{T_f/2T}}{1+e^{T_f/2T}}e^{-\triangle t/T} & \triangle t \in [L, T_f/2] \end{cases} \end{aligned}$$

since $t_{i_0+2} = t_{i_0} + T_f$. $\square$

Remark 1 : Two assumptions are made in the above proof. First is the assumption that when a sinusoidal signal of magnitude R causes forced oscillations, another signal of the same frequency but with a larger amplitude also causes forced oscillations. Secondly, it is assumed that these two external signals lead to forced oscillations with unique solutions. Hence the asssumption that $c(t_{i_0}) = c(t_{i_0'})$. This assumption is reasonable since the same zero initial conditions are imposed in both cases.

Remark 2 : It is well known that subharmonic oscillations with $\nu > 1$, *odd*, is also possible in a closed loop relay feedback system of Figure 1. This can be shown by considering switching instants that satisfy

$$t_{i_0+1} = t_{i_0} + \nu T_f/2$$

and arriving at an equation which is exactly the same as (20) with T_f replaced by νT_f. The analysis will also be the same and the conclusion after (22) will be that subharmonics oscillations of frequencies ω_f/ν are possible only for ν *odd.*

Proposition 1 addresses both the necessary and sufficient conditions for forced oscillations to occur. It is sufficient because it pre-assumes a sufficiently large R which is capable of causing two switches which are $T_f/2$ apart. Necessity comes about because of the assumption on the existence of the two switches which satisfy $t_{i_0+1} = t_{i_0} + T_f/2$. Thus the existence of forced oscillation depends critically on the ability of the system to lock into two switches which satisfy $t_{i_0+1} = t_{i_0} + T_f/2$. All subsequent switchings will occur at every $T_f/2$ and all signals will be periodic. The next section investigates the magnitude requirement on R in order for forced oscillations at any given frequency ω_f to take place.

4 Minimum R for Forced Oscillation

Proposition 1 alludes to the fact that for any ω_f and sufficiently large R, the existence of forced oscillation depends critically on the existence of two switching points, t_{i_0} and t_{i_0+1}, satisfying

$$t_{i_0+1} = t_{i_0} + T_f/2. \tag{23}$$

In order to satisfy the "sufficiently large" condition and (23) in the proposition, some signal conditions must be satisfied. Firstly, t_{i_0} and t_{i_0+1} are switching instants, hence, we have

$$y(t_{i_0}) = c(t_{i_0}) + R\sin\omega_f t_{i_0} = 0 \tag{24}$$
$$y(t_{i_0+1}) = c(t_{i_0+1}) + R\sin\omega_f t_{i_0+1} = 0. \tag{25}$$

At the same time, between t_{i_0} and t_{i_0+1}, the signal must also satisfy

$$y(t) = c(t) + R\sin\omega_f t < 0 \qquad t \in (t_{i_0}, t_{i_0} + T_f/2) \tag{26}$$

so that the output signal, $y(t)$ does not change sign and no additional switching can occur in between the two switching instants. (24)-(26) thus impose restrictions on R, for a given ω_f to satisfy the conditions of Proposition 1. These magnitude conditions for forced oscillations to occur in FOPDT system are given in the following proposition.

Proposition 2. *: In the configuration of Figure 1, for a given plant in (6) and a general $f(t)$ given by $f(t) = R\sin\omega_f t$, the following conditions are necessary and sufficient to achieve forced oscillations at a frequency of ω_f :*

$$\begin{aligned} -K \quad &+ \quad 2K\frac{e^{L/T}}{1+e^{T_f/2T}}e^{-t/T} \\ &+ \quad R\sin(\omega_f t + \phi) < 0 \quad t \in (0, L) \end{aligned} \tag{27}$$
$$\begin{aligned} K \quad &- \quad 2K\frac{e^{L/T}e^{T_f/2T}}{1+e^{T_f/2T}}e^{-t/T} \\ &+ \quad R\sin(\omega_f t + \phi) < 0 \quad t \in (L, T_f/2) \end{aligned} \tag{28}$$

where $\phi \triangleq k\pi + sign(c(t_{i_0}))\sin^{-1}(|c(t_{i_0})|/R)$, $k > 0$ odd.

Proof. : We show that (27) and (28) are equivalent to (24)-(26). From the previous section, (24) implies that

$$\omega_f t_{i_0} = k\pi + sign(c(t_{i_0}))\sin^{-1}(|c(t_{i_0})|/R) = \phi. \tag{29}$$

for k odd. $c(t)$ in (26) can be written as

$$c(t_{i_0} + t) = \begin{cases} -K + 2K\frac{e^{L/T}}{1+e^{T_f/2T}}e^{-t/T} & t \in [0, L] \\ K - 2K\frac{e^{L/T}e^{T_f/2T}}{1+e^{T_f/2T}}e^{-t/T} & t \in [L, T_f/2] \end{cases}$$

Substituting into (26), we get

$$\begin{array}{ll} -K + 2K\frac{e^{L/T}}{1+e^{T_f/2T}}e^{-t/T} + R\sin(\omega_f t_{i_0} + \omega_f t) < 0 & t \in (0, L) \\ K - 2K\frac{e^{L/T}e^{T_f/2T}}{1+e^{T_f/2T}}e^{-t/T} + R\sin(\omega_f t_{i_0} + \omega_f t) < 0 & t \in (L, T_f/2) \end{array}$$

which leads to (27) and (28) respectively after substituting (29). It can also be verified that (24) and (25) are also automatically satisfied because of (29). Hence (27) and (28) are equivalent to (24)-(26) which are also the equivalent conditions of Proposition 1. □

Remark 1 : It can be deduced from (29) that the minimum R must be given by $|c(t_{i_0})|$ in order to guarantee the existence of switch points, t_{i_0}. $c(t_{i_0})$ can be calculated from (17) for a given plant and ω_f.

Remark 2 : For forced oscillations to occur at any particular R and ω_f, (27) and (28) must be satisfied. For any ω_f, the minimum R, denoted as R_{min}, which satisfies these equations will be the minimum amplitude required for forced oscillations at ω_f. Alternatively, for any given R, a range of ω_f can be found for which forced oscillations can occur. Corollary 1 gives conditions equivalent to (27) and (28) when $R = |c(t_{i_0})|$ and allows one to find the range of ω_f which are possible for this R.

Corollary 1. *: For $R = |c(t_{i_0})|$, the range of ω_f at which forced oscillations will take place satisfies the following equation:*

$$\begin{array}{ll} \cos\omega_f t - 1 + \frac{1+e^{T_s/2T}}{1+e^{T_f/2T}}\left(e^{-t/T} - \cos\omega_f t\right) < 0 & t \in (0, L) \\ \cos\omega_f t + 1 - \frac{1+e^{T_s/2T}}{1+e^{T_f/2T}}\left(e^{-t/T}e^{T_f/2T} + \cos\omega_f t\right) < 0 & t \in (L, T_f/2) \end{array}$$
$$\ldots\ldots \tag{30}$$

where $T_s = 2T\log(2e^{L/T} - 1)$ is the self oscillating period given in [8].

Proof. : From (29), it follows that $\phi = \pm\pi/2$, depending on the sign of $c(t_{i_0})$. In (27) and (28), substitute for $2e^{L/T}$ by using its relationship with T_s. The result then follows. □

The inequalities in (30) can be rewritten in terms of L/T, $\alpha \triangleq T_f/T_s$, and $\beta \triangleq (2e^{L/T} - 1)$ as follows :

$$\begin{array}{ll} \cos\theta - 1 + \frac{\beta+1}{1+\beta^\alpha}\left[\beta^{\frac{-\theta\alpha}{\pi}} - \cos\theta\right] < 0 & \theta \in (0, \omega_f L) \\ \cos\theta + 1 - \frac{\beta+1}{1+\beta^\alpha}\left[\beta^{\frac{-\theta\alpha}{\pi}+\alpha} + \cos\theta\right] < 0 & \theta \in (\omega_f L, \pi). \end{array}$$

For a given L/T, the range of α which satisfies the above inequalities can be solved. This range of α holds for all plants with the same L/T ratio. Therefore, this result completely characterises the forced oscillation conditions for plants in (6). By solving the inequalities for plants whose $0.1 \le L/T \le 10$, Figure 2 shows the range of α for which forced oscillation is possible for this set of plants when $R = |c(t_{i_0})|$. With this R, for a given L/T, the range of ω_f is determined by

$$\alpha_l \le \alpha \le \alpha_h \Rightarrow \alpha_l T_s \le T_f \le \alpha_h T_s.$$

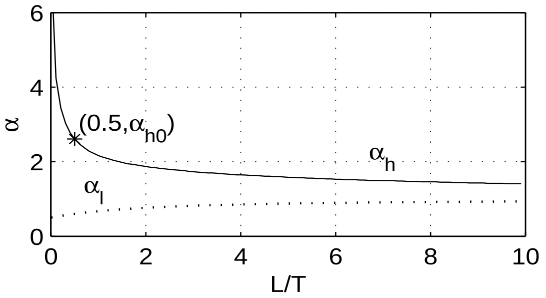

Figure 2: Range of α for different L/T when $R = |c(t_{i_0})|$

As an example, for $L/T = 0.5$, forced oscillations are possible for a range of frequencies corresponding to $\alpha_h = \alpha_{h0} = 2.607$ and $\alpha_l = \alpha_{l0} = 0.623$ with an $R = 0.6069$.

Figure 3, on the other hand, shows the minimum value of R required for forced oscillations to occur at different $T_f = \alpha T_s$ for a plant with $L/T = 0.5$. This plot was obtained by numerically solving for the minimum R which satisfies (27) and (28). The plot of $c(t_{i_0})$ is also given on the same figure to show that above a certain α_{h0}, $R = |c(t_{i_0})|$ cannot achieve any forced oscillations. This can be deduced from this figure when $R_{\min}$ and $|c(t_{i_0})|$ diverges from one another at $\alpha = \alpha_{h0}$.

5 Simulation Results

In this section, two simulation studies demonstrate the results in Propositions 1 and 2. In all cases, the plant was placed in the configuration of Figure 1 and the external forcing signal, $f(t)$ was applied with different values of R. The switching instants, t_i, were tracked and the switching intervals, $T_u/2$, between consecutive switches were recorded.

Example 1 : The plant and $f(t)$ are given by

$$g(s) = \frac{e^{-0.5s}}{s+1} \quad \text{and} \quad f(t) = R \sin 0.5\pi t$$

where $R = |c(t_{i_0})| = 0.6069$ and $T_f = 4$ sec. It can be seen from Figure 4 that at the 8th and 9th switching instants, the first switching interval of 2 sec occured. Subsequently, all switches were exactly 2 sec apart. Figure 5 shows the resulting signals for $c(t)$ (shown as solid line) and $u(t)$ (shown as dash line). $u(t)$ is indeed switching at every 2 sec.

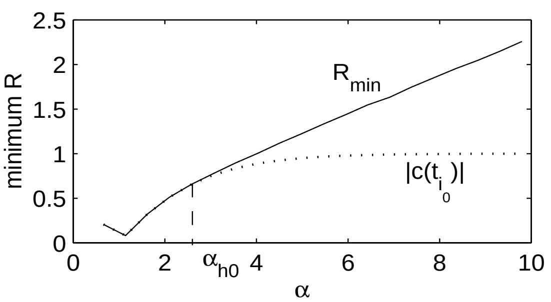

Figure 3: Minimum R required for a range of α.

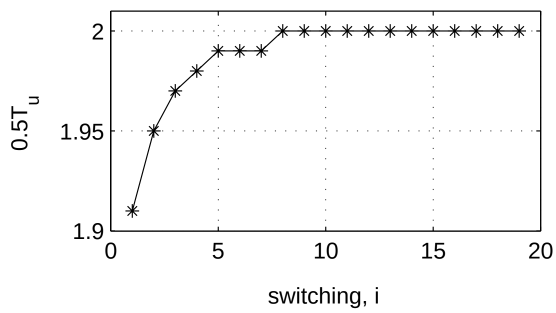

Figure 4: Switching intervals for plant in ex. 1.

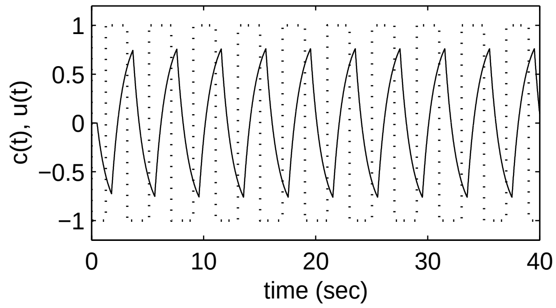

Figure 5: Forced oscillations for plant in ex. 1.

Example 2 : In this example, $g(s)$ is given by

$$g(s) = \frac{e^{-s}}{s+1} \quad \text{and} \quad f(t) = R \sin 0.25\pi t$$

where $R = |c(t_{i_0})| = 0.9022$ and $T_f = 8$ sec. Figure 6 shows the switching intervals. It can be observed that in this case, the switching never attains a steady switching interval of $T_f/2 = 4$ sec. As expected, forced oscillations did not take place as confirmed by Figure 7. This is an example where $R = |c(t_{i_0})|$ is not sufficient to cause forced oscillation to occur.

R is now increased to $1.2|c(t_{i_0})|$. Figure 8 shows that forced oscillation is attained after the 3rd switching instant. This is

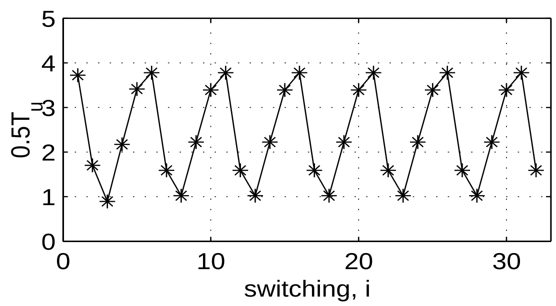

Figure 6: Switching intervals in ex. 2 when $R = |c(t_{i_0})|$.

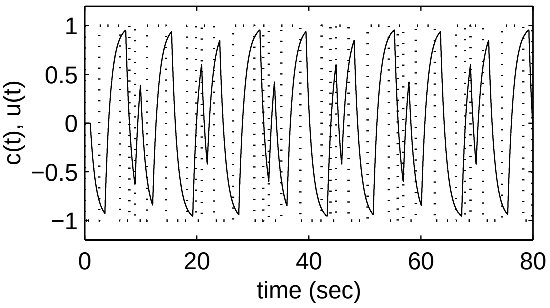

Figure 7: Complex oscillations in ex. 2 when $R = |c(t_{i_0})|$.

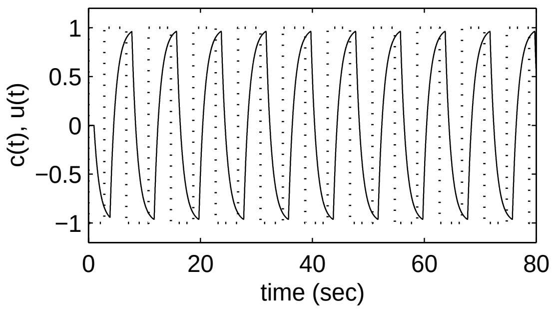

Figure 9: Forced oscillations in ex. 2 when $R = 1.2|c(t_{i_0})|$.

confirmed in Figure 9.

6 Discussions and Conclusions

In conclusion, the main contribution of this paper lies in the derivation of the necessary and sufficient conditions for first order systems to admit forced oscillation solutions. By solving these conditions, the minimum R required for forced oscillations at any frequency can be derived. Furthermore, the minimum R required for forced oscillations in first order systems are completely characterised.

The results in this paper are only a part of a more complete set of results which are currently being written up. The more complete set of results include the following points.

(a) Although the results are shown for FOPDT plants, the principles of the analysis hold for higher order systems. For any n^{th} order system, by writing the switching equations corresponding to (5) for all $(n-1)^{th}$ derivatives of the output, a similar, albeit more complicated, analysis can be carried out to arrive at the forced oscillation conditions. For such higher order systems, all derivatives up to $(n-1)$ have to satisfy symmetric switching conditions.

(b) The results in this paper are specifically for external forcing signals whose frequencies satisfy $L < T_f/2$. Our analysis shows that for this range of frequencies, subharmonic oscillations of frequencies ω_f/ν, $\nu > 1$ are not possible. This is not shown here because of space constraints. For higher frequencies satisfying $T_f < 2L$, subharmonic oscillations are easily obtained.

(c) The relevance of this work is alluded to earlier in the introduction section. The main motivation is to be able to set the required relay amplitudes in a multi-loop system in order to effect a Mode 1 system without the need for further experimentation or adaptation. This leads to the auto-tuning of controllers for multi-loop systems which are not easy to tune initially.

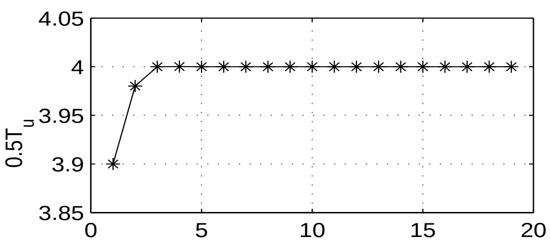

Figure 8: Switching intervals in ex. 2 when $R = 1.2|c(t_{i_0})|$.

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