LOW ORDER $\mathcal{H}_\infty$ CONTROLLER DESIGN: AN LMI APPROACH

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Abstract

In this paper, we consider the $\mathcal{H}_\infty$ controller design problem for both continuous-time and discrete-time linear time-invariant (LTI) systems via low order dynamic output feedback controllers. The existence condition of desired $\mathcal{H}_\infty$ controllers is expressed as a feasibility problem of a bilinear matrix inequality (BMI) with respect to a coefficient matrix defining the controller and a Lyapunov matrix. To solve the BMI, we propose two sufficient conditions which result in linear matrix inequalities (LMIs), by using a block diagonal structure of an equivalent matrix of the Lyapunov matrix in the BMI.

1 Introduction

In the last three decades, the $\mathcal{H}_\infty$ control problem for linear time-invariant systems has been studied extensively. It is well known that when the controller's order is the same as that of the system, the control problem has been solved completely by the algebraic Riccati equation (ARE) approach [1] and the linear matrix inequality (LMI) approach [2, 3], and the computation is also quite easy using the existing softwares (e.g., Robust Control Toolbox & LMI Control Toolbox in MATLAB). However, when the desired order of the controller is smaller than the system's order, there has not been very effective method though various computation algorithms have been proposed up to now [4, 5, 6, 7]. In [4], it has been pointed out that such a control problem is NP-hard in general. Refs. [2, 5] have established some improved algorithms in the framework of LMI, but we have difficulty in dealing with the matrix rank conditions there. In [6], a homotopy-based algorithm has been proposed to deform the controller gradually from the $\mathcal{H}_\infty$ controller of the same order to the one of low order, but the convergence of the algorithm depends greatly on the choice of the initial controller with the same order. Ref. [7] considered the low order $\mathcal{H}_\infty$ controller design problem using the ARE approach. However, the method proposed there is not practicable because the controller's order one can design in that context is related to the rank of the solution of an algebraic Riccati equation.

Recently, Ref. [8] considered an LMI-based design method for low order $\mathcal{H}_\infty$ control problem. Though the derived LMI in that context is sufficient (but not necessary), we think it is quite effective in many cases. We found that though the basic design idea is good, there are some mistakes and inaccurate forms in [8]. For example, the equation $T^{-1}B_2 = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$ was used for a given B_2 in [8], but we see easily that this holds only when the right part of B_2 is zero. For this reason, the authors have applied and corrected the design method in [8] to the low order $\mathcal{H}_\infty$ control problem for discrete-time LTI systems where the matrix B_2 describing how the control input affects the system (or the matrix C_2 describing how the state affects the measurement output) is assumed to have full column (or row) rank [9]. In that context, the existence condition of $\mathcal{H}_\infty$ controllers is also reduced to solving a bilinear matrix equality (BMI) with respect to a coefficient matrix defining the controller and a Lyapunov matrix. To reduce the BMI to an LMI, the authors have proposed there to set an equivalent matrix of the Lyapunov matrix appropriately as block diagonal corresponding to the controller's desired order. Because the structure of the block diagonal matrix can be set freely, one can also specify the controller's order arbitrarily.

In this paper, we extend the results in [9] to low order $\mathcal{H}_\infty$ controller design for both continuous-time and discrete-time LTI systems where B_2 (or C_2) is not assumed to have full column rank (full row rank). We also express the existence condition of desired controllers as a BMI with respect to a coefficient matrix defining the controller and a Lyapunov matrix, and propose to set an equivalent matrix of the Lyapunov matrix appropriately as block diagonal so that we can reduce the BMI to an

LMI. Since B_2 (or C_2) is not full rank, we can construct the $\mathcal{H}_\infty$ controller more flexibly in the sense that we can add a random part to the solution of the LMI. We will use an example to demonstrate the usefulness of our result.

2 Continuous-Time Case

In this section, we consider the continuous-time LTI system described by

$$\left\{\begin{array}{l} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w\,, \end{array}\right. \qquad (1)$$

where $x \in \mathcal{R}^n$ is the state, $w \in \mathcal{R}^r$ is the disturbance input, $z \in \mathcal{R}^p$ is the controlled output, $u \in \mathcal{R}^m$ is the control input, $y \in \mathcal{R}^q$ is the measurement output. The matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21} are constant matrices of appropriate size. Throughout this paper, we assume that the triple (A, B_2, C_2) is stabilizable and detectable.

For the system (1), we consider the following dynamic output feedback controller

$$\left\{\begin{array}{l} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y \\ u = \hat{C}\hat{x} + \hat{D}y\,, \end{array}\right. \qquad (2)$$

where $\hat{x} \in \mathcal{R}^{\hat{n}}$ is the controller's state, $\hat{n} < n$ is the desired order of the controller, and $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are constant matrices to be determined.

The closed-loop system obtained by applying the controller (2) to the system (1) is

$$\left\{\begin{array}{l} \dot{\tilde{x}} = (\tilde{A} + \tilde{B}_2 G \tilde{C}_2)\tilde{x} + (\tilde{B}_1 + \tilde{B}_2 G \tilde{D}_{21})w \\ z = (\tilde{C}_1 + \tilde{D}_{12} G \tilde{C}_2)\tilde{x} + (\tilde{D}_{11} + \tilde{D}_{12} G \tilde{D}_{21})w\,, \end{array}\right. \qquad (3)$$

where $\tilde{x} = [x^T \;\; \hat{x}^T]^T \in \mathcal{R}^{n+\hat{n}}$,

$$\left[\begin{array}{ccc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \end{array}\right] = \left[\begin{array}{cc|c|cc} A & 0_{n\times\hat{n}} & B_1 & 0_{n\times\hat{n}} & B_2 \\ 0_{\hat{n}\times n} & 0_{\hat{n}\times\hat{n}} & 0_{\hat{n}\times r} & I_{\hat{n}} & 0_{\hat{n}\times m} \\ \hline C_1 & 0_{p\times\hat{n}} & D_{11} & 0_{p\times\hat{n}} & D_{12} \\ \hline 0_{\hat{n}\times n} & I_{\hat{n}} & 0_{\hat{n}\times r} & & \\ C_2 & 0_{q\times\hat{n}} & D_{21} & & \end{array}\right] \qquad (4)$$

and

$$G = \left[\begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array}\right]. \qquad (5)$$

Then, our $\mathcal{H}_\infty$ control problem is : *For a specified $\mathcal{H}_\infty$ disturbance attenuation level $\gamma > 0$, design a controller (2) for the system (1) so that the closed-loop system (3) is Hurwitz stable and the $\mathcal{H}_\infty$ norm of the transfer function from w to z is less than γ.* If such a controller exists, we say the system (1) is stabilizable with the $\mathcal{H}_\infty$ disturbance attenuation level γ via a low order controller (2).

Now, we recall the well known Bounded Real Lemma [2] for continuous-time LTI systems.

Lemma 1. *The following statements are equivalent:*

(i) A is Hurwitz stable and $\|C(sI - A)^{-1}B + D\|_\infty < \gamma$.

(ii) There exists a positive definite solution P to the LMI:

$$\left[\begin{array}{ccc} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{array}\right] < 0\,. \qquad (6)$$

Applying this lemma to the closed-loop system (3), we see that our desired low order $\mathcal{H}_\infty$ controller exists if and only if there is a positive definite matrix $\tilde{P}$ such that

$$\left[\begin{array}{ccc} A_{cl}^T \tilde{P} + \tilde{P} A_{cl} & \tilde{P} B_{cl} & C_{cl}^T \\ B_{cl}^T \tilde{P} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{array}\right] < 0\,, \qquad (7)$$

where

$$\begin{array}{rclrcl} A_{cl} & = & \tilde{A} + \tilde{B}_2 G \tilde{C}_2\,, & B_{cl} & = & \tilde{B}_1 + \tilde{B}_2 G \tilde{D}_{21} \\ C_{cl} & = & \tilde{C}_1 + \tilde{D}_{12} G \tilde{C}_2\,, & D_{cl} & = & \tilde{D}_{11} + \tilde{D}_{12} G \tilde{D}_{21}\,. \end{array} \qquad (8)$$

Since the unknown coefficient matrix G is included in $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ and thus in (7), our control problem is reduced to solving the matrix inequality (7) with respect to G and $\tilde{P}$. However, (7) is a BMI with respect to G and $\tilde{P}$, and there is no globally effective method for general BMIs presently [10, 6].

Here, we propose to set an equivalent matrix of the Lyapunov matrix $\tilde{P}$ to be block diagonal appropriately so that the BMI (7) is reduced to an LMI. We now state and prove the first result.

Theorem 1. *Assume $D_{12} = 0$, $rank(B_2) = m_r \le m$. The system (1) is stabilizable with the $\mathcal{H}_\infty$ disturbance attenuation level γ via a low order controller (2) if there exist a positive definite matrix P with block diagonal structure as*

$$P = \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array}\right] \in \mathcal{R}^{(n+\hat{n})\times(n+\hat{n})}\,, \qquad (9)$$

where $P_1 \in \mathcal{R}^{(\hat{n}+m_r)\times(\hat{n}+m_r)}$, $P_2 \in \mathcal{R}^{(n-m_r)\times(n-m_r)}$, and a matrix $W_y \in \mathcal{R}^{(\hat{n}+m_r)\times(\hat{n}+q)}$, satisfying the LMI

$$\left[\begin{array}{ccc} F_{11}^T + F_{11} & F_{12} & \hat{C}_1^T \\ F_{12}^T & -\gamma I & \tilde{D}_{11}^T \\ \hat{C}_1 & \tilde{D}_{11} & -\gamma I \end{array}\right] < 0\,. \qquad (10)$$

Here,

$$\begin{aligned} F_{11} &= P\hat{A} + \begin{bmatrix} W_y \\ 0 \end{bmatrix} \hat{C}_2 \\ F_{12} &= P\hat{B}_1 + \begin{bmatrix} W_y \\ 0 \end{bmatrix} \tilde{D}_{21}\,, \end{aligned} \qquad (11)$$

and

$$\hat{A} = T^{-1}\tilde{A}T\,, \quad \hat{B}_1 = T^{-1}\tilde{B}_1\,, \quad \hat{C}_1 = \tilde{C}_1 T\,, \quad \hat{C}_2 = \tilde{C}_2 T\,, \qquad (12)$$

where $T \in \mathcal{R}^{(n+\hat{n})\times(n+\hat{n})}$ *and* $V \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+m)}$ *are nonsingular matrices satisfying*

$$T^{-1}\tilde{B}_2 V^{-1} = \begin{bmatrix} I_{\hat{n}+m_r} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{(n+\hat{n})\times(\hat{n}+m)}. \qquad (13)$$

If the LMI (10) *is feasible, one of the controller coefficient matrices is computed as*

$$G = V^{-1} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+q)}\,, \qquad (14)$$

where $G_1 = P_1^{-1} W_y \in \mathcal{R}^{(\hat{n}+m_r)\times(\hat{n}+q)}$, *and* $G_2 \in \mathcal{R}^{(m-m_r)\times(\hat{n}+q)}$ *is an arbitrary matrix.*

Proof. First, we define the positive definite matrix $\tilde{P} = (T^{-1})^T P T^{-1}$. Substituting $\hat{A}$, $\hat{B}_1$, $\hat{C}_1$, $\hat{C}_2$ into (10), and then pre- and post-multiplying (10) by $\mathrm{diag}\{(T^{-1})^T, I, I\}$ and $\mathrm{diag}\{T^{-1}, I, I\}$, respectively, we obtain

$$\begin{bmatrix} \tilde{F}_{11}^T + \tilde{F}_{11} & \tilde{F}_{12} & \tilde{C}_1^T \\ \tilde{F}_{12}^T & -\gamma I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0\,, \qquad (15)$$

where

$$\begin{aligned} \tilde{F}_{11} &= \tilde{P}\tilde{A} + (T^{-1})^T \begin{bmatrix} W_y \\ 0 \end{bmatrix} \tilde{C}_2 \\ \tilde{F}_{12} &= \tilde{P}\tilde{B}_1 + (T^{-1})^T \begin{bmatrix} W_y \\ 0 \end{bmatrix} \tilde{D}_{21}\,. \end{aligned} \qquad (16)$$

It is easy to confirm from (9), (13) and (14) that $\begin{bmatrix} W_y \\ 0 \end{bmatrix} = PT^{-1}\tilde{B}_2 G \in \mathcal{R}^{(n+\hat{n})\times(\hat{n}+q)}$. Substituting this equation into (15) and (16) leads to

$$\begin{bmatrix} A_{cl}^T \tilde{P} + \tilde{P} A_{cl} & \tilde{P} B_{cl} & \tilde{C}_1^T \\ B_{cl}^T \tilde{P} & -\gamma I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0. \qquad (17)$$

According to Lemma 1, the closed-loop system (3) is Hurwitz stable, and the $\mathcal{H}_\infty$ norm of the transfer function from w to z is less than γ. ■

In Theorem 1, we have reduced the feasibility problem of the BMI (7) to solving the LMI (10), which can be easily dealt with by existing softwares; for example, the LMI Control Toolbox [12] of MATLAB.

The next theorem deals with the case where the assumption $D_{21} = 0$ holds instead of $D_{12} = 0$.

Theorem 2. *Assume* $D_{21} = 0$, $rank(C_2) = q_r \le q$. *The system* (1) *is stabilizable with the* $\mathcal{H}_\infty$ *disturbance attenuation level* γ *via a low order controller* (2) *if there exist a positive definite matrix* Q *with block diagonal structure as*

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in \mathcal{R}^{(n+\hat{n})\times(n+\hat{n})}\,, \qquad (18)$$

where $Q_1 \in \mathcal{R}^{(\hat{n}+q_r)\times(\hat{n}+q_r)}$, $Q_2 \in \mathcal{R}^{(n-q_r)\times(n-q_r)}$, *and a matrix* $W_y \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+q_r)}$, *satisfying the LMI*

$$\begin{bmatrix} M_{11}^T + M_{11} & \bar{B}_1 & M_{13}^T \\ \bar{B}_1^T & -\gamma I & \tilde{D}_{11}^T \\ M_{13} & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0\,. \qquad (19)$$

Here,

$$\begin{aligned} M_{11} &= \bar{A}Q + \bar{B}_2 \begin{bmatrix} W_y & 0 \end{bmatrix} \\ M_{13} &= \bar{C}_1 Q + \tilde{D}_{12} \begin{bmatrix} W_y & 0 \end{bmatrix}\,, \end{aligned} \qquad (20)$$

and

$$\bar{A} = U^{-1}\tilde{A}U\,, \bar{B}_1 = U^{-1}\tilde{B}_1\,, \bar{B}_2 = U^{-1}\tilde{B}_2\,, \bar{C}_1 = \tilde{C}_1 U\,, \qquad (21)$$

where $U \in \mathcal{R}^{(n+\hat{n})\times(n+\hat{n})}$ *and* $S \in \mathcal{R}^{(\hat{n}+q)\times(\hat{n}+q)}$ *are nonsingular matrices satisfying*

$$S\tilde{C}_2 U = \begin{bmatrix} I_{\hat{n}+q_r} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{(\hat{n}+q)\times(n+\hat{n})}. \qquad (22)$$

If the LMI (19) *is feasible, one of the controller coefficient matrices is computed as*

$$G = \begin{bmatrix} G_3 & G_4 \end{bmatrix} S \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+q)}, \qquad (23)$$

where $G_3 = W_y Q_1^{-1} \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+q_r)}$, *and* $G_4 \in \mathcal{R}^{(\hat{n}+m)\times(q-q_r)}$ *is an arbitrary matrix.*

Proof. We first define the positive definite matrix $\tilde{Q} = UQU^T$. Then, pre- and post-multiplying the matrix inequality (22) by $\mathrm{diag}\{U, I, I\}$ and $\mathrm{diag}\{U^T, I, I\}$, respectively, we obtain

$$\begin{bmatrix} \tilde{M}_{11}^T + \tilde{M}_{11} & \tilde{B}_1 & \tilde{M}_{13}^T \\ \tilde{B}_1^T & -\gamma I & \tilde{D}_{11}^T \\ \tilde{M}_{13} & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0\,, \qquad (24)$$

where

$$\begin{aligned} \tilde{M}_{11} &= \tilde{A}\tilde{Q} + \tilde{B}_2 \begin{bmatrix} W_y & 0 \end{bmatrix} U^T \\ \tilde{M}_{13} &= \tilde{C}_1 \tilde{Q} + \tilde{D}_{12} \begin{bmatrix} W_y & 0 \end{bmatrix} U^T\,. \end{aligned} \qquad (25)$$

It is easy to confirm $\begin{bmatrix} W_y & 0 \end{bmatrix} = G\tilde{C}_2 UQ \in \mathcal{R}^{(\hat{n}+m)\times(n+\hat{n})}$ from (18), (22) and (23). Then, substituting this equation into (24) and (25) results in

$$\begin{bmatrix} \tilde{Q}A_{cl}^T + A_{cl}\tilde{Q} & \tilde{B}_1 & \tilde{Q}C_{cl}^T \\ \tilde{B}_1^T & -\gamma I & \tilde{D}_{11}^T \\ C_{cl}\tilde{Q} & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0\,, \qquad (26)$$

which is equivalent to

$$\begin{bmatrix} A_{cl}^T\tilde{Q}^{-1}+\tilde{Q}^{-1}A_{cl} & \tilde{Q}^{-1}\tilde{B}_1 & C_{cl}^T \\ \tilde{B}_1^T\tilde{Q}^{-1} & -\gamma I & \tilde{D}_{11}^T \\ C_{cl} & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0\,. \tag{27}$$

This implies that the controller (23) stabilizes the system (1) with the $\mathcal{H}_\infty$ disturbance attenuation level γ. ■

Remark 1. Although Theorems 1 and 2 come up with dual forms, they are not equivalent and are supposed to deal with different cases of $D_{12}=0$ or $D_{21}=0$, respectively. Furthermore, the LMI conditions provided by the theorems are sufficient ones. Therefore, even in the case where both $D_{12}=0$ and $D_{21}=0$ hold and thus both theorems can be applied, the LMI condition of one theorem would be satisfied while the other would not. ■

Remark 2. When it is necessary, we can try to obtain a tight $\mathcal{H}_\infty$ disturbance attenuation level by considering the eigenvalue problem (EVP) [11]: "minimize γ, s.t. (10) or (19) with $P>0$ or $Q>0$, respectively". ■

Remark 3. The nonsingular matrix T satisfying (13) is not unique. More precisely, for any T satisfying (13), $\begin{bmatrix} I & * \\ 0 & \Gamma \end{bmatrix} T$ also satisfies (13), where Γ is an arbitrary nonsingular matrix. The same is true for the nonsingular matrix U satisfying (22). However, we can easily prove that the feasibility of the LMI (10) and (19) does not depend on the choice of T and U, respectively. ■

3 Discrete-Time Case

In this section, we consider the discrete-time LTI system described by

$$\begin{cases} x(k+1) = Ax(k) + B_1w(k) + B_2u(k) \\ z(k) = C_1x(k) + D_{11}w(k) + D_{12}u(k) \\ y(k) = C_2x(k) + D_{21}w(k)\,. \end{cases} \tag{28}$$

Here, we assume that all the vectors and the matrices have the same meaning and the same dimension as in (1), except that the vectors are of discrete time.

For this system, we consider the following dynamical output feedback controller

$$\begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y(k) \\ u(k) = \hat{C}\hat{x}(k) + \hat{D}y(k)\,. \end{cases} \tag{29}$$

As in (2), we assume that $\hat{x}(k)\in\mathcal{R}^{\hat{n}}$ is the state of the controller, and $\hat{n}<n$ is the desired order of the controller.

Using the notations defined in the previous section together with $\tilde{x}(k) = [x^T(k) \quad \hat{x}^T(k)]^T$, we describe the closed-loop system by applying the controller (29) to the system (28) as

$$\begin{cases} \tilde{x}(k+1) = A_{cl}\tilde{x}(k) + B_{cl}w(k) \\ z(k) = C_{cl}\tilde{x}(k) + D_{cl}w(k)\,. \end{cases} \tag{30}$$

Now, we recall the Bounded Real Lemma [2] for discrete-time LTI systems.

Lemma 2. *The following statements are equivalent:*

(i) A is Schur stable and $\|C(zI-A)^{-1}B+D\|_\infty < \gamma$.

(ii) There exists a positive definite solution P to the LMI:

$$\begin{bmatrix} -P & PA & PB & 0 \\ A^TP & -P & 0 & C^T \\ B^TP & 0 & -\gamma I & D^T \\ 0 & C & D & -\gamma I \end{bmatrix} < 0\,. \tag{31}$$

Applying this lemma to the closed-loop system (30), we see that the desired low order $\mathcal{H}_\infty$ controller exists if and only if there are matrices G and $\tilde{P}>0$ satisfying

$$\begin{bmatrix} -\tilde{P} & \tilde{P}A_{cl} & \tilde{P}B_{cl} & 0 \\ A_{cl}^T\tilde{P} & -\tilde{P} & 0 & C_{cl}^T \\ B_{cl}^T\tilde{P} & 0 & -\gamma I & D_{cl}^T \\ 0 & C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0\,. \tag{32}$$

Same as (7), this matrix inequality is a BMI with respect to G and $\tilde{P}$, and thus not easy to solve. In this section as well, we consider setting an equivalent matrix of the Lyapunov matrix $\tilde{P}$ to be block diagonal appropriately so that the BMI (32) is reduced to an LMI.

Next, we state two theorems which are analogous to Theorems 1 and 2, respectively. The proofs are omitted since the same ideas in the proofs of Theorems 1 and 2 can be used here for the matrix inequality (32).

Theorem 3. *Assume $D_{12}=0$, $rank(B_2)=m_r\le m$. The system (28) is stabilizable with the $\mathcal{H}_\infty$ disturbance attenuation level γ via a low order controller (29) if there exist a positive definite matrix P structured as in (9) and a matrix $W_y\in\mathcal{R}^{(\hat{n}+m_r)\times(\hat{n}+q)}$ such that the LMI*

$$\begin{bmatrix} -P & F_{11} & F_{12} & 0 \\ F_{11}^T & -P & 0 & \hat{C}_1^T \\ F_{12}^T & 0 & -\gamma I & \tilde{D}_{11}^T \\ 0 & \hat{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0 \tag{33}$$

is satisfied, where all the matrices are defined samely as in Theorem 1.

If the LMI (33) is feasible, one of the controller coefficient matrices is computed as in (14). ■

Theorem 4. *Assume $D_{21}=0$, $rank(C_2)=q_r\le q$. The system (28) is stabilizable with the $\mathcal{H}_\infty$ disturbance* *attenuation level γ via a low order controller* (29) *if there exist a positive definite matrix Q structured as in* (18) *and a matrix $W_y \in \mathcal{R}^{(\hat{n}+m)\times(\hat{n}+q_r)}$ such that the LMI*

$$\begin{bmatrix} -Q & M_{11} & \bar{B}_1 & 0 \\ M_{11}^T & -Q & 0 & M_{13}^T \\ \bar{B}_1^T & 0 & -\gamma I & \tilde{D}_{11}^T \\ 0 & M_{13} & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0 \qquad (34)$$

is satisfied, where all the matrices are defined samely as in Theorem 2.

If the LMI (34) *is feasible, one of the controller coefficient matrices is computed as in* (23). ■

4 Numerical Example

In this section, we present a simple example for the continuous-time LTI system. We consider the system (1) in the case of $n = 8, m = 2, p = 4, q = 2, r = 4$, whose matrices are

$$A = \begin{bmatrix} -1.42 & 2.21 & -0.49 & 3.64 & 0.27 & 1.53 & -0.49 & -3.05 \\ -6.98 & -5.30 & -0.81 & 0.37 & -3.35 & -2.86 & -1.77 & -3.54 \\ 0.94 & 1.60 & -6.34 & 0.79 & -6.13 & 3.09 & -1.42 & 2.02 \\ 3.37 & -1.03 & 5.14 & 1.27 & 1.37 & -1.12 & -6.06 & -2.42 \\ 2.61 & -0.82 & 1.86 & -0.63 & -3.75 & 2.03 & -3.54 & -5.08 \\ 1.64 & 4.91 & 0.58 & 3.37 & -2.57 & -2.43 & -2.10 & -3.02 \\ -1.07 & 0.03 & 1.38 & 3.45 & 1.10 & 2.57 & -3.93 & 4.73 \\ 2.78 & 0.59 & 0.60 & -0.72 & 0.58 & 3.45 & 3.73 & -0.67 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.34 & 1.51 & -1.45 & -0.59 \\ 1.19 & -1.37 & 2.10 & -1.93 \\ -0.17 & 2.74 & 0.47 & 1.22 \\ 1.79 & -0.76 & 3.54 & -0.25 \\ -0.56 & -0.27 & -2.65 & 1.16 \\ -1.19 & 1.22 & 1.70 & 1.35 \\ 0.51 & -0.15 & -0.71 & -1.46 \\ 0.19 & 3.00 & 0.08 & -2.14 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -0.21 & 0.63 \\ 0.77 & -2.31 \\ 1.29 & -3.87 \\ 1.43 & -4.29 \\ 2.52 & -7.56 \\ 1.74 & -5.22 \\ 0.10 & -0.30 \\ 1.89 & -5.67 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.09 & 0.09 & -0.12 & 0.02 & -0.11 & -0.11 & 0 & -0.05 \\ 0.19 & -0.14 & -0.39 & -0.16 & -0.28 & 0.07 & -0.09 & -0.18 \\ 0.04 & 0.11 & -0.16 & 0.12 & 0.09 & 0.18 & -0.07 & -0.07 \\ -0.11 & 0.09 & 0.02 & 0.11 & -0.11 & -0.30 & 0.04 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -5.74 & 11.90 & -6.61 & -7.65 & -1.84 & -2.18 & -0.25 & -2.84 \\ -8.80 & -6.70 & -0.36 & -3.15 & 0.86 & -4.32 & -0.68 & 10.08 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} -0.12 & 0.06 & 0.06 & -0.26 \\ 0.09 & -0.32 & -0.12 & -0.03 \\ 0.37 & 0 & 0.17 & 0.43 \\ 0.32 & -0.17 & -0.09 & 0 \end{bmatrix}, \; D_{12} = 0$$

$$D_{21} = \begin{bmatrix} 3.14 & 1.53 & -1.00 & -1.65 \\ -3.63 & 1.18 & 0.99 & -2.10 \end{bmatrix} \qquad (35)$$

For this system, we set $\gamma = 2.5$ as the desired $\mathcal{H}_\infty$ disturbance attenuation level, and aim to design a low order $\mathcal{H}_\infty$ controller (2) in the cases of $\hat{n} = 5$ and $\hat{n} = 3$. Notice that the values of $\hat{n}$ are less than the order ($n = 8$) of the controlled system, and that the matrix B_2 is not full column rank.

In the case of $\hat{n} = 5$, we solve the LMI in Theorem 1, and then set the arbitrary matrix G_2 as

$$\begin{bmatrix} 1.07 & -1.28 & 0.64 & -1.23 & -0.51 & -0.85 & 0.29 \end{bmatrix}, \qquad (36)$$

to obtain the controller coefficient matrix G as

$$\left[\begin{array}{ccccc|cc} 0.11 & -0.19 & 0.05 & -0.19 & 1.04 & 0.03 & 0.96 \\ 0.29 & -1.30 & 0.05 & -0.29 & -0.31 & 0.01 & 0.21 \\ -0.08 & 0.05 & -1.26 & 0.58 & 0.18 & 0 & -0.06 \\ 0.29 & -0.29 & 0.58 & -1.57 & -0.38 & 0.01 & 0.22 \\ 1.36 & -0.81 & 0.32 & -0.89 & -0.28 & 0.03 & 1.15 \\ \hline -7.10 & 2.58 & -1.00 & 2.55 & -4.99 & 0.66 & -5.67 \\ 17.90 & -3.69 & 0.98 & -3.76 & 16.58 & 0.72 & 16.10 \end{array}\right]. \qquad (37)$$

We can confirm that the closed-loop system is Hurwitz stable and the achieved $\mathcal{H}_\infty$ disturbance attenuation level is $\gamma = 2.0912 < 2.5$.

In the case of $\hat{n} = 3$, we also solve the LMI in Theorem 1, and then set G_2 as

$$G_2 = \begin{bmatrix} -2.55 & -1.26 & -0.52 & -1.46 & 1.18 \end{bmatrix} \qquad (38)$$

to obtain

$$G = \left[\begin{array}{ccc|cc} 1.03 & 0.41 & 1.96 & -0.02 & -0.98 \\ -0.30 & -1.31 & 0.22 & 0 & 0.12 \\ 2.21 & 0.96 & 0.85 & -0.02 & -1.05 \\ \hline 13.26 & 3.79 & 10.79 & 1.27 & -6.55 \\ -31.73 & -7.39 & -30.73 & 0.82 & 15.93 \end{array}\right]. \qquad (39)$$

Then, the closed-loop system is Hurwitz stable and the achieved $\mathcal{H}_\infty$ disturbance attenuation level is $\gamma = 2.0516 < 2.5$. Thus, we have obtained the desired low order $\mathcal{H}_\infty$ controllers.

5 Concluding Remarks

In this paper, we have considered the low order $\mathcal{H}_\infty$ controller design problem for both continuous-time and discrete-time LTI systems where B_2 and C_2 do not have full rank, by using a matrix inequality approach. We express the existence condition of the desired low order $\mathcal{H}_\infty$ controller as a BMI with respect to a coefficient matrix defining the controller and a Lyapunov matrix. To solve the BMI, we have derived two LMIs which deal with the case of $D_{12} = 0$ and $D_{21} = 0$, respectively. The key idea is to set an equivalent matrix of the Lyapunov matrix in the BMI as block diagonal, the block size of which corresponds to the controller's desired order. We have given an example to show the usefulness of the results.

We suggest that the approach proposed in this paper should be practical for mamy control problems concerning matrix inequality approach. For example, we have extended the results in this paper to $\mathcal{H}_\infty$ controller design problems for decentralized control systems [13] and descriptor systems [14, 15].

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