APPLICATION OF UNCERTAIN VARIABLES IN A CLASS OF CONTROL SYSTEMS WITH UNCERTAIN AND RANDOM PARAMETERS

Z. Bubnicki

Institute of Control and Systems Engineering, Wroclaw University of Technology
Wyb. Wyspianskiego 27, 50-370 Wroclaw, Poland
fax: +48 71 320 38 84, e-mail: bubnicki@ists.pwr.wroc.pl

Keywords: uncertain variables, uncertain systems, knowledge-based systems, decision systems, uncertain control.

Abstract

A paper is concerned with a static decision (control) plant described by a relation with two kinds of unknown parameters: uncertain parameters described by certainty distributions and random parameters described by probability distributions. Different versions of the decision making in an open-loop control system are formulated and considered. A simple example illustrates the presented approach.

1 Introduction

There exists a great variety of definitions and formal descriptions of uncertain systems (see e.g. [9, 10, 12, 1]). The idea of so called uncertain variables based on uncertain logics has been introduced and developed as a tool for analysis and decision problems in a class of uncertain systems described by traditional models or by relational knowledge representations. Unknown parameters in these descriptions are considered as uncertain variables characterized by certainty distributions given by an expert and expressing his/her knowledge concerning different approximate values of the parameters [2, 4, 6, 3, 7, 8].

The purpose of this paper is to present a new idea concerning the application of uncertain variables to decision making for decision plants containing two kinds of the unknown parameters in the relational knowledge representation: *uncertain parameters* described by certainty distributions and *random parameters* described by probability distributions. The considerations are limited to static (memory less) plants and open-loop decision (control) systems, but the basic idea may be extended to more complicated cases. In Secs. 2 and 3 a short presentation of the uncertain variables and a basic decision problem for a plant with uncertain parameters are given. Details can be found in [4, 6]. The basic problem is used in the formulations and considerations concerning different versions of the decision problem for a plant with uncertain and random parameters, described in Sec. 4÷7.

2 Uncertain variables

Let us consider a universal set Ω, $\omega \in \Omega$, a vector space $X \subset R^k$ and a function $g: \Omega \to X$. Assume that for the fixed ω the value $\bar{x} = g(\omega)$ is unknown and introduce two soft properties: The property "$\bar{x} \cong x$" which means that "$\bar{x}$ is approximately equal to *x*" or "*x* is the approximate value of $\bar{x}$", and the property "$\bar{x} \,\tilde{\in}\, D_x$" (where $D_x \subseteq X$) which means that "the approximate value of $\bar{x}$ belongs to D_x" or "$\bar{x}$ approximately belongs to D_x". For the fixed *x* a soft property concerning *x* becomes a proposition in multi-valued logic and its logic value belongs to [0,1]. The logic values of our properties are denoted by *v* and are called *certainty indexes*. The variable $\bar{x}$ is called an *uncertain variable*. Two versions of uncertain variables have been introduced.

Definition 1 (uncertain variable): The uncertain variable $\bar{x}$ is defined by the set of values *X*, the function $h(x) = v(\bar{x} \cong x)$ such that $\max h(x) = 1$ (i.e. the certainty index that $\bar{x} \cong x$, given by an expert) and the following definitions:

$$v(\bar{x} \,\tilde{\in}\, D_x) = \begin{cases} \max\limits_{x \in D_x} h(x) & \text{for } D_x \neq \varnothing \\ 0 & \text{for } D_x = \varnothing \,, \end{cases} \tag{1}$$

$$v(\bar{x} \,\tilde{\notin}\, D_x) = 1 - v(\bar{x} \,\tilde{\in}\, D_x)\,,$$

$$v(\bar{x} \,\tilde{\in}\, D_1 \vee \bar{x} \,\tilde{\in}\, D_2) = \max\{v(\bar{x} \,\tilde{\in}\, D_1), v(\bar{x} \,\tilde{\in}\, D_2)\}\,,$$

$$v(\bar{x} \,\tilde{\in}\, D_1 \wedge \bar{x} \,\tilde{\in}\, D_2) = \begin{cases} \min\{v(\bar{x} \,\tilde{\in}\, D_1), v(\bar{x} \,\tilde{\in}\, D_2)\} & \text{for } D_1 \cap D_2 \neq \varnothing \\ 0 & \text{for } D_1 \cap D_2 = \varnothing . \end{cases}$$

The function $h(x)$ is called a *certainty distribution*. □

Definition 2 (*C*-uncertain variable): *C*-uncertain variable $\bar{x}$ is defined by the set of values *X*, the function $h(x) = v(\bar{x} \cong x)$ given by an expert, and the following definitions:

$$v_c(\bar{x} \,\tilde{\in}\, D_x) = \frac{1}{2}[\max_{x \in D_x} h(x) + 1 - \max_{x \in X - D_x} h(x)]\,, \tag{2}$$

$$v_c(\bar{x} \,\tilde{\notin}\, D_x) = 1 - v_c(\bar{x} \,\tilde{\in}\, D_x),$$

$$v_c(\bar{x} \,\tilde{\in}\, D_1 \vee \bar{x} \,\tilde{\in}\, D_2) = v_c(\bar{x} \,\tilde{\in}\, D_1 \cup D_2),$$

$$v_c(\bar{x} \,\tilde{\in}\, D_1 \wedge \bar{x} \,\tilde{\in}\, D_2) = v_c(\bar{x} \,\tilde{\in}\, D_1 \cap D_2). \quad \square$$

The certainty distribution for a particular x evaluates the expert's opinion that $\bar{x} \cong x$. In the case of *C*-uncertain variable the expert's knowledge is used in a better way but the calculations are more complicated.

A *mean value* of the uncertain variable $\bar{x}$ is defined as follows

$$M_x(\bar{x}) = \int_X x h(x) dx \cdot [\int_X h(x) dx]^{-1}. \tag{3}$$

3 Basic decision problem

The uncertain variables may be used in the formulation and solving a decision problem for an uncertain plant ([6]). Let us consider a static (memory less) plant with the input vector $u \in U$ and the output vector $y \in Y$. If the plant is described by a function $y = \Phi(u)$ then the basic decision problem consists in finding the decision u^* such that $\Phi(u^*) = y^*$ where y^* is a desirable output value. Let us assume that the plant is described by a relation $R(u, y) \subset U \times Y$, which is not a function. The description in the form of a relation (a *relational knowledge representation*) means that the plant is nondeterministic, i.e. the fixed input u determines a set of possible outputs

$$D_y(u) = \{y \in Y : (u, y) \in R(u, y)\}.$$

For the requirement $y \in D_y$ where $D_y \subset Y$ is given by a user, the decision problem may consist in determining the largest set of the decisions $D_u \subset U$ such that the implication $u \in D_u \to y \in D_y$ is satisfied.

Consider now the plant described by $R(u, y; x) \subset U \times Y$ where $x \in X$ is an unknown vector parameter which is assumed to be a value of an uncertain variable $\bar{x}$ described by $h(x)$ given by an expert. Now the set of all possible outputs is as follows

$$D_y(u; x) = \{y \in Y : (u, y) \in R(u, y; x)\}. \tag{4}$$

For the requirement $y \in D_y$ we can formulate and solve the following **decision (or control) problem:** Given R, $h(x)$ and D_y, find $\hat{u}$ maximizing the certainty index

$$v(u) = v[D_y(u; \bar{x}) \,\tilde{\subseteq}\, D_y] = v[u \,\tilde{\in}\, D_u(\bar{x})] \tag{5}$$

where $\tilde{\subseteq}, \tilde{\in}$ mean that the properties are satisfied for the approximate value x, and

$$D_u(x) = \{u \in U : D_y(u; x) \subseteq D_y\}. \tag{6}$$

The optimal decision $\hat{u}$ maximizes the certainty index that the set of possible outputs approximately belongs to D_y. It is easy to note that

$$v(u) = v[\bar{x} \,\tilde{\in}\, D_x(u)] = \max_{x \in D_x(u)} h(x) \tag{7}$$

where $D_x(u) = \{x \in X : D_y(u; x) \subseteq D_y\} = \{x \in X : u \in D_u(x)\}$.

Denote by x^* the value maximizing $h_x(x)$. Then the set of the optimal decisions $D_u = \{u \in U : x^* \in D_x(u)\}$ and $v(\hat{u}) = 1$. If $\bar{x}$ is considered as *C*-uncertain variable then, according to (2), one should determine $\hat{u}_c$ maximizing

$$v_c(u) = \frac{1}{2}\{v[\bar{x} \,\tilde{\in}\, D_x(u)] + 1 - v[\bar{x} \,\tilde{\in}\, X - D_x]\}. \tag{8}$$

4 Decision problem for the plant with random parameters

Consider a plant described by $R(u, y; z, x, w) \subset U \times Y$ where $z \in Z$ is a vector of external disturbances which may be measured, $x \in X$ is a value of an uncertain variable $\bar{x}$ considered in Sec. 3, and $w \in W$ is a value of a random variable $\tilde{w}$ described by a probability density $f(w)$. In general w is a vector and W is a vector space. Now the sets in (4) and (6) depend on z and w:

$$D_y(u; z, x, w) = \{y \in Y : (u, y) \in R(u, y; z, x, w)\}, \tag{9}$$

$$D_u(z, x, w) = \{u \in U : D_y(u; z, x, w) \subseteq D_y\}. \tag{10}$$

Then, v and v_c in (5), (7) and (8) depend on z and w:

$$v(u; z, w) = v[D_y(u; z, \bar{x}, w) \,\tilde{\subseteq}\, D_y] = v[u \,\tilde{\in}\, D_u(z, x, w)],$$

$$v(u; z, w) = \max_{x \in D_x(u; z, w)} h(x)$$

where

$$D_x(u; z, w) = \{x \in X : D_y(u; z, x, w) \subseteq D_y\}$$
$$= \{x \in X : u \in D_u(z, x, w)\},$$

and

$$v_c(u; z, w) = \frac{1}{2}\{\max_{x \in D_x(u; z, w)} h(x) + 1 - \max_{x \in X - D_x(u; z, w)} h(x)\}. \tag{11}$$

Consequently, the optimal decisions $\hat{u}$ and $\hat{u}_c$ depend on z and w:

$$\hat{u}(z, w) = \arg\max_{u \in U} v(u; z, w),$$

$$\hat{u}_c(z, w) = \arg\max_{u \in U} v_c(u; z, w).$$

Now two versions of the decision problem may be proposed:

Decision (control) problem – version I: Given R, $h(x)$, $f(w)$ and D_y, find u_1 or u_{c1} maximizing the expected value of $v(u;z,w)$ or $v_c(u;z,w)$, respectively.
Then

$$u_1 = \arg\max_{u \in U} E_w[v(u;z,\widetilde{w})] \stackrel{\Delta}{=} \Psi_1(z)$$

where

$$E_w[v(u;z,\widetilde{w})] = \int_W v(u;z,w) f(w) dw \stackrel{\Delta}{=} G(u,z) . \qquad (12)$$

The decision $u_{c1} = \Psi_{c1}(z)$ is determined in analogous form, with $v_c(u;z,w)$ in the place of $v(u;z,w)$.

Decision (control) problem – version II: Given R, $h(x)$, $f(w)$ and D_y, find u_2 or u_{c2} as the expected value of $\hat{u}(z,w)$ or $\hat{u}_c(z,w)$, respectively.

Then

$$u_2 = E_w[\hat{u}(z,\widetilde{w})] = \int_W \hat{u}(z,w) f(w) dw \stackrel{\Delta}{=} \Psi_2(z) ,$$

$$u_{c2} = E_w[\hat{u}_c(z,\widetilde{w})] = \int_W \hat{u}_c(z,w) f(w) dw \stackrel{\Delta}{=} \Psi_{c2}(z) .$$

The results in versions I and II may be different, i.e. in general $u_1 \neq u_2$ and $u_{c1} \neq u_{c2}$ (see example in Sec. 6). They have different practical interpretations. In version I u_1 (or u_{c1}) is a decision maximizing the mean value of the certainty index that the requirement $y \in D_y$ is approximately satisfied, where the mean certainty index is understood in a probabilistic sense as an expected value. In version II as an optimal decision u_2 (or u_{c2}) we accept the mean value of the decision maximizing the certainty index that the requirement is approximately satisfied for the given *w*. Using the description and names presented in [3, 6] we may say that $\mathrm{KD} = < \hat{u}(z,w) >$ is a *knowledge of the decision making* (or a random decision algorithm in an open-loop system), obtained from the *knowledge of the plant*

$$\mathrm{KP} = < R(u,y;z,x,w),\ h(x) >$$

given by an expert. In versions I and II the different forms of a *determinization* consisting in replacing an uncertain description by a corresponding deterministic model (see [3, 6]) have been applied. In version I the function $v(u;z,w)$ is replaced by $G(u,z)$ and in version II the random decision algorithm $\hat{u}(z,w)$ is replaced by the deterministic control algorithm $\Psi_2(z)$. Two versions of the knowledge-based decision making in open-loop systems are illustrated in Figs. 1 and 2.

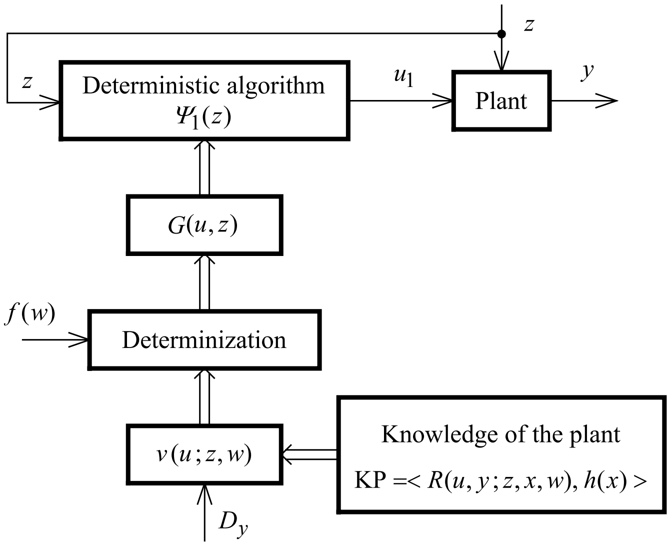

Figure 1: Open-loop control system – version I.

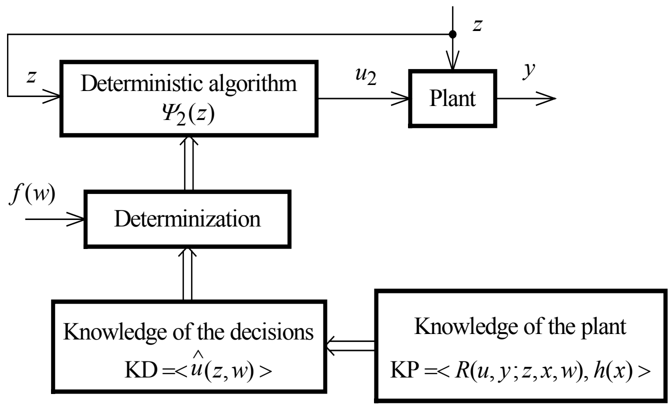

Figure 2: Open-loop control system – version II.

5 Another formulation of the decision problem

It is possible to inverse the order of the considerations concerning the uncertain and random parameters and to introduce another approach to the decision problem, in which the certainty index that $D_y(y;z,w,\bar{x}) \mathrel{\tilde{\subseteq}} D_y$ will be replaced by the probability

$$P[D_y(u;z,x,\widetilde{w}) \subseteq D_y] = \int_{D_w(u;z,x)} f(w) dw \stackrel{\Delta}{=} p(u;z,x) \quad (13)$$

where

$$D_w(u;z,x) = \{w \in W : D_y(u;z,x,w) \subseteq D_y\}$$
$$= \{w \in W : u \in D_u(z,x,w)\},$$

$D_y(u;z,x,w)$ and $D_u(z,x,w)$ are determined by (9) and (10), respectively.

Decision (control) problem – version I: Given *R*, $h(x)$, $f(w)$ and D_y, find u_1 maximizing the mean value of $p(u;z,x)$.

Then, according to (3)

$$u_1 = \arg\max_{u \in U} M_x[p(u;z,\bar{x})]$$

$$= \arg\max_{u \in U} \int_X p(u;z,x)h(x) \cdot [\int_X h(x)dx]^{-1} \triangleq \Psi_1(z). \quad (14)$$

Decision (control) problem – version II: Given R, $h(x)$, $f(w)$ and D_y, find u_2 as a mean value of $\hat{u}(z,x)$ where $\hat{u}(z,x)$ is the decision maximizing the probability (13), i.e.

$$\hat{u}(z,x) = \arg\max_{u \in U} p(u;z,x) .$$

Then

$$u_2 = M_x[\hat{u}(z,\bar{x})] = \int_X \hat{u}(z,x)h(x)dx \cdot [\int_X h(x)dx]^{-1} . \quad (15)$$

The considerations for C-uncertain variables have an analogous form. The decisions (14) and (15) have no evident practical interpretation and the approach presented in this section is not recommended.

6 Example

To illustrate the presented approach let us consider a simple example with a plant without disturbances, described by inequality

$$xu \le y \le \frac{xu}{w}$$

where u, y, x, w are one-dimensional positive variables and $w < 1$. For $D_y = [y_1, y_2]$, according to (10), the set $D_u(x,w)$ is determined by inequality

$$\frac{y_1}{x} \le u \le \frac{wy_2}{x}$$

and the set $D_x(u;w)$ is determined by inequality

$$\frac{y_1}{u} \le x \le \frac{wy_2}{u} .$$

Assume that x is a value of an uncertain variable $\bar{x}$ with triangular certainty distribution:

$$h(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le \frac{1}{2} \\ -2x+2 & \text{for} \quad \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise}. \end{cases}$$

Using (11) it is easy to obtain the following results for the given w ([6]):

$$v_c(u;w) = \begin{cases} \frac{y_2 w}{u} & \text{for} \quad u \ge y_1 + wy_2 \\ 1 - \frac{y_1}{u} & \text{for} \quad y_1 \le u \le y_1 + wy_2 \\ 0 & \text{for} \quad u \le y_1 \end{cases},$$

$$\hat{u}_c(w) = y_1 + wy_2 . \quad (16)$$

Now assume that w is a value of a random variable with rectangular probability density:

$$f(w) = \begin{cases} \beta^{-1} & \text{for} \quad 0 \le w \le \beta \\ 0 & \text{otherwise}, \end{cases}$$

$\beta < 1$.

Then, according to (12) with $v_c(u;w)$ instead of $v(u;w)$, we may obtain the following formula for $G(u) = E_w[v_c(u;\tilde{w})]$:

for $u \le y_1$

$$G(u) = 0 ,$$

for $y_1 \le u \le \beta y_2 + y_1$

$$G(u) = \frac{1}{2\beta}\frac{y_2}{u}\left(\frac{u-y_1}{y_2}\right)^2 + \frac{1}{\beta}\left(1-\frac{y_1}{u}\right)\left(\beta - \frac{u-y_1}{y_2}\right),$$

for $u \ge \beta y_2 + y_1$

$$G(u) = \frac{\beta y_2}{2u} .$$

To obtain u_{c1} the function $G(u)$ should be maximized with respect to u. In version II, using (16) we have $u_{c2} = y_1 + E_w(\tilde{w})y_2 = y_1 + \frac{\beta}{2}y_2$. For the numerical data the functions $G(u)$ are presented in Fig. 3 and the results are as follows:

Data	Results
$\beta = 0.5,\ y_1 = 1,\ y_2 = 4$	$u_{c1} = 3,\ G(u_{c1}) = 0.33,\ u_{c2} = 2$
$\beta = 0.5,\ y_1 = 2,\ y_2 = 12$	$u_{c1} = 8,\ G(u_{c1}) = 0.38,\ u_{c2} = 5$
$\beta = 0.8,\ y_1 = 2,\ y_2 = 12$	$u_{c1} = 7,\ G(u_{c1}) = 0.47,\ u_{c2} = 6.8$

It is easy to show that

$$u_{c1} = \arg\max_u G(u) = \beta y_2 + y_1 , \quad G(u) = \frac{\beta y_2}{2(\beta y_2 + y_1)} .$$

For example, for $\beta = 0.5$, $y_1 = 1$ and $y_2 = 4$ we obtain $u_{c1} = 3$, $G(u) = \frac{1}{3}$ (see Fig. 3).

In version II, using (16) we have $u_{c2} = y_1 + E_w(\tilde{w})y_2 = y_1 + \frac{\beta}{2}y_2 \ne y_{c1}$.

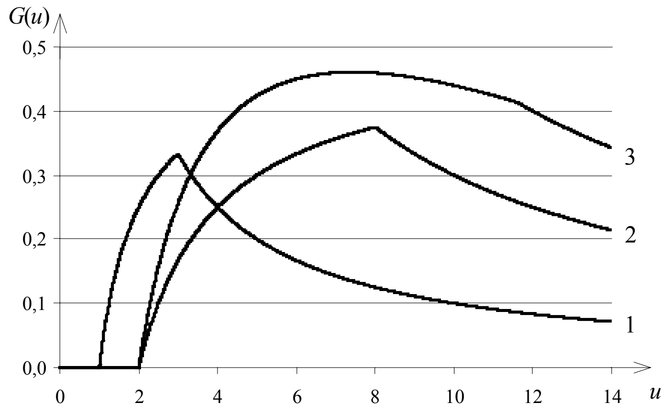

Figure 3: Relationship between the expected certainty index and *u*; 1. $\beta = 0.5$, $y_1 = 1$, $y_2 = 4$, 2. $\beta = 0.5$, $y_1 = 2$, $y_2 = 12$, 3. $\beta = 0.8$, $y_1 = 2$, $y_2 = 12$.

7 Two-level description of the uncertainty

The approaches analogous to those presented in Secs. 4 and 5 may be applied to uncertain systems with two-level description of the uncertainty [8]. Two cases of such a description may be considered.

Case 1

The plant is described by the relation $R(u, y; z, x) \subset U \times Y$ where the unknown vector parameter $x \in X$ is a value of an uncertain variable $\bar{x}$ described by the certainty distribution $h(x; w)$ given by an expert. In this notation $w \in W$ denotes the unknown vector parameter which is assumed to be a value of a random variable $\widetilde{w}$ described by the probability density $f(w)$. In this case the certainty distribution may be considered as a description of the uncertainty on the first level, and the probability density – as a description on the second level (or the second order uncertainty).

The certainty indexes v and v_c in (5), (7) and (8) depend on z and w :

$$v(u; z, w) = \max_{x \in D_x(u;z)} h(x; w)$$

where

$$D_x(u; z) = \{x \in X : D_y(u; z, x) \subseteq D_y\} ,$$

$$D_y(u; z, x) = \{y \in Y : (u, y) \in R(u, y; z, x)\} ,$$

and

$$v_c(u; z, w) = \frac{1}{2}\{ \max_{x \in D_x(u;z)} h(x; w) + 1 - \max_{x \in X - D_x(u;z)} h(x; w)\} .$$

Consequently, the optimal decisions $\hat{u}(z, w)$ and $\hat{u}_c(z, w)$ depend on z and w , and the further considerations are the same as in Sec. 4.

Case 2

The plant is described by the relation $R(u, y; z, w) \subset U \times Y$ where the unknown vector parameter $w \in W$ is a value of a random variable $\widetilde{w}$ described by the probability density $f(w; x)$. In this notation $x \in X$ denotes the unknown vector parameter which is assumed to be a value of an uncertain variable $\bar{x}$ characterized by the certainty distribution $h(x)$ given by an expert. In this case the probability density may be considered as a description of the uncertainty on the first level, and the certainty distribution – as a description on the second level (or the second order uncertainty).

Now, the probability introduced in Sec. 5 is as follows

$$p(u; z, x) = \int_{D_w(u;z)} f(w; x) dw$$

where

$$D_w(u; z) = \{w \in W : D_y(u; z, w) \subseteq D_y\} ,$$

$$D_y(u; z, w) = \{y \in Y : (u, y) \in R(u, y; z, w)\} .$$

Consequently, the optimal decision $\hat{u}(z, x)$ maximizing the probability $p(u; z, x)$ depends on z and x , and the further considerations are the same as in Sec. 5. The considerations for *C*-uncertain variables have an analogous form.

The two-level descriptions of the uncertainty in the cases 1 and 2 have the different interpretations. In the case 1 we consider a group of experts giving the same form of the certainty distributions $h(x, w)$ with the different values of w chosen randomly from a set of variables described by the probability density $f(w)$. In the case 2 the unknown value w in the knowledge representation has been chosen randomly from a set of variables described by the probability density $f(w; x)$ with the unknown parameter w characterized by an expert in the form of the certainty distribution $h(x)$.

8 Conclusions

A method of decision making for a static plant described by a relation with uncertain and random parameters has been presented. Two approaches to the decision problem described in Sec. 4 may be recommended. The similar formulations of the decision problem may be applied for the uncertain plant with tow-level description of the uncertainty.

The determination of the decision in version I may be more complicated than in version II, but the result in version I has better practical interpretation. Numerical examples and simulations have shown that the parameters in the certainty distribution $h(x)$ given by an expert may have a significant influence on the quality of the control. It is then reasonable to apply an adaptation consisting in self-adjusting of these parameters in the system with a real plant or with its simulator.

It is worth noting that the formulations and solutions of the decision problem under consideration may be presented for a discrete case, i.e. for a case with finite sets X and W : $X = \{x_1, x_2, ..., x_m\}$, $W = \{w_1, w_2, ..., w_n\}$. Then the probability density $f(w)$ is replaced by the probability distribution

$$P(\widetilde{w} = w_i) \stackrel{\Delta}{=} p_i , \qquad i = 1,...,n$$

and the integrals in the formulas for mean values are replaced by the sums. For example, the formulas analogous to (3) and (12) have the forms

$$M_x(\overline{x}) = \sum_{j=1}^{m} x_j h(x_j) \cdot [\sum_{j=1}^{m} h(x_j)]^{-1} ,$$

$$E_w[v(u; z, \widetilde{w})] = \sum_{i=1}^{n} v(u; z, w_i) p_i ,$$

respectively.

The idea presented in the paper may be extended to dynamical systems and systems with a distributed knowledge [5, 8], for the stability analysis [7] and for the recognition problem [11], described by a knowledge representation with uncertain and random parameters.

Acknowledgements

This work was supported by the Polish State Committee for Scientific Research under Grant No. 7 T11A 039 20.

References

[1] Z. Bubnicki. "General approach to stability and stabilization for a class of uncertain discrete nonlinear systems", *International Journal of Control*, **73**, pp. 1298-1306, (2000).

[2] Z. Bubnicki. "Uncertain variables and their applications for a class of uncertain systems", *International Journal of Systems Science*, **32**, pp. 651-659, (2001).

[3] Z. Bubnicki. "A unified approach to descriptive and prescriptive concepts in uncertain decision systems", *Proc. European Control Conference ECC 2001*, Porto, Portugal, (2001).

[4] Z. Bubnicki. "Uncertain variables and their application to decision making", *IEEE Trans. on SMC, Part A: Systems and Humans*, **31**, no. 6, pp. 587-596, (2001).

[5] Z. Bubnicki. "Application of uncertain variables to decision making in a class of distributed computer systems", *Proc. of 17th IFIP World Computer Congress,* **IIP**, Montreal, pp. 261-264, (2002).

[6] Z. Bubnicki. *Uncertain Logics, Variables and Systems*. Springer-Verlag; London, Berlin, (2002).

[7] Z. Bubnicki. "Stability and stabilization of discrete systems with random and uncertain parameters", *Proc. of 15th IFAC World Congress*, Barcelona, (2002).

[8] Z. Bubnicki. *Analysis and Decision Making in Uncertain Systems*. Springer-Verlag; London, Berlin, (2004) (in press).

[9] G. J. Klir, T. A. Folger. *Fuzzy Sets, Uncertainty, and Information*. Prentice-Hall, Englewood Cliffs, NJ, (1988).

[10] R. Kruse, E. Schwecke, J. Heinsohn. *Uncertainty and Vagueness in Knowledge Based Systems: Numerical Methods*. Springer-Verlag; Berlin, (1991).

[11] M. Szala. "Two-level pattern recognition in a class of knowledge-based systems", *Knowledge-Based Systems*, **15/1-2**, pp. 95101, (2002).

[12] Y.Y. Yager. "Fuzzy modeling for intelligent decision making under uncertainty", *IEEE Trans. on SMC, Part B: Cybernetics*, **30**, pp. 60-70, (2000).