A PIECEWISE SMOOTH SWITCHING CONTROL ALGORITHM FOR CHAINED SYSTEMS*

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Abstract

In this paper, a simple control approach is proposed for the multi-chain systems. This method is based on a switching algorithm of two sets of smooth steering laws. Global exponential convergence of all states is guaranteed. More importantly, the convergence rate is given explicitly. Two concrete methods are proposed for the design of steering laws. One is based on the backstepping design, another is based on the use of Riccati equation. Extension of this approach to some driftless systems is also discussed.

This approach is very flexible in the sense that the requirement on the generator is very weak. Therefore, there is a great freedom in the determination of the generator which may be used to satisfy other control specifications.

1 Introduction

As a typical class of nonholonomic systems, the chained systems has been studied extensively and many methods have been proposed. Time-varying methods are proposed in [8, 10]. A feedforward steering method is proposed in [7]. Exponential stabilization methods are proposed in [12, 3]. An elegant discontinuous control is proposed in [1]. In [2] a relatively simple iterative method is proposed by combining backstepping with time-varying techniques. In [9] a switching control method is proposed in which the generator is switched between a positive constant and a negative constant periodically. See also [13, 14] and the book of Sastry[11].

Through these works it has been made clear that theorectically speaking the chained systems can be stabilized relatively easily. So the focus of the next stage should be on the physical feasibility of control laws. It is in this aspect that there are some issues need to be addressed. First of all, it is assumed in these previous works that the system can take any path freely in the whole space. However, in practical problems such as the parking of cars the path of the system is usually constrained. Therefore, these methods may not be applied directly to such practical problems. Secondly, the input of the model is velocity physically, hence it is at least piecewise continuously differentiable. So in order to construct a control law that is physically feasible, the control input must possess this property. It has to be admitted that this viewpoint is lacking in previous researches. To put it concretely, the generator v_0 is determined purely mathematically in order to prove stability, without any consideration on the dynamics of v_0. Moreover, in most of these methods other than [1, 9], the control laws are complicated in general.

In this paper, a control approach is proposed for the multi-chain systems, which is quite simple in structure and is able to resolve the problems mentioned above. The proposed approach is based on the switching of 2 sets of smooth steering control laws and the switching of steering laws occurs only at the time instants when the generator is zero. Therefore, the steering input is continuous and piecewise smooth. Hence all inputs are physically implementable. Global exponential convergence of all states is guaranteed. More importantly, the convergence rate is given explicitly.

For the design of steering control laws, two concrete methods are proposed. One is based on the backstepping design, another is based on the use of Riccati equation. The extension of this approach to a class of driftless systems is also discussed.

This approach is very flexible in the sense that the requirement on the generator is extremely weak. So there is a great freedom in the determination of the generator which may be used to satisfy other control specifications, such as the avoidance of obstacles.

In the sequel, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalues for a square matrix A, and $\kappa(A)$ denotes the condition number $\lambda_{\max}(A)/\lambda_{\min}(A)$.

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2 Switching Control Algorithm

The following $n+1$ states system

$$\begin{aligned} \dot{z}_0 &= v_0 \\ \dot{z}_1 &= z_2 v_0 \\ &\vdots \\ \dot{z}_{n-1} &= z_n v_0 \\ \dot{z}_n &= v_1 \end{aligned} \tag{1}$$

gives the well-known one-chain system. Here, v_0 is called the generator and v_1 the steering input.

If a state vector $z := [z_1, \ldots, z_n]^T$ is defined and v_1 is set as v, then the one-chain system can be expressed as the following driftless system

$$\begin{cases} \dot{z}_0 = v_0 \\ \dot{z} = Azv_0 + Bv \end{cases} \tag{2}$$

in which

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3}$$

Obviously, $(A,\ B)$ is controllable. Meanwhile, the multi-chain system

$$\begin{aligned} \dot{z}_0 &= v_0 \\ \dot{z}_{1k} &= z_{2k} v_0 \\ &\vdots \\ \dot{z}_{(n_k-1)k} &= z_{n_k k} v_0 \\ \dot{z}_{n_k k} &= v_k, \quad k = 1, \ldots, m \end{aligned} \tag{4}$$

can also be expressed as (2) with $z = [Z_1^T, \cdots, Z_m]^T$, $Z_k = [z_{1k}, \ldots, z_{n_k k}]^T$, $v = [v_1, \cdots, v_m]^T$ and

$$A = \text{blk diag}[A_1 \ \cdots \ A_m],\ B = \text{blk diag}[B_1 \ \cdots \ B_m]. \tag{5}$$

Here $A_k,\ B_k$ are compatible matrices with the same structures as those of (3) and n_k rows. Therefore, $(A,\ B)$ is also controllable in this case.

For this reason, the steering control of this driftless system (2) is considered in this section. The obtained result will be specialized to chained systems in the following sections.

As in the chained systems, v_0 is called the generator and v called the steering input in the driftless system (2). It will be shown that this system can be controlled by a steering control law in the form $v = F(z)v_0$.

Now let v have the following structure

$$v = uv_0. \tag{6}$$

Then (2) becomes

$$\begin{aligned} \dot{z}_0 &= v_0 &(7)\\ \dot{z} &= (Az + Bu)v_0. &(8) \end{aligned}$$

Assumption (A, B) is controllable.

Subject to this assumption, there exist state feedback gains F_p and F_n such that $(A - BF_p)$ and $-(A - BF_n)$ are stabilized. Hence, for any compatible matrices $Q_p,\ Q_n > 0$, the following two Lyapunov equations have unique positve definite solutions P_p and P_n respectively

$$(A - BF_p)^T P_p + P_p(A - BF_p) + Q_p = 0 \tag{9}$$
$$-(A - BF_n)^T P_n - P_n(A - BF_n) + Q_n = 0. \tag{10}$$

The following theorem is derived by using the stabilizing feedback gains F_p and F_n.

Theorem 1 *Assume that $(A - BF_p)$ and $-(A - BF_n)$ are stable.*
(1) Let $u = -F_p z$ when $v_0 > 0$. Then

$$\begin{aligned} \|z(t)\| &\leq \sqrt{\kappa(P_p)}\|z(0)\| e^{-\frac{\lambda_{\min}(Q_p P_p^{-1})}{2} \int_0^t |v_0| ds} \\ &= \sqrt{\kappa(P_p)}\|z(0)\| e^{-\frac{\lambda_{\min}(Q_p P_p^{-1})}{2} |z_0(t) - z_0(0)|}. \end{aligned}$$

(2) Let $u = -F_n z$ when $v_0 < 0$. Then

$$\begin{aligned} \|z(t)\| &\leq \sqrt{\kappa(P_n)}\|z(0)\| e^{-\frac{\lambda_{\min}(Q_n P_n^{-1})}{2} \int_0^t |v_0| ds} \\ &= \sqrt{\kappa(P_n)}\|z(0)\| e^{-\frac{\lambda_{\min}(Q_n P_n^{-1})}{2} |z_0(t) - z_0(0)|}. \end{aligned}$$

(Proof) (1) Set a candidate of Lyapunov function as $V_p(z) = z^T P_p z$. First of all, substitution of $u = -F_p z$ into Eq.(8) yields $\dot{z} = (A - BF_p)zv_0$. Then, differention of $V_p(z)$ along the trajectory and substitutions of this equation as well as Lyapunov equation (9) lead to

$$\dot{V}_p(z) = -z^T Q_p z \cdot |v_0|.$$

Since the eigenvalues of $P_p^{-1/2} Q_p P_p^{-1/2}$ are equal to those of $Q_p P_p^{-1}$, one has $z^T Q_p z \geq \lambda_{\min}(Q_p P_p^{-1})V$. Therefore,

$$\dot{V}_p \leq -\lambda_{\min}(Q_p P_p^{-1})|v_0| V_p$$

holds. Thus from the Comparison Principle

$$V_p(z(t)) \leq V_p(z(0)) e^{-\lambda_{\min}(Q_p P_p^{-1}) \int_0^t |v_0| ds}$$

is obtained. Further, the first norm bound on z is derived from this inequality by using $\lambda_{\min}(P_p)\|z(t)\|^2 \leq V_p(z(t))$ and $V_p(z(0)) \leq \lambda_{\max}(P_p)\|z(0)\|^2$.

(2) In this case, there holds with respect to $V_n(z) = z^T P_n z$ the following equation

$$\dot{V}_n(z) \ = \ -z^T Q_n z \cdot |v_0|.$$

Therefore, the first norm bound is derived by an analogous argument as above.

Finally, in both cases the second norm bounds follow from the fact that

$$\int_0^t |v_0(s)|ds = \left|\int_0^t v_0(s)ds\right| = |z_0(t) - z_0(0)|$$

because v_0 does not change sign and $v_0 = \dot{z}_0$. ◇

If $\int_0^t |v_0(s)|ds$ is a $\mathcal{K}_\infty$ class function of time t, then the last n states are stabilized in an exponential sense. However, in this process the boundedness of z_0 must be ensured. Also, the stabilization of z_0 must be achieved in some way. Therefore, a switching control algorithm is proposed for this purpose.

Algorithm 1 *Determine a finite interval* $[z_0^{\min},\ z_0^{\max}]$ *which is the admissible region of* z_0. *Control the system (2) by the algorithm below:*

1. *Given any initial position* $z_0(0)$*, initial velocity* $v_0(0)$*, norm bound* $\epsilon > 0$ *and finite steering period* $T > 0$.
   - *(a) If* $v_0(0) > 0$ *go to Step (c), if* $v_0(0) < 0$ *go to Step (b), if* $v_0(0) = 0$ *and* $z_0(0)$ *is close to* $z_0^{\max}$ *go to Step (b), if* $v_0(0) = 0$ *and* $z_0(0)$ *is close to* $z_0^{\min}$ *go to Step (c).*
   - *(b) Construct a generator* $v_0(t)$ *that is at least a* C^1 *function with* sgn $(v_0) = -1$ *and* $v_0(T) = 0$ *so that* $z_0(T)$ *is brought close to* $z_0^{\min}$. *Apply this* v_0 *and the corresponding steering input* v *to the system. If* $\|z(T)\| < \epsilon$ *go to Step 5, otherwise go to Step 2.*
   - *(c) Construct a generator* $v_0(t)$ *that is at least a* C^1 *function with* sgn $(v_0) = +1$ *and* $v_0(T) = 0$ *so that* $z_0(T)$ *is brought close to* $z_0^{\max}$. *Apply this* v_0 *and the corresponding steering input* v *to the system. If* $\|z(T)\| < \epsilon$*, go to Step 5. Otherwise, go to Step 3.*
2. *Reset* $t = T$ *as* $t = 0$. *Construct a generator* $v_0(t)$ *that is at least a* C^1 *function with* sgn $(v_0) = +1$ *and* $v_0(T) = 0$ *so that* $z_0(T)$ *is brought close to* $z_0^{\max}$. *Apply this* v_0 *and the corresponding steering input* v *to the system. If* $\|z(T)\| < \epsilon$ *go to Step 5.*
3. *Reset* $t = T$ *as* $t = 0$. *Construct a generator* $v_0(t)$ *that is at least a* C^1 *function with* sgn $(v_0) = -1$ *and* $v_0(T) = 0$ *so that* $z_0(T)$ *is brought close to* $z_0^{\min}$. *Apply this* v_0 *and the corresponding steering input* v *to the system. If* $\|z(T)\| < \epsilon$*, go to Step 5.*
4. *Return to Step 2.*
5. *Construct a linear feedback* $v_0 = -kz_0$ $(k > 0)$ *so that* z_0 *is stabilized. Apply this* v_0 *and the corresponding steering input* v *to the system.* ◇

Steps (1)-(4) are referred to as the *steering phase* in the sequel.

Remark 1 *Note that the steering control laws are switched only when* $v_0 = 0$. *Therefore, all inputs are continuous. In the case of parking, this means the steering velocity is switched after the car stops.*

In this algorithm, the admissible region of z_0 *is prescribed and* z_0 *only moves within the admissible region in the steering phase. By using this property, the avoidance of obstacles can be achieved. For example, in the parking the car should not collide with other cars and/or the garage. This is guaranteed by setting suitable* $z_0^{\min}$ *and* $z_0^{\max}$ *(refer to Fig. 1). Moreover, by using switching control the car can be parked without producing unrealistically large steering angle.* ◇

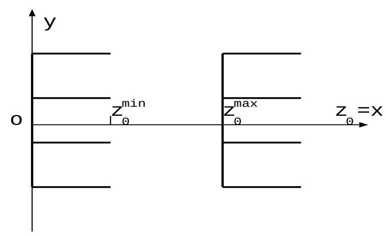

Figure 1: 2-lane parking lot (bold frame shows the parking space)

The following theorem shows that exponential convergence of all states is achieved by combining Algorithm 1 and the steering laws of Theorem 1.

Theorem 2 *Let* m_v *be the mean of* $|v_0|$ *in the steering phase and define constants*

$$\begin{aligned} c &= \frac{1}{2}\min\left\{\lambda_{\min}(Q_pP_p^{-1}),\ \lambda_{\min}(Q_nP_n^{-1})\right\} \\ \theta &= \frac{1}{T}\ln\sigma, \quad \sigma = \max\left\{\sqrt{\kappa(P_p)},\ \sqrt{\kappa(P_n)}\right\}. \end{aligned}$$

When $m_v > \theta/c$ *holds, Algorithm 1 achieves the exponential convergence of all states if the steering laws of Theorem 1 are applied to system (2). The convergence rate is greater than* $m_vc - \theta$.

(Proof) Let the total steering period be pT. Note that for $(k-1)T < t \le kT$ the state $z(t)$ satisfies the norm bound

$$\|z(t)\| \le \sigma\|z((k-1)T)\|e^{-c\int_{(k-1)T}^t |v_0(s)|ds}$$

in the kth switching no matter $v_0 > 0$ or $v_0 < 0$. Repeated applications of this norm bound yield

$$\|z(pT)\| \le \|z(0)\|e^{-(cm_v-\theta)pT}.$$

Therefore, under the given condition the exponential convergence of z is guaranteed. As for z_0, its convergence rate can be assigned arbitrarily in the last stage. ◇

Remark 2 *The convergence condition $m_v > \theta/c$ is equivalent to $(\sigma)^{1/c} < e^{m_v T}$. Since $m_v T$ is the average displacement of z_0 coordinate in the steering phase, this condition implies that the exponential convergence is guaranteed if a sufficiently long displacement of z_0 is allowed. However, this condition may be rather conservative since it is derived for the worst case.* ◇

This theorem indicates that when the proposed switching control algorithm is applied, the control design of driftless system (2) is reduced to the state feedback stabilization design of the following two linear systems

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{x} &= -Ax - Bu.\end{aligned}$$

The latter is much simpler. Any linear control method can be applied. Two of them will be discussed concretely in the following sections for the chained systems.

3 Optimal Control Based Design

Since (A, B) is controllable, the following two Riccati equations have respectively unique positive definite solutions for any compatible matrces $Q > 0$ and $R > 0$

$$A^T P_p + P_p A - P_p B R^{-1} B^T P_p + Q = 0 \quad (11)$$
$$(-A^T)P_n + P_n(-A) - P_n B R^{-1} B^T P_n + Q = 0. \quad (12)$$

Now, define two feedback matrices $F_p,\ F_n$ as

$$F_p = R^{-1} B^T P_p, \quad F_n = -R^{-1} B^T P_n. \quad (13)$$

Then, $(A - BF_p)$ and $-(A - BF_n)$ are stable and $F_p,\ F_n$ provide the feedback gains for the steering laws of Theorem 1 and the convergence rate can be computed using matrices $P_p,\ P_n$ and

$$Q_p = P_p B R^{-1} B^T P_p + Q,\ Q_n = P_n B R^{-1} B^T P_n + Q.$$

For one-chain systems, the feedback gains is computed as above. Meanwhile, for the multi-chain system (4), since $A,\ B$ matrices have a block diagonal structure, the design of optimal gains can be separated into that of each chain. That is, decentralized control is possible and this could yield a bettere estimate of convergence rate.

For $k = 1, \ldots, m$, solve Riccati equations

$$A_k^T P_{pk} + P_{pk} A_k - \frac{1}{r_k} P_{pk} B_k B_k^T P_{pk} + Q_k = 0 \quad (14)$$

$$(-A_k^T)P_{nk} + P_{nk}(-A_k) - \frac{1}{r_k} P_{nk} B_k B_k^T P_{nk} + Q_k = 0 \quad (15)$$

where $Q_k > 0,\ r_k > 0$. And define feedback vectors $f_{pk},\ f_{nk}$ accordingly as

$$f_{pk} = \frac{1}{r_k} B_k^T P_{pk}, \quad f_{nk} = -\frac{1}{r_k} B_k^T P_{nk}. \quad (16)$$

Then the overall feedback gains are given by

$$\begin{aligned}F_p &= \text{blk diag}[f_{p1}\ \cdots\ f_{pm}] \\ F_n &= \text{blk diag}[f_{n1}\ \cdots\ f_{nm}].\end{aligned}$$

Further, define the following matrices

$$\begin{aligned}Q_{pk} &= \frac{1}{r_k} P_{pk} B_k B_k^T P_{pk} + Q_k > 0 \\ Q_{nk} &= \frac{1}{r_k} P_{nk} B_k B_k^T P_{nk} + Q_k > 0.\end{aligned}$$

Corollary 1 *Let m_v be the mean of $|v_0|$ in the steering phase and define constants*

$$\begin{aligned}c &= \frac{1}{2} \min_{1 \le k \le m} \min\left\{\lambda_{\min}(Q_{pk} P_{pk}^{-1}),\ \lambda_{\min}(Q_{nk} P_{nk}^{-1})\right\} \\ \theta &= \frac{1}{T} \ln \sigma, \quad \sigma = \max_{1 \le k \le m} \max\left\{\sqrt{\kappa(P_{pk})},\ \sqrt{\kappa(P_{nk})}\right\}.\end{aligned}$$

When $m_v > \theta/c$ holds, Algorithm 1 achieves the exponential convergence of all states if the steering law

$$v = \begin{cases} -F_p z v_0 & v_0 > 0 \\ -F_n z v_0 & v_0 < 0 \end{cases} \quad (17)$$

is applied to system (4). The convergence rate is greater than $m_v c - \theta$.

(Proof) This is proved by applying

$$v_k = \begin{cases} -f_{pk} Z_k v_0 & v_0 > 0 \\ -f_{nk} Z_k v_0 & v_0 < 0 \end{cases}$$

to each chain, invoking Theorem 1 and using $\|z(t)\|^2 = \sum_{i=1}^m \|Z_k(t)\|^2$. ◇

4 Backstepping Based Design

An advantage of this approach is that the convergence rate is determined by the design parameters directly.

4.1 One-Chain System

The steering control law and the corresponding Lyapunov function are constructed recursively according to the following algorithm.

Algorithm 2 *For $i = 1, \ldots, n$, construct linear functions $L_i(z_1, \ldots, z_i)$ and quadratic functions $V_i(z_1, \ldots, z_i)$ as follows:*
1. Set

$$\begin{aligned}&x_1 = z_1,\ L_1(z_1) = c_1 \text{sgn}\ (v_0) z_1 \\ &V_1(z_1) = \frac{1}{2} x_1^2.\end{aligned}$$

2. For $i = 2, \ldots, n$, let

$$x_i = z_i + L_{i-1}(z_1, \ldots, z_{i-1})$$
$$L_i(z_1, \ldots, z_i) = L_{i-1}(z_2, \ldots, z_i) + x_{i-1} + c_i \text{sgn }(v_0) x_i$$
$$V_i(z_1, \ldots, z_i) = V_{i-1}(z_1, \ldots, z_{i-1}) + \frac{1}{2} x_i^2.$$

◇

Obviously, $x := [x_1, \ldots, x_n]^T$ is a linear function of z and can be expressed as

$$x = S(\text{sgn }(v_0)) z \tag{18}$$

by using a square matrix $S(\text{sgn }(v_0))$. Investigation of Algorithm 2 shows that $S(\text{sgn }(v_0))$ is a lower triangular matrix with all diagonal entries equal to 1. Thus, it is nonsingular. For simplicity, matrix $S(\text{sgn }(v_0))$ will be denoted by

$$S(\text{sgn }(v_0)) = S_\pm = \begin{cases} S_-, & v_0 < 0 \\ S_+, & v_0 > 0 \end{cases} \tag{19}$$

in the sequel.

The theorem below shows that the steering law is indeed obtained by the recursive computation above.

Theorem 3 *Suppose v_0 has the same sign and $c_i > 0$ ($\forall$ i). Then the steering law*

$$v_1 = -L_n(z_1, \ldots, z_n) v_0 \tag{20}$$

guarantees the following norm bounds

$$\begin{aligned} \|z(t)\| &\leq \sigma \|z(0)\| e^{-c \int_0^t \ |v_0(s)| ds} \\ &= \sigma \|z(0)\| e^{-c|z_0(t) - z_0(0)|} \end{aligned}$$

in which $\sigma = \max\left\{\sqrt{\kappa(S_-^T S_-)}, \sqrt{\kappa(S_+^T S_+)}\right\}$ *and* $c = \min_{1 \leq i \leq n} \{c_i\} > 0$.

(Proof) It is proved by induction that

$$\begin{aligned} \dot{V}_n &= -|v_0| \sum_{k=1}^n c_k x_k^2 + x_n[v_1 + L_n(z_1, \ldots, z_n) v_0] \\ &= -|v_0| \sum_{k=1}^n c_k x_k^2. \end{aligned}$$

From this equation and the definition of V_n, we have $\dot{V}_n \leq -2c|v_0|V_n$. Applying the Comparison Principle, it is derived that $V_n(t) \leq V_n(0) e^{-2c \int_0^t \ |v_0(s)| ds}$. Therefore, from $V_n = \|x\|^2/2$ one has

$$\|x(t)\| \leq \|x(0)\| e^{-c \int_0^t \ |v_0(s)| ds}.$$

As $\lambda_{\min}(S_\pm^T S_\pm) \|z\|^2 \leq \|x\|^2 = z^T S_\pm^T S_\pm z \leq \lambda_{\max}(S_\pm^T S_\pm)$ $\|z\|^2$, the norm bounds on z are obtained immediately. ◇

The convergence rate is provided by the corollary below. Its proof is the same as that of Theorem 2.

Corollary 2 *Let m_v be the mean of $|v_0|$ over the steering phase and define a constant $\theta = \ln \sigma / T$. If $m_v > \theta / c$, then Algorithm 1 realizes the exponential convergence of all states with a rate greater than $c m_v - \theta$ when the steering laws of Theorem 3 are applied.* ◇

4.2 Multi-Chain System

The result is readily extended to multi-chain systems (4).

Algorithm 3 *For $k = 1, \ldots, m$ and $i = 1, \ldots, n_k$, construct linear functions $L_{ik}(z_{1k}, \ldots, z_{ik})$ according to the following algorithm:*
1. Set

$$x_{1k} = z_{1k}, \ L_{1k}(z_1) = c_{1k} \text{sgn }(v_0) z_{1k}.$$

2. For $i = 2, \ldots, n_k$, let

$$\begin{aligned} x_{ik} &= z_{ik} + L_{(i-1)k}(z_{1k}, \ldots, z_{(i-1)k}) \\ L_{ik}(z_{1k}, \ldots, z_{ik}) &= L_{(i-1)k}(z_{2k}, \ldots, z_{ik}) + x_{(i-1)k} \\ &\quad + c_{ik} \text{sgn }(v_0) x_{ik}. \end{aligned}$$

◇

Define state vectors $X_k = [x_{1k}, \ldots, x_{n_k k}]^T$ and $x = [X_1^T, \ldots, X_m^T]^T$. Then the following relationship

$$X_k = S_{k,\pm} Z_k, \quad x = S_\pm z \tag{21}$$

holds in which $S_{k,\pm}$ is a nonsingular lower triangular matrix with a structure like the matrix $S_\pm$ of Eq. (19). Meanwhile, $S_\pm$ is a block diagonal matrix with $S_{k,\pm}$ as the kth diagonal block.

The following theorem holds. The proof is omitted.

Theorem 4 *Suppose v_0 has the same sign and $c_{ik} > 0$ ($\forall i, k$). Then for multi-chain system (4) the steering laws*

$$v_k = -L_{n_k k}(z_{1k}, \ldots, z_{n_k k}) v_0, \ k = 1, \ldots, m \tag{22}$$

achieve the norm bound

$$\|z(t)\| \leq \sigma \|z(0)\| e^{-c \int_0^t |v_0(s)| ds}$$

where $\sigma = \max_{1 \leq k \leq m} \max\left\{\sqrt{\kappa(S_{k,-}^T S_{k,-})}, \ \sqrt{\kappa(S_{k,+}^T S_{k,+})}\right\}$ *and* $c = \min_{1 \leq k \leq m} \min_{1 \leq i \leq n_k} \{c_{ik}\} > 0$.

Similar to Corollary 2, there holds

Corollary 3 *Let m_v be the mean of $|v_0|$ in the steering phase and define a constant $\theta = \ln \sigma / T$. If $m_v > \theta / c$, then the exponential convergence of all states is guaranteed for system (4) by Algorithm 1 when the steering law of Theorem 4 is applied. The convergence rate is greater than $m_v c - \theta$.*

5 Extension to Driftless Systems

Finally, for driftless system

$$\begin{cases} \dot{z}_0 = v_0 \\ \dot{z} = f(z)v_0 + g(z)v, \end{cases} \tag{23}$$

the following results can be derived. Here $g(z)$ is a matrix function and v is a vector input.

The following theorem can be proved based on Lyapunov converse theorem.

Theorem 5 *Algorithm 1 and the following steering law*

$$v = \begin{cases} u_1(z)v_0, & v_0 > 0 \\ u_2(z)v_0, & v_0 < 0 \end{cases}$$

guarantees the global/local exponential convergence of system (23) if the following conditions hold.

(1) There exist a control law $u = u_1(x)$ stabilizing the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

globally/locally exponentially and a control law $u = u_2(x)$ stabilizing the nonlinear system

$$\dot{x} = -f(x) - g(x)u$$

globally/locally exponentially.

(2) $f(x) + g(x)u_i(x)$ $(i = 1,\ 2)$ is continuously differentiable to the 1st order and has bounded Jacobian matrix $\partial(f + gu_i)/\partial x$ globally/locally.

(3) The admissible region of z_0 is sufficiently wide.

6 Conclusion

In this paper, a simple feedback control method has been proposed for multi-chain systems. This approach guarantees global exponential convergence and does not contain any singularities. The approach is based on a switching algorithm of two sets of steering control laws. Each steering law is given by a product of the generator and a linear time-invariant function of states.

This approach has the following distinctive features. First of all, the steering laws are switched only when the generator v_0 is zero, so all control inputs are continuous and piecewise smooth, therefore physically feasible. Secondly, it is revealed that the convergence of the switching algorithm is essentially determined by the admissible displacement of state z_0. Thirdly, the control law has an explicit convergence rate. Further, the constraint on the generator is very weak so that the proposed approach may be applied to stabilization problems with constraints such as obstacle avoidance.

Two concrete methods have been proposed for the design of the steering control laws. One is based on the backstepping design and another based on optimal control. Lastly, the extension to a class of driftless systems has also been discussed.

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