TURNPIKE THEOREMS BY A VALUE FUNCTION APPROACH

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Abstract

For a problem of calculus of variations in infinite horizon, linear with respect to the derivative, we use the theory of viscosity solutions to obtain a unique characterization of the value function by an Hamilton-Jacobi equation. This approach allows to extend in the scalar case known results on turnpike properties.

1 Introduction

In this paper we consider a problem of calculus of variations in infinite horizon whose objective $J[.]$ is given by

$$J[x(.)] = \lim_{T \to +\infty} \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt$$

where δ is a positive number and $l(.)$ is a real valued function on $\mathbb{R} \times \mathbb{R}$, linear w.r.t. his second argument. Our interest is the maximization of $J[.]$ on the paths $x(.)$ with fixed initial condition $x(0) = x_0$, for which the velocities respect some inequality constraints, like

$$\alpha \leq \dot{x}(t) \leq \beta \quad \forall t \geq 0$$

(α, β being real numbers).
Our goal in this paper is to study the so called Turnpike property which asserts, roughly speaking, in this framework, that there exists a particular solution $\bar{x}(.)$ (the "turnpike") such that, from any initial condition, an optimal trajectory reaches as fast as possible the path $\bar{x}(.)$ and coincides with this solution for any future time. To our knowledge it does not exist many theoretical results in this setting.
We underline that several papers on turnpikes are found in the literature but only few of them are relevant with our problem. These papers can be classified in two categories, depending on whether the Euler first order condition :

$$l_x(x(t), \dot{x}(t)) - l_{\dot{x}}(x(t), \dot{x}(t)) + \delta l_{\dot{x}}(x(t), \dot{x}(t)) = 0 \qquad (1)$$

is a differential equation, the regular case that we don't consider here, or an algebraic equation, a case we call singular and that is the aim of the present paper.
In the regular case, the Turnpike property is obtained when from all initial conditions it is optimal to reaches (asymptotically) a particular equilibrium of the problem, that is to say an optimal constant solution of (1). This situation can be compared with the one presented below, the equilibrium being $\bar{x}(.)$.
In this paper we will consider a lagrangian given by

$$l(x, p) = A(x) + B(x)p$$

for $A(.)$ end $B(.)$ two functions. In this setting (1) becomes

$$C(x) := A'(x) + \delta B(x) = 0$$

and can only possess constant solutions. Therefore from most of the initial conditions, there does'nt exist a solution of (1). If emating from $\bar{x}$ such a solution $\bar{x}(.) = \bar{x}$ exists and is optimal, for instance if the sufficient optimality conditions are satisfied (concavity of the objective and transversality condition), then this solution is what we called below a turnpike. The problem is then to characterize the optimal solutions (if any), emanating from x_0 that are not solution of (1). It is relevant to introduce the Most Rapid Approach Paths (MRAPs), which are the admissible trajectories that join x_0 to $\bar{x}$ as quickly as possible and then rest at $\bar{x}$ for all future time. Following a method proposed by Miele [14], which is based on the Green theorem, one can obtain sufficient conditions for the optimality of the MRAPs [12] when the Euler equation $C(x) = 0$ possesses only one solution $\overline{x}$. More precisely if for the unique solution of (1), $\bar{x}$, the following condition

$$\forall x, \quad C(c)(\overline{x} - c) \geq 0$$

is fulfilled, the MRAPs are optimal.
Without loss of generality, we shall take $\alpha = -1$ and $\beta = 1$. In this paper we propose a new optimality condition of the MRAPs which is necessary and sufficient. We consider more general growth assumptions than the usual ones when one deals with absolutely continuous paths. Our approach is based on a characterization of the value function of a particular Hamilton-Jacobi equation, in terms of viscosity solutions ([13]). This approach allows also to consider the case of a multiplicity of singular solutions of the Euler equation, and therefore provides a criterion for the choice of the turnpikes in competition, depending on the initial condition. In the last section of the paper, an example which exhibits the different possible occurrences of turnpikes (one or several) is given.

2 Statement of the problem and assumptions

Let us consider the following set

$$Adm(x_0) = \left\{ x(.) : [0, \infty[\to \mathbb{R}, AC, \left| \begin{array}{l} x(0) = x_0, \\ \dot{x}(t) \in [-1, +1] \; a.e. \end{array} \right. \right\}$$

whose elements are called the admissible paths. We also consider the functional, when it converges, given by :

$$J[x(.)] = \int_0^{\to\infty} e^{-\delta t}[A(x(t)) + B(x(t))\dot{x}(t)]dt \qquad (2)$$

and we are interested by the following optimal control problem :

$$\max_{x(.)\in Adm_{x_0}} \quad J[x(.)] \qquad (3)$$

We denote by V the value function associated to this problem :

$$V(x_0) = \sup_{x(.)\in Adm_{x_0}} \quad J[x(.)] \qquad (4)$$

We assume that
$(\mathbf{H_1})$: $A(.)$ is twice differentiable and $B(.)$ is differentiable.
$(\mathbf{H_2})$: There exists two real numbers $k > 0$ and $\gamma < \delta$ such that for all x

$$\max\left(\,|A(x)|\,,\,|A'(x)+\delta B(x)|\,,\,|A''(x)+\delta B'(x)|\,\right) \le k e^{\gamma|x|}$$

3 The Hamilton-Jacobi Equation

The uniqueness of (generalized) solutions of first order partial differential equations, defined on unbounded sets is known to be a hard question and can be obtained only for well chosen classes of functions (see for instance [1]). For this reason we do not characterize the value function V itself, but a transformation of it, denoted Z in the sequel, which is solution of a particular Hamilton-Jacobi equation, for which we are able to state a result of unique characterization.

Proposition 3.1 *Under the assumption (H_1) et (H_2) the function Z defined by*

$$Z(x) = e^{-\eta\sqrt{x^2+1}}\left(V(x) - \frac{A(x)}{\delta}\right)$$

where η satisfies $\gamma < \eta < \delta$, is the unique viscosity solution in the class of B.U.C. (bounded uniformly continuous) functions of the following Hamilton-Jacobi equation :

$$\begin{aligned}\delta Z(x) - \Bigg| Z'(x) + \eta\frac{x}{\sqrt{x^2+1}}Z(x) \\ + \frac{e^{-\eta\sqrt{x^2+1}}}{\delta}(A'(x)+\delta B(x))\Bigg| = 0\end{aligned} \qquad (5)$$

Proof : The proof is split in four lemmas, for which we do not give proofs because of lack of space.

Lemma 3.2 *Let*

$$E(T) = \int_0^T e^{-\delta t}\left[A'(x(t)) + \delta B(x(t))\right]\dot{x}(t)dt.$$

The problem (3) is equivalent to the problem

$$\max_{Adm_{x_0}} E(\infty) \qquad (6)$$

Moreover if $W(x_0)$ denotes the value function of the problem (6),then

$$V(x_0) - \frac{A(x_0)}{\delta} = \frac{W(x_0)}{\delta} \ge 0, \quad \forall x_0.$$

Lemma 3.3 *Let Z be defined by*

$$Z(x) = e^{-\eta\sqrt{x^2+1}}\left(V(x) - \frac{A(x)}{\delta}\right)$$

where η satisfies $\gamma < \eta < \delta$. Then Z is B.U.C.

Lemma 3.4 *The function $Z(.)$ satisfies the following equation*

$$\begin{aligned} Z(x_0)e^{\eta\varphi(x_0)} &= \\ \sup_{Adm_{x_0}} &\left\{\frac{1}{\delta}\int_0^T e^{-\delta t}[A'(x(t)) + \delta B(x(t))]\dot{x}(t)dt \right. \\ &\left. + e^{-\delta T}Z(x(T))e^{\eta\varphi(x(T))}\right\}\end{aligned} \qquad (7)$$

(Dynamic programming equation for the value function (4) associated to the problem (3))

Lemma 3.5 *The function $Z(.)$ is a viscosity solution of (5).*

Lemma 3.6 *The function $Z(.)$ is the unique solution in the set of B.U.C. functions of the Hamilton-Jacobi equation (5).*

4 Turnpike

For $\bar{x}$ a solution of the Euler equation

$$C(x) := A'(\bar{x}) + \delta B(\bar{x}) = 0$$

an admissible path which links x_0 and $\bar{x}$ as rapidly as possible, is called MRAP($x_0, \bar{x}$). We call basin of $\bar{x}$ the following set:

$$\mathcal{B}(\bar{x}) := \{x_0 \ \text{ s.t. } \ MRAP(x_0,\bar{x}) \ \text{ is optimal}\}$$

When $\mathcal{B}(\bar{x})$ is not empty, $\bar{x}$ is called a turnpike. When $\mathcal{B}(\bar{x})$ is not reduced to the singleton $\{\bar{x}\}$, then we also call $\bar{x}$ a "true" turnpike. We denote by E the set of the solutions of the Euler equation:

$$E := \{x \ \text{ s.t. } \ C(x) = 0\}$$

We also define the following function S, which is playing an important role in the following

$$S(x_0,\bar{x}) = \int_{x_0}^{\bar{x}} (A'(y) + \delta B(y))e^{-\delta|x_0-y|}dy.$$

Proposition 4.1 *Let us assume (H_1),(H_2), and suppose that the cardinal of E is finite. Then the two following assertions are equivalent:*
(i) For any x_0, there exists a turnpike $\bar{x} \in E$ (i.e. there exists $\bar{x}$ such that $x_0 \in \mathcal{B}(\bar{x})$)
(ii) For any x, we have

$$T(x) := \max_{\bar{x}\in E} S(x,\bar{x}) \ge 0 \qquad (8)$$

Moreover the value function of the problem is given by $V(x) = (A(x) + T(x))/\delta$.

Proof : Assume that the subset E has finite cardinality, and assume (8).
Let x_0 be an initial condition x_0 and $\bar{x} \in \arg\max_{\bar{\xi}\in E} S(x_0, \bar{\xi})$. The path $\hat{x}(.) := MRAP(x_0, \bar{x})$ is given by :

- if $x_0 \geq \bar{x}$, $\quad \hat{x}(t) = \left| \begin{array}{lll} x_0 - t & \text{if} & t \leq x_0 - \bar{x} \\ \bar{x} & \text{if} & t > x_0 - \bar{x} \end{array} \right.$

- if $x_0 \leq \bar{x}$, $\quad \hat{x}(t) = \left| \begin{array}{lll} x_0 + t & \text{if} & t \leq \bar{x} - x_0 \\ \bar{x} & \text{if} & t > \bar{x} - x_0 \end{array} \right.$

When $x_0 \geq \bar{x}$, the value of the objective along this path is :

$$J[\hat{x}()] = \frac{A(x_0)}{\delta} + \int_0^\infty e^{-\delta t}[A'(\hat{x}(t)) + \delta B(\hat{x}(t))]\dot{\hat{x}}(t)dt$$
$$= \frac{A(x_0)}{\delta} - \frac{1}{\delta}\int_0^{x_0-\bar{x}} e^{-\delta t}[A'(x_0 - t) + \delta B(x_0 - t)]dt$$

With the new variable $\xi = x_0 - t$, it can be rewritten

$$J[\hat{x}()] = \frac{A(x_0)}{\delta} + \frac{1}{\delta}\int_{x_0}^{\bar{x}} e^{-\delta(x_0-\xi)}[A'(\xi) + \delta B(\xi)d\xi$$

When $x_0 \leq \bar{x}$, we then obtain

$$J[\hat{x}()] = \frac{A(x_0)}{\delta} + \frac{1}{\delta}\int_0^{\bar{x}-x_0} e^{-\delta t}[A'(x_0 + t) + \delta B(x_0 + t)]dt$$

and with $\xi = x_0 + t$,

$$J[\hat{x}()] = \frac{A(x_0)}{\delta} + \frac{1}{\delta}\int_{x_0}^{\bar{x}} e^{-\delta(\xi-x_0)}[A'(\xi) + \delta B(\xi)]d\xi$$

Then in these two cases we can write

$$J[\hat{x}()] = \frac{A(x_0)}{\delta} + \frac{S(x_0, \bar{x})}{\delta} = \frac{A(x_0)}{\delta} + \frac{T(x_0)}{\delta}$$

Therefore the $MRAP(x_0, \bar{x})$ are optimal paths for any x_0 if and only if the function

$$x \mapsto \frac{A(x)}{\delta} + \frac{T(x)}{\delta}$$

is the value function that we have characterized in our preceding proposition, that is to say

$$Z(x) = e^{-\eta\phi(x)}\frac{T(x)}{\delta}.$$

The proof of this result is then given by the two following lemmas:

Lemma 4.2 *The function* $Z(x) = e^{-\eta\phi(x)}\dfrac{T(x)}{\delta}$ *is B.U.C.*

Proof of the lemma : Let $x \in \mathbb{R}$, we have for a particular $\bar{x} \in E$:

$$Z(x) = \frac{1}{\delta}\int_x^{\bar{x}} e^{-\delta|x-\xi|}e^{-\eta\varphi(x)}[A'(\xi) + \delta B(\xi)]d\xi$$

From assumption H_2, we obtain

$$\begin{aligned} |Z(x)| &\leq \frac{1}{\delta}k\int_x^{\bar{x}} e^{-\delta|x-\xi|}e^{-\eta\varphi(x)}e^{\gamma|\xi|}d\xi \\ &= \frac{k}{\delta}\int_x^{\bar{x}} e^{-\delta|x-\xi|}e^{-\eta\varphi(x)+\gamma|x|}e^{\gamma(|\xi|-|x|)}d\xi \end{aligned}$$

from

$$|\xi| - |x| \leq |\xi - x| \quad \text{and} \quad \varphi(x) \geq |x|\,,$$

we finally have

$$\begin{aligned} |Z(x)| &\leq \frac{k}{\delta}\int_x^{\bar{x}} e^{-\delta|x-\xi|}e^{(\gamma-\eta)|x|}e^{\gamma|x-\xi|}d\xi \\ &= \frac{k}{\delta}\int_x^{\bar{x}} e^{(\gamma-\delta)|x-\xi|}e^{(\gamma-\eta)|x|}d\xi \end{aligned}$$

This expression is bounded by the value at $\bar{x}$ or at x_0 depending on whether $\bar{x} \leq x_0$ or $\bar{x} \geq x_0$, because $\gamma - \delta < 0$ and $\gamma - \eta < 0$. Therefore $Z(.)$ is bounded. ∎

Now we study the differentiability w.r.t. x of the new function W defined by

$$W(x, \bar{\xi}) = e^{-\eta\varphi(x)}\frac{S(x, \bar{\xi})}{\delta}$$

When $x < \bar{\xi}$, we have

$$\begin{aligned} W(x, \bar{\xi}) &= \frac{1}{\delta}\int_x^{\bar{\xi}} e^{\delta(x-\xi)}e^{-\eta\varphi(x)}[A'(\xi) + \delta B(\xi)]d\xi \\ &= \frac{e^{\delta x}e^{-\eta\varphi(x)}}{\delta}\int_x^{\bar{\xi}} e^{-\delta\xi}[A'(\xi) + \delta B(\xi)]d\xi \end{aligned} \tag{9}$$

which is derivable and whose derivative is given by :

$$\begin{aligned} \frac{\partial W}{\partial x}(x, \bar{\xi}) = \frac{\delta - \eta\varphi'(x)}{\delta}e^{\delta x-\eta\varphi(x)}\int_x^{\bar{\xi}} e^{-\delta\xi}[A'(\xi) + \delta B(\xi)]d\xi \\ -\frac{e^{\delta x}e^{-\eta\varphi(x)}}{\delta}e^{-\delta x}[A'(x) + \delta B(x)] \end{aligned}$$

When $x > \bar{\xi}$, we have

$$\begin{aligned} W(x, \bar{\xi}) &= \frac{1}{\delta}\int_x^{\bar{\xi}} e^{-\delta(x-\xi)}e^{-\eta\varphi(x)}[A'(\xi) + \delta B(\xi)]d\xi \\ &= \frac{e^{-\delta x}e^{-\eta\varphi(x)}}{\delta}\int_x^{\bar{\xi}} e^{\delta\xi}[A'(\xi) + \delta B(\xi)]d\xi \end{aligned} \tag{10}$$

which is also derivable, with :

$$\begin{aligned} \frac{\partial W}{\partial x}(x, \bar{\xi}) = \frac{-\delta - \eta\varphi'(x)}{\delta}e^{-\delta x-\eta\varphi(x)}\int_x^{\bar{\xi}} e^{\delta\xi}[A'(\xi) + \delta B(\xi)]d\xi \\ -\frac{e^{-\delta x}e^{-\eta\varphi(x)}}{\delta}e^{\delta x}[A'(x) + \delta B(x)] \end{aligned}$$

At $x = \bar{\xi}$, we compute the following limits

$$\lim_{x \to \bar{\xi}^-} \frac{\partial W}{\partial x}(x, \bar{\xi}) = \lim_{x \to \bar{\xi}^+} \frac{\partial W}{\partial x}(x, \bar{\xi}) = -\frac{e^{-\eta\varphi(\bar{\xi})}}{\delta}(A'(\xi) + \delta B(\xi))$$

therefore W is also differentiable at $\bar{\xi}$. Moreover we observe that, for all x:

$$\frac{\partial W}{\partial x}(x, \bar{\xi}) = \frac{-\text{sgn}(x - \bar{\xi})\delta - \eta\varphi'(x)}{\delta} \times \int_x^{\bar{\xi}} e^{-\delta|x-\xi|} e^{-\eta\varphi(x)}[A'(\xi) + \delta B(\xi)] d\xi - \frac{e^{-\eta\varphi(x)}}{\delta}(A'(x) + \delta B(x)) \tag{11}$$

and we obtain

$$\left| \frac{\partial W}{\partial x}(x, \bar{\xi}) \right| \leq \frac{\delta + \eta}{\delta} \int_x^{\bar{\xi}} e^{-\delta|x-\xi|} e^{-\eta\varphi(x)} k e^{\gamma|\xi|} d\xi + \frac{e^{-\eta\varphi(x)}}{\delta} k e^{\gamma|x|}$$

This last expression is bounded because $\gamma - \eta < 0$ and $\gamma - \delta < 0$, the exponential being bounded by their values at x or at $\bar{\xi}$ depending on whether $x \leq \bar{\xi}$ or $x \geq \bar{\xi}$.
Therefore we conclude that W is uniformly continuous with respect to x. But Z is given by

$$Z(x) = \max_{\bar{\xi} \in E} W(x, \bar{\xi}), \quad \forall x \in \mathbb{R} \tag{12}$$

which establishes that Z is B.U.C. ∎

Lemma 4.3 *The function Z is a viscosity solution of the Hamilton-Jacobi equation (5)*

Proof of the lemma :
From (11) we immediately obtain

$$\frac{\partial W}{\partial x}(x, \bar{\xi}) + \frac{1}{\delta} e^{-\eta\varphi(x)}[A'(x) + \delta B(x)] + \eta\varphi'(x) W(x, \bar{\xi}) = -\text{sgn}(x - \bar{\xi})\delta W(x, \bar{\xi}) \tag{13}$$

We now have to consider two cases:

1. $\arg\max_{\bar{\xi}} W(x, \bar{\xi}) = \{\bar{x}\}$
2. $\arg\max_{\bar{\xi}} W(x, \bar{\xi})$ not given by a single $\bar{x}$

In the first case, at any x such that $\arg\max_{\bar{\xi}} W(x, \bar{\xi}) = \{\bar{x}\}$, the function Z is differentiable and we have

$$Z'(x) = \frac{\partial W}{\partial x}(x, \bar{x})$$

From (12) and (13), we deduce

$$\left| Z'(x) + \frac{1}{\delta} e^{-\eta\varphi(x)}[A'(x) + \delta B(x)] + \eta\varphi'(x) Z(x) \right| = \delta\, |Z(x)|$$

therefore Z is an classical solution of the Hamilton-Jacobi equation (5) as soon as $Z(x) \geq 0$.

In the second case: Z is no more differentiable, and its sub-differentials are :

$$\begin{aligned} D^+ Z(x) &= \emptyset, \\ D^- Z(x) &= \overline{\text{co}} \left\{ \frac{\partial W}{\partial x}(x, \bar{x}) \text{ , } \bar{x} \in \arg\max_{\bar{\xi} \in E} W(x, \bar{\xi}) \right\} \\ &= \left\{ \sigma\delta Z(x) - \eta\varphi'(x) Z(x) - \frac{e^{-\eta\varphi(x)}}{\delta}[A'(x) + \delta B(x)] \text{ , } \sigma \in [-1, 1] \right\} \end{aligned}$$

It then suffices to prove that Z is a viscosity super-solution for the Hamilton-Jacobi equation (5).
For $p^- \in D^- Z(x)$, we have

$$\delta Z(x) - \left| p^- + \frac{1}{\delta} e^{-\eta\varphi(x)}[A'(x) + \delta B(x)] + \eta\varphi'(x) Z(x) \right| = \delta Z(x)(1 - \sigma)$$

which is non negative for all $\sigma \in [-1, 1]$ if and only if $Z(x) \geq 0$. ∎

This ends the proof of our second proposition. Indeed Z is a viscosity solution of (5) if it is non negative, that is to say if and only if $T(x) \geq 0$, $\forall x \in \mathbb{R}$. ∎

Remark : We observe that this proposition gives a generalization of the standard result, obtained with the help of the Green theorem, which asserts, when E is reduced to a unique $\bar{x}$, that $MRAP(x_0, \bar{x})$ is optimal from any initial condition x_0 if the following (sufficient) condition is satisfied

$$(\bar{x} - x)(A'(x) + \delta B(x)) \geq 0, \quad \forall x \in \mathbb{R}. \tag{14}$$

This last condition just implies our condition (8) when E is a singleton :

$$S(x, \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}.$$

5 An example

We now give an example for which the Euler equation is singular and admits more than one stationary solutions, none of them satisfying (14). From (8), we deduce the optimality of $MRAP(x_0, \bar{x})$ for one or several turnpikes.

Let $0 < a < b$, we consider

$$\max_{x(.)} \int_0^{\to\infty} e^{-t}x^2(t)\left[2\dot{x}(t)(a+b-x(t)) - ab\right] dt$$
$$\text{with } \dot{x}(t) \in [-1,+1] \text{ a.e.}$$

The associated Euler equation

$$A'(x) + \delta B(x) = -2x(a-x)(b-x) = 0$$

possesses three stationary solutions (see Figure 1).

We observe that $\bar{x} = 0$ and $\bar{x} = b$ satisfies (only locally) the classical condition (14). From Proposition 4.1, we prove now that depending on the values of a and b,

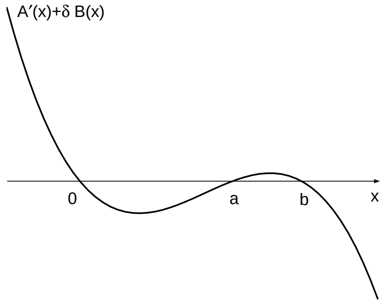

Figure 1:

- $MRAP(x_0, 0)$ or $MRAP(x_0, b)$ is optimal for any initial condition x_0,
- there exists $x^* \in]0, b[$ such that $MRAP(x_0, 0)$ (resp. $MRAP(x_0, b)$) is optimal for $x_0 \le x^*$ (resp. $x_0 \ge x^*$).

In this example we have

$$A(x) = -abx^2, \quad B(x) = 2x^2(a+b-x) \quad \text{et} \quad \delta = 1.$$

Then the Euler equation is :

$$C(x) = -2x(ab + (a+b)x - x^2) = -2x(a-x)(b-x),$$

which admits three solutions : $\bar{x} \in \{0, a, b\}$. It is easy to verify the following properties :

$$\begin{cases} x \in [0,b] \Rightarrow S(x,a) \le 0 \\ x \le a \Rightarrow \{\ S(x,0) \ge 0 \ \text{ et } \ S(x,b) \ge S(0,b)\ \} \\ x \ge b \Rightarrow S(x,0) \ge S(b,0) \\ x \ge a \Rightarrow S(x,b) \ge 0 \end{cases}$$

Therefore we can conclude by the Proposition 4.1 that :
1. $MRAP(x_0, a)$ is never optimal for any initial condition x_0.
2. $MRAP(x_0, 0)$ is optimal for all x_0 as soon as $S(b, 0) \ge 0$.
3. $MRAP(x_0, b)$ is optimal for all x_0 as soon as $S(0, b) \ge 0$.
4. If $x \to \max\{S(x,0), s(x,b)\}$ is non negative for all $x \in [0,b]$, then $MRAP(x_0, 0)$ or $MRAP(x_0, b)$ is optimal.

For $x \in [0,b]$, we compute the following functions

$$S(x,0) = e^{-x}\int_0^x 2y(a-y)(b-y)e^y dy,$$

$$S(x,b) = -e^x\int_x^b 2y(a-y)(b-y)e^{-y} dy$$

with the help of a symbolic computation software :

$$\begin{cases} S(x,0) = & 2\left[x^3 - (a+b+3)x^2 + (ab + 2(a+b+3))\right. \\ & \left.(x - 1 + e^{-x})\right] \\ S(x,b) = & 2\left[-x^3 + (a+b-3)x^2 + (-ab + 2(a+b-3))\right. \\ & \left.(x + 1 + e^{x-b}) + (b^2 - 4a + 2b + 12)e^{x-b}\right] \end{cases} \tag{15}$$

Then for different values of a et b, we obtain the three following cases :

1. For $a = 2$ et $b = 3$, we have

$$S(b,0) = 2(22e^{-3} - 1) > 0.$$

and we can conclude that the paths that are going with a most rapid velocity to $\bar{x} = 0$ are optimal (see Figure 2).

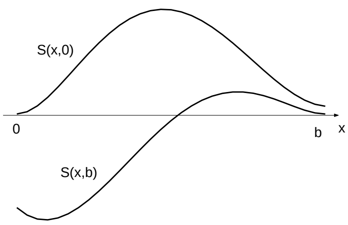

Figure 2:

2. For $a = 1$ et $b = 4$, we obtain

$$S(0,b) = 64e^{-4} > 0.$$

and we can conclude that the paths that are going with a most rapid velocity to $\bar{x} = b$ are optimal (see Figure 3).

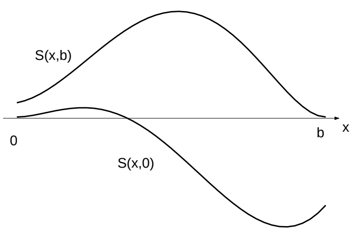

Figure 3:

3. For $a = 2$ et $b = 5$, we obtain

$$\begin{cases} S(b,0) = 2(30e^{-5} - 5) < 0 \\ S(0,b) = 2(37e^{-5} - 2) < 0 \end{cases}$$

but we observe that $\max\{S(x,0), s(x,b)\}$ is non negative on $[0,b]$. Let $x^* \in]0, b[$ be such that $S(x^*, 0) = S(x^*, b)$ (see Figure 4). Then there is a competition between the two turnpikes

$\bar{x} = 0$ and $\bar{x} = b$: for $x_0 \leq x^*$ (resp. $x_0 \geq x^*$), it is optimal to go as quickly as possible to $\bar{x} = 0$ (resp. $\bar{x} = b$). Let us

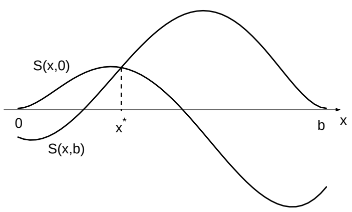

Figure 4:

underline that in this last case there is no longer uniqueness of the turnpike for the initial condition $x_0 = x^*$, which also corresponds to a non-differentiability point of the value function.

6 Conclusion

For singular scalar problems of calculus of variation with infinite horizon, we have obtain a new necessary and sufficient condition for the optimality of the MRAPs. This condition generalizes the standard one which is only sufficient and valid only in the case of a unique solution of the singular Euler equation. Our result is established with the help of the viscosity solutions of a particular Hamilton-Jacobi equation associated to the problem. Our condition also applies when one is dealing with singular Euler equations that possess more than one solutions.

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