STOCHASTIC OPTIMAL CONTROL OF DYNAMIC SYSTEMS UNDER GAUSSIAN AND POISSON EXCITATIONS

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Keywords: Dynamic programming, Stochastic optimal control, Nonlinear control

Abstract

A mathematical pendulum under Poisson and Gaussian excitations is considered. A bounded in magnitude control force is applied to the system in order to minimize mean system's response energy. An optimal control law for the Boltza cost function may be found via Dynamic Programming approach, resulting in the Hamilton-Jacobi-Bellman (HJB) equation. Solution to the nonlinear HJB equation has been derived in two steps, as suggested by the recently introduced method of Hybrid Solution. Influence of viscous damping on synthesis of an optimal control law is investigated.

1. INTRODUCTION

Problems of stochastic optimal control are very important in different areas of science and engineering. One way of handling these problems is the Dynamic Programming approach, which reduces a problem of synthesis of optimal control law to solution of the Hamilton-Jacobi-Bellman (HJB) nonlinear, partial differential equation (Fleming, *et. al*., 1975; Kolmanovskii, *et. al*., 1996). Because of the specific structure of the latter, only few analytical solutions are known today (Bensoussan, 1988; Bratus, 1975; Chernousko, *et. al*., 1978). The viscosity solution to the HJB equation has been intensively studied by (Lions, 1982). Systems composed with continuous and discrete parts have been studied by (Bensoussan and Menaldi, 2000). Most recent results in discontinuous solutions to the HJB equation are discussed by (Bardi and Capuzzo-Dolcetta, 1997).
A new Hybrid Solution method was introduced and successfully implemented as an alternative way of finding an optimal control law (Bratus *et. al*., 2000). An optimal control law for a single-degree-of-freedom (SDOF) system subjected to Gaussian white noise excitation was derived, using the characteristic approach (Melikyan A.A. 1998). An exact analytical solution to the Lagrange cost function has indicated that, for a steady-state response a dry friction provides the optimal control law for a system's response energy reduction (Iourtchenko, 2000). This method was also successfully applied to a MDFO system (Bratus, *et. al*., 2000).
In this paper, authors apply the foregoing method to a SDOF system subjected to Gaussian and Poisson excitation. Although an exact analytical solution within an "outer" domain shows that dry friction law is not the optimal one, it still can be used as an approximate one for small values of Poisson's noise intensity. A problem of optimal control of a damped SDOF system subjected to Gaussian white noise is also considered.

2. PROBLEM FORMULATION

Consider a mathematical pendulum subjected to Gaussian white noise and Poisson excitations, governing by the following equation of motion

$$\ddot{x} + \Omega^2 x = u + \sigma(t)\xi(t) + \gamma\eta(t) \tag{1}$$

where $\xi(t)$ and $\eta(t)$ are independent Gaussian and Poisson processes, the latter has a constant arrival rate λ, $|u| \le R,\ R > 0$ is a control force. Consider a problem of minimization of a mean system's response energy, where the Bellman function is taken as

$$\begin{aligned} H(x_1, x_2, t) = \min_{|u|\le R} & \left\{ a_1 \left\langle \int_0^T \frac{1}{2}\left(x_1^2\Omega^2 + x_2^2\right) dt \right\rangle \right\} \\ & + \left\{ a_2 \left\langle \frac{1}{2}\left(x_1^2(T)\Omega^2 + x_2^2(T)\right) \right\rangle \right\} \end{aligned} \tag{2}$$

with a_1 and a_2 being some positive constants. $\sigma^2(t)$ and $\gamma - const$ are intensities of Gaussian white noise and Poisson noise respectively. The Bellman function, defined by the latter expression should satisfy the following HJB equations

$$\frac{\partial H}{\partial \tau} = L(H) - R\left|\frac{\partial H}{\partial x_2}\right| + \frac{\sigma^2}{2}\frac{\partial^2 H}{\partial x_2^2} + P(\lambda, \gamma) \tag{3}$$

where, backward time $\tau = T - t$ was introduced, as well as operation of minimization was performed

$$\min_{|u|\le R}\left\{u\frac{\partial H}{\partial x_2}\right\} = -R\left|\frac{\partial H}{\partial x_2}\right|, u = -R\,\mathrm{sgn}\left(\frac{\partial H}{\partial x_2}\right) \tag{4}$$

The following notation has been used in formula (3) for one-sided Poisson process

$$L(H) = x_2\frac{\partial H}{\partial x_1} - \Omega^2 x_1\frac{\partial H}{\partial x_2} + \frac{a_1}{2}\left(\Omega^2 x_1^2 + x_2^2\right)$$
$$P(\lambda,\gamma) = \lambda\left[H(x_1, x_2+\gamma,\tau) - H(x_1,x_2,\tau)\right]$$

The difference-differential equation (3) has to be solved with the following initial condition

$$H(x_1,x_2,0) = \frac{a_2}{2}\left(\Omega^2 x_1^2 + x_2^2\right)$$

Solution to the Gaussian excitation only may be found in (Bratus *et. al.*, 2000) for Mayer and in (Iourtchenko, 2000) for Lagrange cost functions. Solution to the Boltz cost function $(a_1 \ne 0, a_2 \ne 0)$ may be developed as a linear combination of the above solutions

$$H_B(x_1,x_2,\tau) = a_1 H_L + a_2 H_M$$

3. ONE-SIDED POISSON PROCESS

Lets consider the HJB equation for one-sided Poisson process (3). To derive solutions to Lagrange problem, it is sufficient to obtain solution to Mayer cost function and integrate this solution with respect to explicitly entering the solution backward time τ from zero to τ (Iourtchenko, 2000). Therefore, solution to Mayer problem only will be derived here.

Statement 1. The following function

$$\begin{aligned} H_M(x_1,x_2,\tau) &= \frac{1}{2}\left[x_2 - \frac{R\,\mathrm{sgn}(x_2)-\lambda\gamma}{\Omega}\sin\Omega\tau\right]^2 \\ &+\frac{1}{2}\left[x_1\Omega + \frac{R\,\mathrm{sgn}(x_2)-\lambda\gamma}{\Omega}(1-\cos\Omega\tau)\right]^2 + \\ &\frac{\gamma^2\lambda\tau}{2} + \sigma_1(\tau), \sigma_1(\tau) = \frac{1}{2}\int_0^\tau \sigma^2(t)dt \end{aligned} \tag{5}$$

provides an exact analytical solution within the "outer" domain D_+, D_-, defined as

$$\begin{aligned} D_+ &= \left\{\begin{array}{l} x_1,x_2,\tau : x_2 \ge \dfrac{R-\lambda\gamma}{\Omega}, \\ x_2 < \min\left[-\dfrac{R+\lambda\gamma}{\Omega},\gamma\right]; R-\lambda\gamma > 0 \end{array}\right\} \\ D_+ &= \left\{\begin{array}{l} x_1,x_2,\tau : x_2 > 0, \\ x_2 < \min\left[-\dfrac{R+\lambda\gamma}{\Omega},-\gamma\right]; R-\lambda\gamma < 0 \end{array}\right\} \end{aligned} \tag{6}$$

Both D_+ and D_- domains are non symmetric with respect to the line $x_2 = 0$. Furthermore, D_- is shifted downwards and starts from $x_2 = 0+$ for large values of the Poisson noise intensity. Within these domains, the dry friction control law is the optimal one.

4. DAMPED OSCILLATOR UNDER GAUSSIAN WHITE NOISE

All mechanical systems possess a certain energy dissipation mechanism. One of these is a viscous damping, described by the equation (1). It seems natural to presume that, its influence should be negligible for synthesis of optimal control law, if a damping coefficient α is small. However, it is still interesting to confirm or refute this assumption, and give, if possible, some estimates for values of damping coefficient. These estimates will provide us with information, needed to make a decision on keeping damping term in the HJB equation or not. For this purpose, solution to the following HJB equation

$$\frac{\partial H}{\partial \tau} = L(H) - R\left|\frac{\partial H}{\partial x_2}\right| + \frac{\sigma^2}{2}\frac{\partial^2 H}{\partial x_2^2} - 2\alpha\frac{\partial H}{\partial x_2} \tag{7}$$

condition $\alpha << \Omega$. The terminal cost function is considered first ($a_1 = 0$). Let's introduce a set of new variables

$$\begin{aligned} \overline{x}_1 &= k\left[x_1 + \frac{R\,\mathrm{sgn}(x_2)}{\Omega^2}\right]; \overline{x}_2 = \\ &= x_2 + \frac{\alpha}{k}\overline{x}_1, k^2 = \Omega^2 - \alpha^2 \end{aligned} \tag{8}$$

Statement 2. The following function

$$\begin{aligned} H_M^\alpha &= \frac{e^{-2\alpha\tau}}{2k^2}\Big[\left(\overline{x}_1^2+\overline{x}_2^2\right)\Omega^2 + \alpha^2\left(\left(\overline{x}_1^2-\overline{x}_2^2\right)\cos 2k\tau\right. \\ &\left.+2\overline{x}_1\overline{x}_2\sin 2k\tau\right) - \alpha k\left(2\overline{x}_1\overline{x}_2\cos 2k\tau - \left(\overline{x}_1^2-\overline{x}_2^2\right)\sin 2k\tau\right]\\ &+R\,\mathrm{sgn}(x_2)\frac{e^{-\alpha\tau}}{k}\left[\overline{x}_1\cos k\tau + \overline{x}_2\sin k\tau\right] + \frac{R^2}{2\Omega^2} + \sigma_3(\tau), \\ \sigma_3(\tau) &= \frac{1}{2k^2}\times \\ &\times\int_0^\tau \sigma^2(\chi)e^{-2\alpha\chi}\left[\Omega^2 - \alpha^2\cos 2k\chi - \alpha k\sin 2k\chi\right]d\chi \end{aligned} \tag{9}$$

provides an analytical solution to the HJB equation (12) within the “outer” domain Γ_3, defined as

$$\Gamma_3 = \left\{ x_1, x_2, \tau : |x_2| \geq \max\left[\frac{R}{\Omega} e^{\alpha\tau} + O(\alpha) \right] \right\} \quad (10)$$

It is easy to show that the solution (9) reduces to the one for zero value of a damping coefficient and with an absence of Poisson noise.

5. NUMERICAL RESULTS

To solve the HJB equation numerically, a bounded state-space domain has to be selected first. It should satisfy conditions (6) or (10) for the corresponding problems over all period of simulation time. The HJB equation is solved within the “inner” domain numerically by finite difference method with an exact analytical solution imposed as boundary conditions.

All numerical calculations are performed for constant values of the following parameters $\Omega = 1$, $\sigma^2 = 2$. It is convenient to introduce three dimensionless parameters for the system without viscous damping. Namely, parameter $\mu = R/\sigma\sqrt{\Omega}$ that was introduced in (Bratus, *et al., 2000)* and two new parameters $\rho = \lambda\gamma^2/\sigma^2$ and $\varphi = \lambda T^n = 2\pi\lambda/\Omega$, where T^n is a natural period of the corresponding conservative system (1). First of these two parameters is a ratio of Poisson to Gaussian white-noise intensities. The second parameter is clearly a number of “events” occurring within a system’s natural period. Gaussian white noise is dominant in the system when value of parameter ρ is small and vice versa. Numerical simulation of a system (1) without Poisson noise, corresponding to the HJB equation (2) without last term, is presented in the beginning. Switching lines, defined by equality $\partial H/\partial x_2 = 0$ completely define the optimal control law based on expression (4).

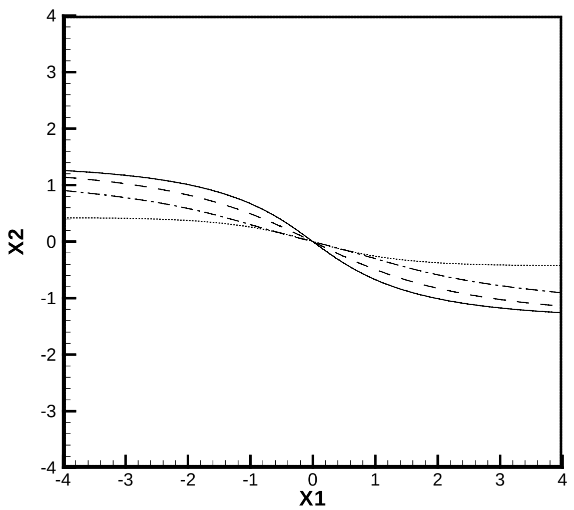

Figure 1. τ=0 (solid line), τ=$\pi/4$ (dashed line), τ=$\pi/2$ (dash-dot line), τ=π (dotted line).

Switching lines in phase plane at different instants of backward time τ and $\mu = 1.414$ are demonstrated in Figure 1. The value of the derivative is positive everywhere above the corresponding line and negative below it.

Let’s consider the case of one-sided Poisson excitations. This problem is defined by the equation (2) with corresponding initial conditions for the Mayer cost function. We will be interested to consider the case when $R - \lambda\gamma < 0$. The results of numerical simulation are presented in Figure 2 for different values of backward time τ and $\mu = 0.71, \rho = 0.75, \varphi = 3\pi$.

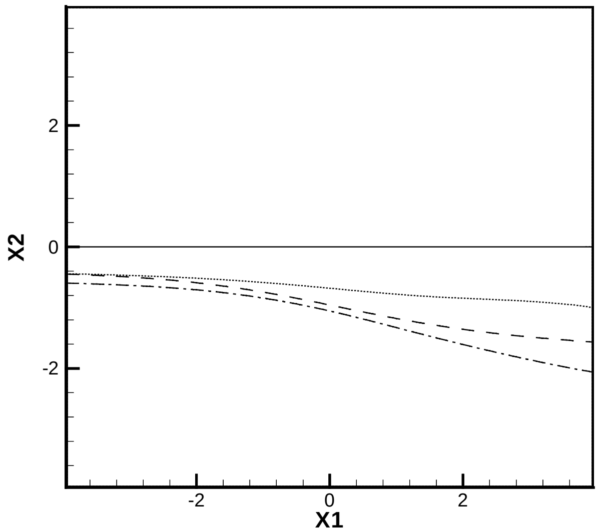

Figure 2. Switching lines for τ=0 (solid line), $\tau = \pi/4$ (dashed line), $\tau = \pi/2$ (dash-dot line) and $\tau = 3\pi/4$ (dotted line)

It is seen that switching lines are shifted significantly downwards from the line $x_2 = 0$. It also should be stressed that, switching lines for $x_1 < 0$ much closer to each other then for $x_1 > 0$, resulting in a non-skew-symmetric optimal control law.

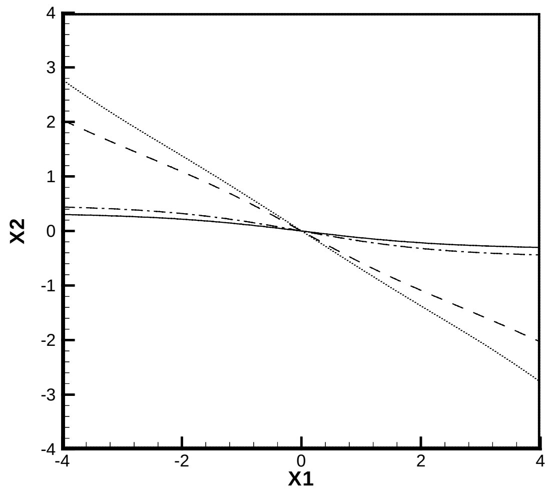

Figure 3. $\tau = \pi/4$: α=0 (solid line), α=0.3 (dashed line), $\tau = \pi/2$: α=0 (dash-dot line), α=0.3 (dotted line).

Finally, the influence of damping coefficient will be discussed here for the problem defined by equation (7). According to the paper by (Dimentberg *et. al.*, 2000) a value of an equivalent damping coefficient for system with dry friction may be expressed as $\alpha_{eq} = 2R\Omega/\pi$. The numerical results for values of $\alpha << \alpha_{eq}$ showed that the difference in position of corresponding switching lines is negligible, as it was expected. Thus, for the system with a small value of damping coefficient the optimal control law may be taken in the same form as for a system without damping term in the HJB equation. The results of numerical simulation for a case of $\alpha \approx \alpha_{eq}$ (same order of magnitude) are presented in Figure 3 for values of $\alpha = 0.3, R = 0.5$ and two values of backward time τ. Figure 3 clearly demonstrates how far the corresponding switching lines lie apart from each other for zero and nonzero damping coefficient. Apparently, if the applied control force has the same order of magnitude as a coefficient of system's viscous damping, an optimal control law cannot be substituted by one for a system without damping and has to be calculated through the HJB equation with an analytical solution served as the boundary conditions.

6. CONCLUSIONS

The problem of stochastic optimal control of a SDOF system under Gaussian and Poisson excitations has been considered. Solution to the corresponding HJB equations has been derived based on newly developed Hybrid solution method. Numerical simulations have shown that the presence of Poisson noise may significantly change the optimal control law. However, for small values of Poisson's noise intensity, a dry friction may be used as an approximate control law to simplify great computational efforts. Moreover, for a damped SDOF system, numerical simulation indicated that relatively small values of linear damping coefficient do not change a lot the optimal control law. However, it is not the true for the case of relatively large values of that coefficient.

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