ON SYSTEMS AND CONTROL CONCEPTS IN LINEAR INTEREST RATE THEORY

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Abstract

This contribution discusses the concept of stochastic controllability within the framework of linear interest rate models of Heath-Jarrow-Morton-Musiela (HJMM) type that may be represented by an infinite dimensional stochastic differential equation. Despite the fact that not all such models are controllable, we nonetheless investigate the possibility of influencing the drift term of the aforementioned differential equation by a particular choice of control function. As a consequence, the primary purpose of our study is to determine necessary and sufficient conditions for the stochastic controllability of a special subclass of the aforementioned models. In particular, we find a control that transfers the said model from an arbitrary interest rate to any other interest rate in the state space of forward rate curves. In order to address this problem we introduce deterministic and stochastic controllability operators related to such interest rate models and solve a linear regulator problem associated with the minimum energy principle.

1 Introduction and Preliminaries

Many situations in modern financial economics involve the use of continuous time systems and control theory. In particular, problems in stochastic control impact such areas as portfolio selection (see, for instance, [12] and [15]) option pricing ([6] and [10] and the references contained therein), loan management (see, for instance, [16]) and insurance theory (see [9] and [14] and many others). In this contribution, we consider a connection between interest rate theory and stochastic control. In the Heath-Jarrow-Morton-Musiela (HJMM) interest rate model (see [8], [13] and [4]) the evolution of the forward rate curve is explained in terms of a stochastic structure. This model can be considered to be a unification in terms of the family of Itô processes indexed by the continuum of the maturities of all continuous interest rate models. The stochastic differential equations that arise in this case involve interest rates that may be regarded as a field of random variables that changes with respect to the parameters of time t and maturity T.

The study of bond markets are underscored by the term structure of interest rates that are by nature infinite dimensional and generally not directly observable. Emperically it is necessary to devise curve fitting methods for the daily estimation of the term structure. The aim of this paper is to investigate how infinite dimensional models of the term structure of interest rates can be controlled and regulated by analytic means. In particular, we study the system theoretic concept of controllability as it pertains to linear interest rate models of Heath-Jarrow-Morton-Musiela (HJMM) type. Our motivation for considering this problem is that we would like to develop a better understand of the evolution of interest rates in time. Also, we know that controllability has a role to play in minimality that in turn has been shown in [3] to be of considerable consequence for linear interest rate models. In particular, it is known that such finite-factor term structure models are useful for practitioners. Of the two groups of practitioners in the fixed income market, namely the fund managers and the interest rate option traders, the latter have a special interest in the low dimensionality of the interest rate model since the number of factors usually equals the number of instruments one needs to hedge in the model. It is known that the daily adjustment of huge numbers of instruments becomes infeasible due to transaction costs.

Next, we provide a few preliminaries about notation and terminology. $\mathrm{Dom}\ A$ denotes the domain of the (bounded linear) operator A and $R(\lambda, A)$ is the notation used for its resolvent $(\lambda I - A)^{-1}$. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For $\mathcal{X}$ a separable Hilbert space, we denote the space of equivalence classes of all functions from $[0, T]$ to $\mathcal{X}$ that are Lebesque measurable and square integrable with respect to the Lebesque measure by $L_2([0, T]; \mathcal{X})$. $L_2^{\mathcal{F}}([0, T]; \mathcal{X})$ is the space of $\mathcal{F}_t$-adapted, $\mathcal{X}$-valued measurable processes $\psi(t, \omega)$ on $[0, T]$ such that $\mathbf{E} \int_0^T ||\psi(t, \omega)||^2 < \infty$. The notation $\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}$ is used for the triangular set over $[0, T]$. $\mathcal{B}_2(\Delta, \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ denotes the class of all $\mathcal{L}(\mathcal{X}, \mathcal{Y})$-valued functions on Δ that are strongly measurable and square integrable with respect to the Lebesque measure on Δ. For $s \leq r \leq T$, we denote the controllability operator corresponding to a specific type of stochastic and deterministic system by Π_r^T and Γ_s^T, respectively.

2 The HJMM Interest Rate Model

In this section, we provide a brief description of the HJMM interest rate model and decide on the most economic Hilbert space to be considered as a state space of forward rate curves.

2.1 Basic Description

As was described in [8], the HJM interest rate model for the forward curve $x \mapsto r(t,x)$ is fixed by the structure of its volatility σ and the market price of risk. In this case, $r(t,x)$ is the notation used to denote the forward rate at time t with maturation date $t+x$. In this model, we consider a default free, frictionless bond market with perfectly divisible bonds on a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathbf{P})$. Furthermore, denote the price at time t of a zero coupon bond maturing at $t+x$ by $p(t,x)$, where x is time to maturity and

$$p(t,x) = \exp\{-y(t,x)\},$$

where the period yield $y(t,x)$ is defined by

$$y(t,x) = \int_0^x r(t,s)ds.$$

The expression $\int_0^x \cdot(t,s)ds$ denotes integration with respect to time to maturity x. Also, $r(t,x)$ the forward rate contracted at t maturing at $t+x$ has the form

$$r(t,x) = -\frac{\partial \log p(t,x)}{\partial x}.$$

Moreover, we denote the short rate by $R(t)$, where $R(t) = r(t,0)$. As is well-known the HJMM approach addresses the question of the modelling of the dynamics for the entire forward rate curve. Here the yield curve r is the state variable rather than the short rate R.

As regards notation, in the ensuing discussion the forward rate at time t with maturation date $t+x$, is simply denoted by $r(t)$. From [7], we know that every classical HJM model can more or less be realized as a stochastic differential equation (SDE) of the form

$$\begin{cases} dr(t) &= (Ar(t) + D(t))\,dt + \sigma(r(t))dW(t), \\ r(0) &= r^*(0), \end{cases} \tag{2.1}$$

where W is an m-dimensional Wiener process, $\sigma(r(t))dW(t) = \sum_{j=1}^{m} \sigma_j(r(t))dW_j(t)$ and the initial curve $\{r^*(0,x):\ x\geq 0\}$ is interpreted as the observed forward rate curve. This equation evolves on some open convex subset $\mathcal{U}$ in a separable Hilbert state space $\mathcal{H}$ of forward rate curves. More specifically, we have for $A,\ D \in \mathcal{L}(\mathcal{U},\mathcal{H})$ and σ that

$$A = \frac{\partial}{\partial x} : \mathrm{Dom}\, A \subset \mathcal{H} \to \mathcal{H};\ \ D : \mathcal{U} \subset \mathcal{H} \to \mathcal{H};$$
$$\sigma = (\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_m) : \mathcal{U} \subset \mathcal{H} \to \mathcal{H}^m,$$

respectively. Also, the model must be arbitrage-free which leads to the existence of an equivalent local martingale measure $\mathbf{Q} \sim \mathbf{P}$. In this case $D(t)$ in the drift term from (2.1) can be written in terms of the volatility σ and be specified as

$$D(t) = \sigma(r(t)) \int_0^x \sigma(r(s))ds. \tag{2.2}$$

Here (2.1) is commonly referred to as the **HJMM equation** and (2.2) is called the **HJMM drift condition**. This means that the pricing formula for interest rate sensitive contingent claims only depend on σ. Furthermore, the deterministic counterpart of the stochastic linear HJMM interest rate model (2.1) may be represented as

$$\begin{cases} dr(t) &= (Ar(t) + D(t))\,dt, \\ r(0) &= r^*(0). \end{cases} \tag{2.3}$$

We assume that our state space $\mathcal{H}$ is separable. Furthermore, from [7] we know that $\mathcal{H}$ is continuously embedded in $C([0,\infty);\mathbf{R})$. In other words, for any choice of $x \in [0,\infty)$ the pointwise evaluation $r \longmapsto r(x)$ is a linear functional on $\mathcal{H}$ that is continuous. Furthermore, $\mathcal{H}$ contains the constant function 1. We also insist that the family of right shifts

$$S_t r = r(t+x) \text{ for } t \in [0,\infty) \tag{2.4}$$

forms a strongly continuous semigroup on $\mathcal{H}$ with generator $\frac{\partial}{\partial x}$. Furthermore, we may assume that the domain of $\frac{\partial}{\partial x}$ has the form

$$\left\{ h \in \mathcal{H} \cap C^1([0,\infty);\mathbf{R}) :\ \frac{\partial}{\partial x} h \in \mathcal{H} \right\}.$$

3 Controllability of Linear HJMM Interest Rate Models

In this section we investigate the possibility of influencing the HJMM interest rate model (2.1) by postmultiplying $D(t)$ in the drift term (2.2) by a positive function $u : [0,\infty) \to \mathcal{U}$, where $\mathcal{U}$ is the open convex subset described earlier.

3.1 Partially Observable HJMM Interest Rate Models

It is often not possible to observe the interest rate directly. For this reason, at the outset, we choose a partially observable infinite dimensional HJMM model of the form

$$\begin{cases} d\widetilde{r}(t) &= (A\widetilde{r}(t) + D(t)u(t))\,dt + \sigma' d\widetilde{W}(t), \\ dz(t) &= C\widetilde{r}(t) + Fd\widehat{W}(t), \\ r(0) &= r^*(0), \end{cases} \tag{3.5}$$

where $D(t)$ in (2.1) acting on u that belongs to some admissible control set $\mathcal{U}_{ad} \subset \mathcal{U}$, σ', C and F are given by

$$D(t) : \mathcal{U}_{ad} \subset \mathcal{H} \to \mathcal{H},\ \ \sigma' : Q^{1/2}\mathcal{E} \to \mathcal{H},$$
$$C : \mathcal{U} \subset \mathcal{H} \to \mathbf{R}^m,\ \ F : \mathbf{R}^m \to \mathbf{R}^m,$$

respectively. The process z usually stands for an observable quantity such as a stock price or an index or a combination of the two. For the sake of our analysis we will investigate the situation where $\widetilde{W}(t)$ is an m-dimensional Wiener process on some separable Hilbert space $\mathcal{E}$ with covariance operator $\widetilde{Q}$ and $\widehat{W}(t)$ is a vector-valued Wiener process on $\mathbf{R}^m$ with covariance operator Q. Also, the volatility σ' in (3.5) is a Hilbert Schmidt operator from $\widetilde{Q}^{1/2}\mathcal{E}$ into $\mathcal{H}$ with the Hilbert-Schmidt norm $||\cdot||_2$. Furthermore, we assume that $D(t)$ is a bounded linear operator from $\mathcal{U}_{ad}$ with generic element $u(t)$ to $\mathcal{H}$. Also, Q and F are invertible and Q^{-1}, F and $F^{-1} \in \mathcal{L}(\mathbf{R}^m)$. For each $z \in L_2(\mathcal{Z}_t, \mathcal{U})$ in (3.5) from Theorem 5.6 of [11] we know that there exists a process $\psi(\cdot) \in L_2^{\mathcal{Z}}([0,T]; \mathcal{L}(\mathbf{R}^k, \mathbf{R}^n))$ with the property that $z \in L_2(\mathcal{Z}_t, \mathcal{U})$ is given by

$$z = \mathbf{E}(z|\mathcal{Z}_t) = \mathbf{E}z + \int_0^T \psi(s)dW(s). \tag{3.6}$$

3.2 Admissible Control Sets

The set-up of the admissible control set $\mathcal{U}_{ad}$ is described below. Consider a $\mathbf{Q}$-measurable function z on Ω to $L_2([0,T];\mathcal{U})$. For $0 < t \leq T$, denote by z_t the restriction of z to $[0,t] \times \Omega$ and by $\Sigma(z_t)$ denote the Σ-algebra generated by $z_t(s),\ 0 \leq s \leq t$. Let $\mathcal{U}_{z_t} = L_2(\Omega, \Sigma(z_t), \mathbf{Q}, \mathcal{U})$. The set

$$\mathcal{U}_{ad} = \int^{\oplus} \mathcal{U}_{z_t} dt$$
$$= \{z \in L_2([0,T];\mathcal{U}) : z_t \in \mathcal{U}_{z_t} \text{ for almost every } t \in [0,T]\}$$

is the Hilbertian sum of subspaces $\mathcal{U}_{z_t}$ which by [2] is independent of the choice of control. It follows that for $u \in \mathcal{U}_{ad}$ that the stochastic system (3.5) is well-defined and u is a feedback control that obeys laws of the type where $u(t) = \psi(t, z_t)$ is admissible for ψ being measurable, nonanticipative and satisfying a uniform Lipschitz condition.

Next, we introduce the Σ-algebras $\mathcal{Z}_t = \Sigma(z_t)$ and $\mathcal{Z}_t^0 = \Sigma(z_t^0)$, generated by $z_t(\cdot)$ and $z_t^0(\cdot)$, respectively. Also, we consider an important subclass $\mathcal{U}_{ad}^g \subset \mathcal{U}_{ad}$ (see [5] for more details) defined by a linear feedback on the observation

$$\mathcal{U}_{ad}^g = \left\{ \quad u(t) = w(t) + \int_0^t K(t,s)dz(s) \quad \right\} \subset \mathcal{U}, \tag{3.7}$$

where $w(\cdot) \in L_2([0,T];\mathcal{U})$ and $K(\cdot,\cdot) \in \mathcal{B}_2(\Delta, \mathcal{L}(\mathbf{R}^k, \mathcal{U}))$. We contend that this choice of the control u transfers (3.5) from an arbitrary interest rate r_0 to an arbitrary element h of $\mathcal{U}$.

3.3 An Appropriate Subclass of HJMM Interest Rate Models

From [17] it is easy to deduce that if $u(\cdot) \in \mathcal{U}_{ad}$ then for $\widetilde{r}(t)$ from (3.5) the Kalman filter

$$r(t) = \mathbf{E}\{\widetilde{r}(t)|\mathcal{Z}_t\} = \mathbf{E}\{\widetilde{r}(t)|\mathcal{Z}_t^0\}$$

is the mild solution of the linear HJMM interest rate model

$$\left\{ \begin{array}{rcl} dr(t) &=& (Ar(t) + D(t)u(t))\, dt + \sigma(r(t))dW(t), \\ r(0) &=& r^*(0), \end{array} \right. \tag{3.8}$$

where $D(t)$ and $u(t)$ are given by (2.2) and (3.7), respectively. Also, we have that

$$\sigma(r(\cdot)) \quad = \quad P(\cdot)C^*(FQF^*)^{-1}F. \tag{3.9}$$

We assume that the linear operator P may be chosen in such a way that the volatility (3.9) is (locally) Lipschitz continuous in r. From [7], in this case, we have that σ and $D(t)$ in (3.8) are (locally) bounded.

The general problem that is discussed in the rest of this section may be stated as follows.

Can we find a control u that drives the HJMM interest rate model given by (3.8) from an arbitrary interest rate r_0 to any other interest rate h contained in the open convex subset $\mathcal{U}$ of the separable state space $\mathcal{H}$ of forward rate curves ?

Strong solutions of (3.8) are very seldom encountered in the context of interest rate models. Under the conditions specified in the previous paragraph (see [17]) for more details), we are able to write the mild solution $r(t; r_0, u)$ of (3.8) explicitly as

$$\begin{aligned} r(t; r_0, u) \quad &= \quad S_t r^*(0) + \int_0^t S_{t-s}D(s)u(s)ds \\ &\quad + \int_0^t S_{t-s}\sigma(r(s))dW(s). \end{aligned} \tag{3.10}$$

In addition, we note that a deterministic counterpart of (3.8) may be given as

$$\begin{array}{rcl} \dfrac{dr_d(t)}{dt} &=& Ar_d(t) + D(t)v(t), \\ r_d(0) &=& r_d^*(0), \end{array} \tag{3.11}$$

with a solution of the form

$$r_d(t) \quad = \quad S_t r_d^*(0) + \int_0^t S_{t-s}D(s)v(s)ds. \tag{3.12}$$

3.4 Controllability Operators for Linear Interest Rate Models

Our strategy for solving the controllability problem for the linear HJMM interest rate model described above is based on an analysis of the stochastic controllability operator

$$\Pi_s^T : L_2(\mathcal{Z}_T, \mathcal{U}) \to L_2(\mathcal{Z}_T, \mathcal{U})$$

corresponding to (3.8) and its solution (3.10) given by

$$\Pi_s^T\{\cdot\} = \int_s^T S_{T-t}D(t)D(t)^* S_{T-t}^* \mathbf{E}\{\cdot|\mathcal{Z}_t\}dt, \tag{3.13}$$

where S_t and $D(t)$ are given by (2.4) and (2.2), respectively. Moreover, the related deterministic controllability operator Γ_s^T corresponding to (3.11) and its solution (3.12) has the form

$$\Gamma_s^T\{\cdot\} = \int_s^T S_{T-t}D(t)D(t)^* S_{T-t}^* dt. \tag{3.14}$$

Next, we state and prove an important lemma.

Lemma 3.1 *For each $z \in L_2(\mathcal{Z}_t, \mathcal{U})$ there exists a process $\psi(\cdot) \in L_2^{\mathcal{Z}}([0,T]; \mathcal{L}(\mathbf{R}^k, \mathbf{R}^n))$ with the properties that*

1. *the stochastic controllability operator*

$$\Pi_0^T z = \Gamma_0^T \mathbf{E}z + \int_0^T \Gamma_s^T \psi(s)dW(s); \tag{3.15}$$

2. *for all $\lambda > 0$, we have that*

$$R(\lambda_1, -\Pi_0^T)z = R(\lambda_1, -\Gamma_0^T)\mathbf{E}z + \int_0^T R(\lambda_1, -\Gamma_s^T)\psi(s)dW(s). \quad (3.16)$$

Proof. The proofs of these facts are as follows.

1. We note that a consideration of the definition of the stochastic controllability operator Π_s^T in (3.13), the stochastic version of Fubini's Theorem and (3.6) leads to the required representation

$$\begin{aligned}\Pi_0^T z &= \Gamma_0^T \mathbf{E}z \\ &+ \int_0^T S_{T-t}D(t)D(t)^* S_{T-t}^* \int_0^t \psi(s)dW(s)dt \\ &= \Gamma_0^T \mathbf{E}z \\ &+ \int_0^T \int_s^T S_{T-t}D(t)D(t)^* S_{T-t}^* dt \psi(s)dW(s).\end{aligned}$$

2. The proof makes use of (3.6) and (3.15).

□

3.5 Regulator Problem for Linear HJMM Interest Rate Models

We define a linear regulator problem related to the minimum energy principle. The problem is to minimize

$$J(u) = \mathbf{E}||r(T;r_0,u) - h||^2 + \lambda \mathbf{E}\int_0^T ||u(t)||^2 dt, \quad (3.17)$$

over all $u(\cdot) \in \mathcal{U}_{ad}$, where $r(T;r_0,u)$ is a stochastic interest rate; $h \in L_2(\mathcal{Z}_T, \mathcal{U})$ and $\lambda > 0$ are parameters; and h has the representation

$$h = \mathbf{E}(h) + \int_0^T h(s)dW(s).$$

Next, we investigate the existence of a unique optimal control $u^\lambda(\cdot) \in \mathcal{U}_{ad}$ at which the functional (3.17) takes on its minimum value.

Lemma 3.2 *There exists a unique optimal control $u^\lambda(\cdot) \in \mathcal{U}_{ad}$ at which* (3.17) *takes on its minimum value and*

$$\begin{aligned}u^\lambda(t) &= -D(t)^* S_{T-t}^* \quad (3.18)\\ &\times \{R(\lambda, -\Gamma_0^T)(S_T r_0 - \mathbf{E}(h)) \\ &+ \int_0^t R(\lambda, -\Gamma_0^T)\left[S_{T-s}\sigma(r(s)) - h(s)\right]dW(s)\};\end{aligned}$$

$$\begin{aligned}r(T, r_0, u^\lambda) - h &= \lambda R(\lambda, -\Gamma_0^T)(S_T r_0 - \mathbf{E}(h)) \quad (3.19)\\ &+ \int_0^t \lambda R(\lambda, -\Gamma_s^T)[S_{T-s}\sigma(r(s)) - h(s)]dW(s).\end{aligned}$$

Proof. The problem of minimizing the functional (3.17) has a unique solution $u^\lambda(\cdot) \in \mathcal{U}_{ad}$ which is completely characterized by the stochastic maximum principle (see [1]) and has the following form.

$$u^\lambda(t) = -\lambda^{-1}D(t)^* S_{T-t}^* \mathbf{E}\left\{r(T, r_0, u^\lambda) - h|\mathcal{Z}_t\right\}. \quad (3.20)$$

Using this in (3.10) we have that

$$\begin{aligned}r(T, r_0, u^\lambda) &= S_T r_0 + \int_0^T S_{T-s}\sigma(r(s))dW(s) \\ &\quad -\lambda^{-1}\Pi_0^T(r(T, r_0, u^\lambda) - h).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\lambda r(T, r_0, u^\lambda) \\ &= \lambda\left(S_T r_0 + \int_0^T S_{T-s}\sigma(r(s))dW(s)\right) \\ &\quad -\Pi_0^T(r(T, r_0, u^\lambda) - h)\end{aligned}$$

which implies that

$$\begin{aligned}&(\lambda I + \Pi_0^T)r(T, r_0, u^\lambda) \\ &= \lambda\left(S_T r_0 + \int_0^T S_{T-s}\sigma(r(s))dW(s)\right) + \Pi_0^T h.\end{aligned}$$

Consequently, it follows from Lemma 3.1 that

$$\begin{aligned}&r(T, r_0, u^\lambda) - h = \lambda(\lambda I + \Pi_0^T)^{-1} \\ &\left(S_T r_0 + \int_0^T S_{T-s}\sigma(r(s))dW(s)\right) \\ &+(\lambda I + \Pi_0^T)^{-1}(\lambda I + \Pi_0^T - \lambda I)h - h\end{aligned}$$

Thus (3.19) holds. Substituting (3.19) into (3.20) we obtain (3.18) in the following way.

$$\begin{aligned}u^\lambda(t) &= -\lambda^{-1}D(t)^* S_{T-t}^* \mathbf{E} \\ &\times \left\{\lambda R(\lambda, -\Gamma_0^T)(S_T r_0 - \mathbf{E}h) + \int_0^T \lambda R(\lambda, -\Gamma_s^T)\right. \\ &\left.(S_{T-s}\sigma(r(s)) - h(s))\, dW(s)|\mathcal{Z}_t\right\}.\end{aligned}$$

This proves the lemma. □

3.6 Complete and Approximate Controllable HJMM Interest Rate Models

Suppose for the definitions of complete and approximate controllability of the linear HJMM interest rate model that the reachability subspace

$$\mathcal{R}(t, r_0) = \{r(t, r_0, u) : u \in \mathcal{U}_{ad}\}.$$

Definition 3.3 *The linear HJMM interest rate model* (2.1) *is* **completely controllable** *on $[0, T]$ if for* (3.8) *all the points in $L_2(\mathcal{Z}_T, \mathcal{U})$ can be reached from the initial interest rate r_0 at time T, i.e., if $\mathcal{R}(T, r_0) = L_2(\mathcal{Z}_T, \mathcal{U})$. Also,* (2.1) *is* **approximately controllable** *on $[0, T]$ if for* (3.8) $\overline{\mathcal{R}(T, r_0)} = L_2(\mathcal{Z}_T, \mathcal{U})$.

Lemma 3.4 *1. If the stochastic HJMM interest rate model in* (2.1) *is approximately controllable on* $[0, T]$ *then its deterministic counterpart* (2.3) *is approximately controllable on every* $[s, T],\ 0 \leq s < T$.

2. If the deterministic model (2.3) *is small time approximately controllable on every* $[s, T],\ 0 \leq s < T$ *then the stochastic HJMM interest rate model in* (2.1) *is small time approximately controllable on* $[0, T]$.

Proof.

1. By approximate controllability we have

$$\mathbf{E}||\lambda R(\lambda, -\Pi_0^T)z||^2 \to 0.$$

From (3.19) in Lemma 3.1 we conclude that

$$\mathbf{E}||\lambda R(\lambda, -\Pi_0^T)z||^2 = ||\lambda R(\lambda, -\Gamma_0^T)\mathbf{E}z||^2 \quad (3.21)$$
$$+\mathbf{E}\sum_{j=1}^{k} D_j(t)\int_0^T ||\lambda R(\lambda, -\Gamma_s^T)\psi_j(s)||^2 ds \to 0$$

which for all $\psi(\cdot) \in L_2^{\mathcal{Z}}([0, T]; \mathcal{L}(\mathbf{R}^k, \mathcal{U}))$ has the result that

$$\mathbf{E}\sum_{j=1}^{k} D_j(t)\int_0^T ||\lambda R(\lambda, -\Pi_s^T)\psi_j(s)||^2 ds \to 0.$$

This implies that a subsequence $\{\lambda_k\}$ exists such that for all $h \in \mathcal{U}$

$$||\lambda_k R(\lambda_k, -\Gamma_s^T)h||^2 \to 0, \text{ almost everywhere on } [0, T].$$

Because of the continuity of $R(\lambda, -\Gamma_s^T)$ this property holds for all $0 \leq s < T$ and the result follows.

2. By small time controllability we have

$$||\lambda R(\lambda, -\Gamma_s^T)h|| \to 0 \text{ as } \lambda \to 0^+.$$

Since by the Lebesque Dominated Convergence Theorem

$$\sum_{j=1}^{k} D_j(t)||\lambda R(\lambda, -\Pi_s^\tau)\psi_j(s)||^2 \leq ||\psi_j(s)||^2$$

it follows from (3.21) that

$$\mathbf{E}||\lambda R(\lambda, -\Pi_0^T)z||^2 \to 0 \text{ as } \lambda \to 0^+.$$

□

We note that in the case where $S_t = e^{At}$ the generator is analytic and hence we have the following result.

Theorem 3.5 *The linear stochastic HJMM interest rate model* (2.1) *is approximately controllable on* $[0, T]$ *if and only if it is small time approximately controllable.*

Proof. This result follows from the definitions of approximate and small time approximate controllability and Lemma 3.4. □

3.7 Stochastically Controllable Linear HJMM Interest Rate Models

Next, we define the stochastic controllability of linear HJMM interest rate models. For this, we have to introduce the set

$$\mathcal{A}_\epsilon^p(t, r_0) =$$
$$\{h \in \mathcal{U} : \exists u \in \mathcal{U}_{ad},\ \mathbf{Q}(||r(t, r_0, u) - h||^2 \leq \epsilon) \geq p\}$$

and $\mathcal{A}(T, r_0) = \cap_{\epsilon > 0,\ 0 \leq p \leq 1}\mathcal{A}_\epsilon^p(T, r_0)$. Moreover, it is a well-known fact that $\overline{\mathcal{A}(T, r_0)} = \mathcal{A}(T, r_0)$. The following definition is an important one.

Definition 3.6 *The linear stochastic HJMM interest rate model* (2.1) *is* **stochastically controllable** *if for* (3.8)

$$\overline{\mathcal{A}(T, r_0)} = \mathcal{A}(T, r_0) = \mathcal{U}.$$

Theorem 3.7 *The linear stochastic HJMM interest rate model* (2.1) *is approximately controllable on* $[0, T]$ *if and only if it is small time stochastically controllable with the control set* $\mathcal{U}_{ad}^g$.

Proof. If (2.1) is approximately controllable then by Lemma 3.4 it is approximately controllable on each $[s, T]$ and hence $\lambda R(\lambda, -\Gamma_s^T) \to 0$ strongly as $\lambda \to 0^+$. Furthermore, Lemma 3.2 claims that for any fixed $h \in \mathcal{U}$ that is nonrandom there exists a Guassian control (3.18) in $\mathcal{U}_{ad}^g$ with the property that

$$r(T, r_0, u^\lambda) - h = \lambda R(\lambda, -\Gamma_0^T)(S_T r_0 - \mathbf{E}h)$$
$$+ \int_0^T \lambda R(\lambda, -\Gamma_s^T)S_{T-s}\sigma(r(s))dW(s).$$

We conclude that $\mathbf{E}||r(T, r_0, u^\lambda) - h||^2 \to 0$ as $\lambda \to 0^+$. In this case, the small time stochastic controllability of the HJMM interest rate model in (2.1) with control set $\mathcal{U}_{ad}^g$ is a consequence of Chebychev's inequality.

Conversely, let $h \in \mathcal{U}$ and consider

$$\{\epsilon_n : \epsilon_n > 0,\ \epsilon_n \to 0\} \text{ and } \{p_n : 0 \leq p_n \leq 1,\ p_n \to 1\}.$$

In this case, we are assured of the existence of a sequence $\{u^n\} \in \mathcal{U}_{ad}^g$ with the property that

$$\mathbf{Q}\{||r(T, r_0, u^n) - h||^2 \leq \epsilon\} \geq p_n.$$

From this we deduce that for any $\epsilon > 0$ there is a number N such that $0 < \epsilon_n < \epsilon^2$. Furthermore, we have that

$$\mathbf{Q}\{||r(T, r_0, u^n) - h||^2 > \epsilon\} \leq 1 - p_n,$$

for all $n > N$. This inequality suggests that $r(T, r_0, u^n)$ converges to h in probability which in turn implies that for any $\epsilon > 0$, it follows that

$$\mathbf{Q}\{||r(T, r_0, u^n) - h|| > \epsilon\} \to 0, \text{ as } n \to \infty.$$

In this case, for all $x \in \mathcal{U}$ we have

$$\lim_{n\to\infty} \mathbf{E}e^{i\langle r(T, r_0, u^n), x\rangle} = \mathbf{E}e^{i\langle h, x\rangle}.$$

Since $r(T, r_0, u^n)$ is a Guassian random variable (as the solution (3.10) of (3.8) corresponding to the Guassian control u^n), from the convergence of characteristic functions and the guassian properties of $r(T, r_0, u^n)$ and h it follows that

$$\mathbf{E}e^{i\langle r(T,r_0,u^n),x\rangle} = e^{i\langle m_n,x\rangle - 1/2\langle \Lambda_n x,x\rangle}$$

and $\lim_{n\to\infty} e^{i\langle m_n,x\rangle-1/2\langle\Lambda_n x,x\rangle} = e^{i\langle h,x\rangle}$. As a result we may deduce that for all $x \in \mathcal{U}$ we have

$$\langle m_n, x\rangle \to \langle h, x\rangle \text{ and } \langle \Lambda_n x, x\rangle \to 0 \text{ as } n \to \infty,$$

where $m_n = \mathbf{E}r(T, r_0, u^n)$ and $\Lambda_n = \text{cov}\ (r(T, r_0, u^n))$. Convergence in the first instance results in the sequence $\{m_n\} \in \mathcal{U}$ converging weakly to h in $\mathcal{U}$. Mazur's Theorem implies that we can construct the sequence

$$h_n = \sum_{i=1}^{n} c_i^n m_i,\ c_i^n \geq 0,\ \sum_{i=1}^{n} c_i^n = 1,\ i = 1,\ 2,\ \ldots$$

of convex combinations of $m_i = \mathbf{E}r(T, r_0, u^i)$, $i = 1,\ 2,\ \ldots,\ n$ so that h_n converges to h in the strong topology of $\mathcal{U}$. Next, we write $\widetilde{u}^n = \sum_{i=1}^{n} c_i^n u_i,\ n = 1,\ 2,\ \ldots$. It follows directly that $\widetilde{u}^n \in \mathcal{U}_{ad}$. Because of the affineness of (3.8), it follows that $h_n = \sum_{i=1}^{n} c_i^n \mathbf{E}r(T, r_0, u^i) = \mathbf{E}r(T, r_0, \widetilde{u}^n)$. Next, if $\widetilde{v}^n = \mathbf{E}\widetilde{u}^n \in \mathcal{V}$ then $h_n = y(T, r_0, \widetilde{u}^n)$ and as a result

$$\lim_{n\to\infty} ||y(T, r_0, \widetilde{u}^n) - h|| = \lim_{n\to\infty} ||h_n - h|| = 0.$$

By the equivalence in Lemma 3.4, the result holds. □

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