A METHOD TO DETERMINE A FLAT OUTPUT AND THE PARAMETRIZATION OF THE SOLUTION OF SOME SYSTEMS DESCRIBED BY PARTIAL DIFFERENTIAL EQUATIONS

Ch. Fleck [*], **D. Abel.**

[*] Aachen University of Technology, Institute of Automatic Control, Steinbachstr. 54, D-52074 Aachen, Germany, Ch.Fleck@irt.rwth-aachen.de

Keywords: flatness, Taylor-expansion, distributed-parameter systems; open loop control, heat transfer.

Abstract

This paper presents a method to determine the flat output of some one dimensional boundary controlled distributed systems.
Once the flat output determined, the parametrization of the solution is achieved using a Taylor-expansion in the space coordinate variable of the flat output. The procedure is illustrated by two problems related to heat transfer, described respectively by a parabolic and a hyperbolic partial differential equation.

1 Introduction

One problem of open loop control is, using a model of a process, to define the time shape of the manipulated variable such that the controlled variable takes a desired trajectory between two stationary states.
This problem simplifies a lot if the system is flat. For this kind of system, all the state variables and inputs are parametrized by the so called flat output. So if the flat output is also the output of the system, then the time shape of the manipulated variable is directly defined by injecting the desired trajectory of the controlled variable in the parametrization.
The concept of flatness has first be defined [1] for systems gouverned by ordinary differential equations and could be extended to boundary controlled one dimensional distributed systems [4].
For the last case the flat output is a function of the solution of the pde in a fixed space coordinate. From the knowledge of the author there is no method to find out the flat output of such a system. Some cases have been studied, specially parabolic equations and can help by similarity to determine the flat output of some other systems. Concerning the parametrization some results are known: for parabolic equation [3], a power series expansion in the space coordinate of the flat output is used and its coefficients are determined by substituting this series in the pde. For hyperbolic equations [5], Mikusiński operators are used but can only be applied for linear pde.

In this paper, we present a simple condition to determine the space location of the flat output for some one dimensional pde. The parametrization of the solution is then achieved using a Taylor-expansion about the space coordinate of the flat output.

2 The method

Consider the following example taken from [4]:

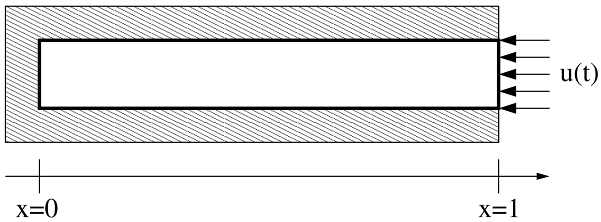

Figure 1: Heat conduction in a rod

A rod is heated on its surface $x = 1$ and insulated on its other surfaces. The heat conduction equation is given by:

$$\frac{\partial T(x,t)}{\partial t} = a \cdot \frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

The boundary conditions are given by:

$$\begin{aligned} &\forall t && \tfrac{\partial T}{\partial x}(0,t) = 0 \\ &\forall t > 0 && T(1,t) = u(t) \end{aligned} \tag{2}$$

The rod is supposed to be in a stationary state for $t < 0$, so the initial condition can be set at zero: $T(x,0) = 0$.
Transforming into the Laplace domain, equation (1) becomes:

$$s \cdot \hat{T}(x,s) = \frac{\partial \hat{T}^2}{\partial x^2}(x,s) \tag{3}$$

The solution of this ordinary differential equation under consideration of the boundary condition (2) is then given by

$$\hat{T}(x,s) = \frac{\cosh\left(x \cdot \sqrt{\frac{s}{a}}\right)}{\cosh\left(\sqrt{\frac{s}{a}}\right)} \cdot \hat{u}(s) \tag{4}$$

With

$$\hat{T}(0,s) = \frac{\hat{u}(s)}{\cosh\left(\sqrt{\frac{s}{a}}\right)} \tag{5}$$

equ. (4) can be rewritten as

$$\hat{T}(x,s) = \cosh\left(x \cdot \sqrt{\frac{s}{a}}\right) \cdot \hat{T}(0,s) \tag{6}$$

Setting $y(s) = \hat{T}(0,s)$ and transforming back into the time domain gives the parametrization of the solution

$$T(x,t) = \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i \cdot \frac{x^{2\cdot i}}{(2\cdot i)!} \cdot y^{(i)}(t) \tag{7}$$

$$u(t) = \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i \cdot \frac{y^{(i)}(t)}{(2\cdot i)!} \tag{8}$$

$y_f(t)$ is a flat output because it determine, by a series involving infinitely many derivatives of it (equ.8,7), the solution of (1).
On the basis of this example, we can state that:

A one dimensional distributed system with one boundary control is flat, if the substitution for one given boundary condition with one in $x = x_f$ which equals $y_f(t)$ fixes as many boundary conditions in the point x_f as necessary to solve the system.
The flat output is then $y_f(t)$ and its space coordinate is given by x_f

To determine the parametrization, the solution $T(x,t)$ is expanded in a Taylor-series about the point $x = x_f$, and using the pde of the system all the derivatives of the solution with respect to the space coordinate x are replaced by a function of derivatives of the flat output.

In the example above, substituting $T(1,t) = u(t)$ with $T(0,t) = y_f(t)$ alloys with $\frac{\partial T}{\partial x}(0,t) = 0$ to solve (4) with only boundary conditions in $x = 0$. Thus $T(0,t)$ is a flat output.
The Taylor-expansion of $T(x,t)$ about the point $x = 0$ is given by:

$$T(x,t) = T(0,t) + \frac{\partial T}{\partial x}(0,t) \cdot x + \frac{\partial^2 T}{\partial x^2}(0,t) \cdot \frac{x^2}{2!} + \ldots \tag{9}$$

with $T(0,t) = y_f(t)$ and $\frac{\partial T}{\partial x}(0,t) = 0$. The pde (4) states that derivating twice with respect to x is equivalent to derivating ones with respect to t :

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t} \tag{10}$$

Thus for $m \in \mathbb{N}$ the following relation is valid:

$$\frac{\partial^{2m} T}{\partial x^{2m}} = \frac{1}{a^m}\frac{\partial^m T}{\partial t^m},\ \frac{\partial^{2m+1}}{\partial x^{2m+1}} = \frac{1}{a^m}\frac{\partial^m}{\partial t^m}\left(\frac{\partial T}{\partial x}\right) \tag{11}$$

Calculating the limit of this expressions as x approaches 0 gives:

$$\begin{aligned} &\tfrac{\partial^{2m} T}{\partial x^{2m}}(0,t) = \tfrac{1}{a^m}\tfrac{\partial^m T}{\partial t^m}(0,t) = \tfrac{1}{a^m}\tfrac{d^m y_f(t)}{dt^m} \\ &\tfrac{\partial T^{2m+1}}{\partial x^{2m+1}}(0,t) = \lim_{x\to 0}\tfrac{1}{a^m}\tfrac{\partial^m}{\partial t^m}\left(\tfrac{\partial T}{\partial x}\right) = \tfrac{1}{a^m}\tfrac{\partial^m}{\partial t^m}\left(\lim_{x\to 0}\tfrac{\partial T}{\partial x}\right) = 0 \end{aligned} \tag{12}$$

Consequently the parametrization is given by

$$T(x,t) = \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^n \cdot \frac{x^{2\cdot n}}{(2\cdot n)!} \cdot y^{(n)}(t) \tag{13}$$

It is worth to notice that the since $\frac{\partial T}{\partial x}(0,t) = 0$ the solution of (4) is even

$$T(x,t) = T(-x,t)$$

which implies directly that:

$$\forall m \in \mathbb{N} \quad \frac{\partial T^{2m+1}}{\partial x^{2m+1}}(0,t) = 0$$

3 The one dimensional heat transfer equation

We consider again the one dimensional heat equation for simple shapes like the rod, the cylinder or the sphere heated on their surfaces.

The general equation for this shapes is

$$\frac{1}{a} \cdot \frac{\partial T}{\partial t}(r,t) = \frac{\partial^2 T}{\partial r^2}(r,t) + \frac{m}{r} \cdot \frac{\partial T}{\partial r}(r,t) \tag{14}$$

with $m = 0$ for the rod, $m = 1$ for the cylinder and $m = 2$ for the sphere.

We consider the system to be in a stationary state for $t < 0$, $T(x,0) = 0$ and the following boundary conditions:

$$T(1,t) = T(-1,t) = u(t) \tag{15}$$

This conditions imply that the solution of (14) is even. Due to this symmetry, we get another boundary condition which is:

$$\frac{\partial T}{\partial r}(0,t) = 0 \tag{16}$$

This condition implies also that all the odd derivatives of the temperature $T(r,t)$ with respect to r in the position $x = 0$ must vanish:

$$\frac{\partial^3 T}{\partial r^3}(0,t) = \frac{\partial^5 T}{\partial r^5}(0,t) = \ldots \frac{\partial^{2\cdot n+1} T}{\partial r^{2\cdot n+1}}(0,t) = 0 \tag{17}$$

The pde is of order 2 so we need two boundary conditions at same space coordinate to get a flat output:
Here it is obvious: If $T(0,t) = y_f(t)$ is given, then because of $\frac{\partial T}{\partial r}(0,t) = 0$ the system is determined. Thus $y_f(t) = T(0,t)$ is a flat output.

To determine the parametrization of the solution of the pde system (14), the temperature $T(r,t)$ is expanded in a Taylor series about the position $r = 0$:

$$\begin{aligned} T(r,t) = \quad & T(0,t) + \tfrac{\partial T(0,t)}{\partial r} \cdot \tfrac{r}{1!} + \tfrac{\partial^2 T(0,t)}{\partial r^2} \cdot \tfrac{r^2}{2!} \\ & + \tfrac{\partial^3 T(0,t)}{\partial r^3} \cdot \tfrac{r^3}{3!} + \ldots + \tfrac{\partial^n T(0,t)}{\partial r^n} \cdot \tfrac{r^n}{n!} + \ldots \end{aligned} \tag{18}$$

Due to the symmetry (equ 16-17), this expression simplifies to

$$\begin{aligned} T(r,t) &= T(0,t) + \tfrac{\partial^2 T(0,t)}{\partial r^2} \cdot \tfrac{r^2}{2!} + \ldots \\ &= \sum_{n=0}^{\infty} \tfrac{\partial^{2n} T}{\partial r^{2n}}(0,t) \cdot \tfrac{r^{2n}}{(2n)!} \end{aligned} \tag{19}$$

The next step in to find a relation between the derivatives of the temperature with respect to x in the position $x = 0$ and the derivatives of the flat output $y_f(t)$.

Derivating

$$\frac{\partial^2 T}{\partial r^2}(r,t) = -\frac{m}{r} \cdot \frac{\partial T}{\partial r}(r,t) + \frac{1}{a} \cdot \frac{\partial T}{\partial t}(r,t) \tag{20}$$

with respect to x $(2n)$ times becomes

$$\frac{\partial^{2n+2} T}{\partial r^{2n+2}}(r,t) = -m \cdot \frac{\partial^{2n}}{\partial r^{2n}}\left(\frac{1}{r} \cdot \frac{\partial T}{\partial r}(r,t)\right) + \frac{\partial^{2n}}{\partial r^{2n}}\left(\frac{\partial T}{\partial t}(r,t)\right) \tag{21}$$

Therefore

$$\begin{aligned} &\tfrac{\partial^{2n+2} T}{\partial r^{2n+2}}(0,t) \\ &= \lim_{x \to 0} -m \tfrac{\partial^{2n}}{\partial r^{2n}}\left(\tfrac{1}{r} \cdot \tfrac{\partial T}{\partial r}(r,t)\right) + \tfrac{\partial^{2n}}{\partial r^{2n}}\left(\tfrac{\partial T}{\partial t}(r,t)\right) \\ &= -m \cdot \lim_{r \to 0} \tfrac{\partial^{2n}}{\partial r^{2n}}\left(\tfrac{1}{r} \cdot \tfrac{\partial T}{\partial r}(r,t)\right) + \tfrac{\partial}{\partial t}\left(\tfrac{\partial^{2n}}{\partial r^{2n}}(0,t)\right) \\ &= -m \cdot \lim_{r \to 0} \tfrac{\partial^{2n}}{\partial r^{2n}}\left(\tfrac{1}{r} \cdot \tfrac{\partial T}{\partial r}(r,t)\right) + \tfrac{\partial}{\partial t}\left(\tfrac{\partial^{2n}}{\partial r^{2n}}(0,t)\right) \end{aligned} \tag{22}$$

A simple calculation shows that

$$\lim_{r \to 0} \frac{\partial^{2n}}{\partial r^{2n}}\left(\frac{1}{r} \cdot \frac{\partial T}{\partial r}(r,t)\right) = \frac{1}{(2n+1)} \cdot \frac{\partial^{2n+2} T}{\partial r^{2n+2}}(0,t) \tag{23}$$

implying the following recursive relation

$$\frac{\partial^{2n+2} T}{\partial r^{2n+2}}(0,t) = \frac{2n+1}{2n+1+m} \cdot \frac{\partial}{\partial t} \frac{\partial^{2n} T}{\partial r^{2n}}(0,t) \tag{24}$$

from which we get

$$\frac{\partial^{2n} T}{\partial r^{2n}}(0,t) = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(1+m) \cdot \ldots \cdot (2n-1+m)} \cdot \frac{\partial^n T}{\partial t^n}(0,t) \tag{25}$$

Finally

$$T(r,t) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!} \cdot f(m) \cdot y^{(n)}(t) \tag{26}$$

$$\begin{array}{ll} \text{Rod} & T(r,t) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!} \cdot y^{(n)}(t) \\ \text{Cylinder} & T(r,t) = \sum_{n=0}^{\infty} \left(\frac{r}{2}\right)^{2n} \frac{y^{(n)}(t)}{(n!)^2} \\ \text{Sphere} & T(r,t) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n+1)!} \cdot y^{(n)}(t) \end{array} \tag{27}$$

This procedure can be extended, for the same kind of boundary conditions, to other shapes and to nonlinear heat equations (e.g with temperature depending heat capacity).

This example shows that the symmetry of the solution of the heat equation plays a important role to figure out the flat output and the parametrization of the solution.

4 The heat exchanger equation

This example is taken from [5].

Co-current Heat Exchanger

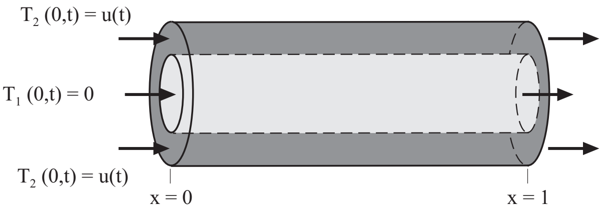

Figure 2: Co-current heat Exchanger

The co-current heat exchanger can be described by the following system of first order pde's [2]:

$$\begin{aligned} \frac{\partial T_1}{\partial t}(x,t) + v_1 \cdot \frac{\partial T_1}{\partial x}(x,t) = a_1 \cdot (T_2(x,t) - T_1(x,t)) \\ \frac{\partial T_2}{\partial t}(x,t) + v_2 \cdot \frac{\partial T_2}{\partial x}(x,t) = a_2 \cdot (T_1(x,t) - T_2(x,t)) \end{aligned} \tag{28}$$

$T_1(x,t)$, $T_2(x,t)$, v_1 and v_2 denotes the temperature profile and the velocity of, respectively, the fluid 1 and 2. This system is considered to be in a stationary state for $t < 0$ meaning the initial conditions are vanishing

$$T_1(x,0) = 0 \text{ and } T_2(x,0) = 0 \quad t < 0 \quad 0 \le x \le 1 \tag{29}$$

The purpose of this system is to control the temperature profile $T_1(x,t)$ of the fluid 1 using as manipulated variable the temperature at the boundary $x = 0$, $T_2(0,t)$ of the fluid 2. Thus the boundary conditions are taken to be

$$T_1(0,t) = 0 \text{ and } T_2(0,t) = u(t) \tag{30}$$

The dimension of the manipulated variable ist one. Applying the main result of this paper, we will be able to determine a flat output in a point $x = x_f$ if by adding a boundary condition in x_f equals to $y_f(t)$ we have enough boundary condition in this point to solve the system (28).

Frrom the first boundary condition in $x = 0$ we get

$$T_1(0,t) = 0 \text{ and } \frac{\partial T_1}{\partial t}(0,t) = 0 \tag{31}$$

Therefore the system of pde (28) in $x = 0$ becomes

$$\frac{\partial T_1}{\partial x}(0,t) = \frac{a_1}{v_1} \cdot T_2(0,t) \tag{32}$$

and

$$\frac{\partial T_2}{\partial x}(0,t) = -\frac{a_2}{v_2}\cdot T_2(0,t) - \frac{1}{v_2}\cdot\frac{\partial T_2}{\partial t}(0,t) \tag{33}$$

Thus $T_2(0,t)$ or $\frac{\partial T_1}{\partial x}(0,t)$ are possible flat outputs for this system:
If $T_2(0,t)$ is given then the first equation (32) implies that $\frac{\partial T_1}{\partial x}(0,t)$ is determined. If $T_2(0,t)$ is given, so is $\frac{\partial T_2}{\partial t}(0,t)$ and the second equation (33) implies that $\frac{\partial T_2}{\partial x}(0,t)$ is determined.
We choose $y_f(t) = T_2(0,t)$ as a flat output.
To determine the parametrization of the solution of the pde system (28), we assume as in the case of parabolic pde that the temperature $T_1(x,t)$ and $T_2(x,t)$ can be expanded in a Taylor series about the position of the flat output $x = 0$:

$$T_1(x,t) = T_1(0,t) + \frac{\partial T_1}{\partial x}(0,t)\cdot x + \frac{\partial^2 T_1}{\partial x^2}(0,t)\cdot\frac{x^2}{2!} + \ldots \tag{34}$$

$$T_2(x,t) = T_2(0,t) + \frac{\partial T_2}{\partial x}(0,t)\cdot x + \frac{\partial^2 T_2}{\partial x^2}(0,t)\cdot\frac{x^2}{2!} + \ldots \tag{35}$$

The next step is to determine an expression between the derivative of the temperatures with respect to x in the position $x = 0$ and the flat output.
For that we derive the system (28) $n-1$ times with respect to x:

$$\begin{aligned}\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial T_1}{\partial x}(x,t)\right) &= -\frac{1}{v_1}\cdot\frac{\partial}{\partial t}\left(\frac{\partial^{n-1}T_1}{\partial x^{n-1}}(x,t)\right)\\ &+\frac{a_1}{v_1}\cdot\left(\frac{\partial^{n-1}T_2(x,t)}{\partial x^{n-1}} - \frac{\partial^{n-1}T_1(x,t)}{\partial x^{n-1}}\right)\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial T_2}{\partial x}(x,t)\right) = -\frac{1}{v_2}\cdot\frac{\partial}{\partial t}\left(\frac{\partial^{n-1}T_2}{\partial x^{n-1}}(x,t)\right) +\\ \frac{a_2}{v_2}\cdot\left(\frac{\partial^{n-1}T_1(x,t)}{\partial x^{n-1}} - \frac{\partial^{n-1}T_2(x,t)}{\partial x^{n-1}}\right)\end{aligned} \tag{37}$$

Defining

$$f_i(t) = \frac{\partial^i T_1}{\partial x}(0,t) \quad g_i(t) = \frac{\partial^i T_2}{\partial x^i}(0,t) \tag{38}$$

we get from equ.((36) and (37) the following recursion formulae:

$$f_i(t) = -\frac{1}{v_1}\cdot\frac{df_{i-1}(t)}{dt} + \frac{a_1}{v_1}\cdot(g_{i-1}(t) - f_{i-1}(t))$$

$$g_i(t) = -\frac{1}{v_2}\cdot\frac{dg_{i-1}(t)}{dt} + \frac{a_2}{v_2}\cdot(f_{i-1}(t) - g_{i-1}(t))$$

with

$$f_0(t) = 0 \text{ and } g_0(t) = y_f(t)$$

Thus the Taylor expansion of $T_1(x,t)$ (34) and $T_2(x,t)$ (35) becomes

$$T_1(x,t) = \sum_{i=0}^{\infty} f_i(t)\cdot\frac{x^i}{i!} \tag{39}$$

$$T_2(x,t) = \sum_{i=0}^{\infty} g_i(t)\cdot\frac{x^i}{i!} \tag{40}$$

This results are illustrated by a simulation.
$y_f(t)$ is chosen to be a polynomial of class C^{11}. The Taylor-expansion for $T_1(x,t)$ and $T_2(x,t)$ were broken after 10 terms so that the continuity of the temperatures profiles is insured. To compare the solution given by the Taylor expansion, the system (28) was solved numericaly with Maple using the boundary condition

$$T_1(0,t) = 0 \text{ and } T_2(0,t) = y_f(t) \tag{41}$$

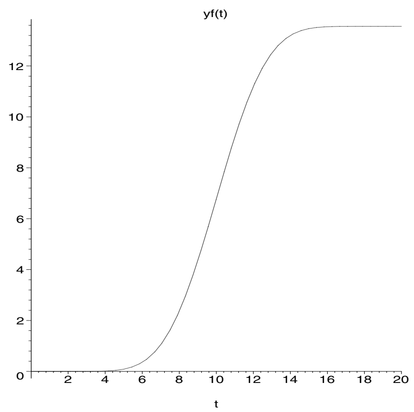

Figure 3: $y_f(t) = T_2(0,t)$

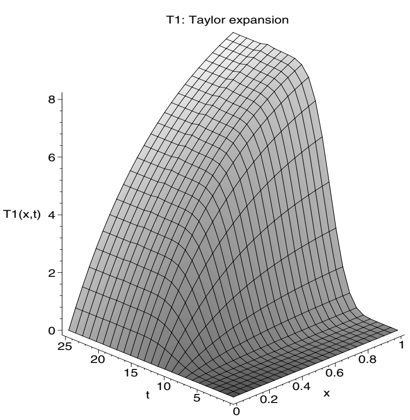

Figure 4: $T_1(x,t)$: Taylor-Expansion

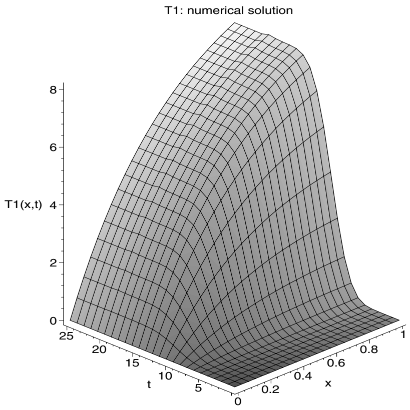

Figure 5: $T_1(x,t)$: Numerical result

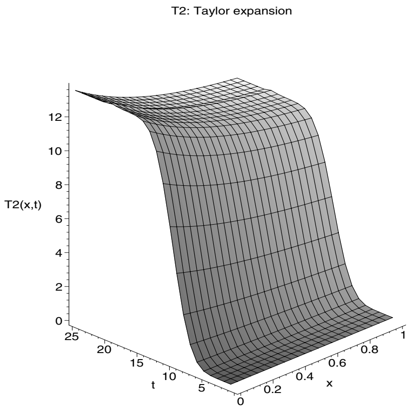

Figure 6: $T_2(x,t)$: Taylor-Expansion

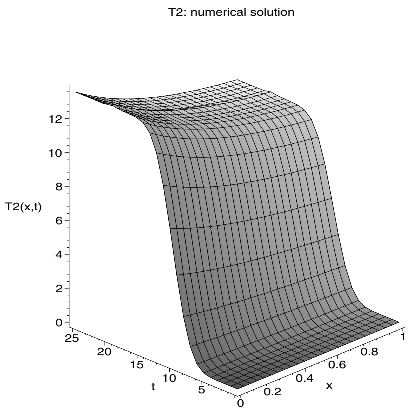

Figure 7: $T_2(x,t)$: Numerical result

The temperatures profiles calculated with the Taylor-expansion agree with the corresponding one calculated numerically by Maple quite well.

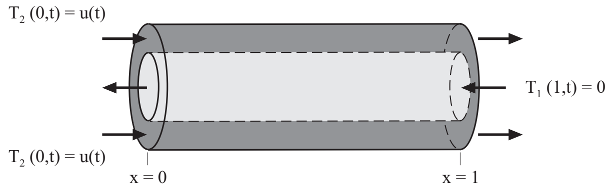

Figure 8: Counter-current heat Exchanger

4.0.1 Counter current Heat Exchanger

The equations describing a counter-current heat exchanger are given by:

$$\begin{aligned}\frac{\partial T_1}{\partial t}(x,t) + v_1 \frac{\partial T_1}{\partial x}(x,t) &= a_1 \cdot (T_2(x,t) - T_1(x,t)) \\ \frac{\partial T_2}{\partial t}(x,t) - v_2 \frac{\partial T_2}{\partial x}(x,t) &= a_2 \cdot (T_1(x,t) - T_2(x,t))\end{aligned} \tag{42}$$

The initial conditions are set at 0

$$T_1(x,0) = 0 \; T_2(x,0) = 0 \;\; t < 0 \;\; 0 \le x \le 1 \tag{43}$$

and the boundary conditions are given by

$$T_1(1,t) = 0 \text{ and } T_2(0,t) = u(t) \tag{44}$$

Like in the co-current case we get from the first boundary condition

$$\frac{\partial T_1}{\partial t}(1,t) = 0 \tag{45}$$

and the system of pde (42) becomes in $x = 1$

$$\begin{aligned}\tfrac{\partial T_1}{\partial x}(1,t) &= \tfrac{a_1}{v_1} T_2(1,t) \\ \tfrac{\partial T_2}{\partial x}(1,t) &= \tfrac{a_2}{v_2} T_2(1,t) + \tfrac{1}{v_2} \cdot \tfrac{\partial T_2}{\partial t}(1,t)\end{aligned} \tag{46}$$

Therefore similarly to the co-current case, possible flat outputs of this systems are $\frac{\partial T_1}{\partial x}(1,t)$ or $T_2(1,t)$.

The parametrization is then given by

$$\begin{aligned}T_1(x,t) &= \sum_{i=0}^{\infty} f_i(t) \cdot \tfrac{(x-1)^i}{i!} \\ T_2(x,t) &= \sum_{i=0}^{\infty} g_i(t) \cdot \tfrac{(x-1)^i}{i!}\end{aligned} \tag{47}$$

with

$$\begin{aligned}f_n(t) &= \tfrac{a_1}{v_1} \cdot (g_{n-1}(t) - f_{n-1}(t)) - \tfrac{1}{v_1} \cdot \tfrac{df_{n-1}(t)}{dt} \\ g_n(t) &= -\tfrac{a_2}{v_2} \cdot (f_{n-1}(t) - g_{n-1}(t)) + \tfrac{1}{v_2} \cdot \tfrac{dg_{n-1}(t)}{dt}\end{aligned} \tag{48}$$

and if $y_f(t) = \frac{\partial T_1}{\partial x}(1,t)$

$$f_0(t) = 0 \; f_1(t) = y_f(t) \;\; g_0(t) = \frac{v_1}{a_1} f_1(t) \tag{49}$$

if $y_f(t) = T_2\,(1,t)$

$$f_0\,(t) = 0 \;\; g_0\,(t) = y_f(t) \;\; f_1\,(t) = \frac{a_1}{v_1} \cdot g_0\,(t) \tag{50}$$

The method can also be applied if equ.(42) is nonlinear (the coefficients $a_i(T(x,t))$ are a function of the temperature) or time-varying (the coefficients $v_i(t)$ are a function of the time.

5 Conclusion

We presented a method to figure out the flat output and the parametrization of the solution for some pde. This method was also applied to many other systems (moving boundary problems, heat equation with a heat-generation term, telegraph equation, etc...) with success independantly if the system were linear or not.
As it will be shown in further papers, the parametrisation of the solution with the Taylor-expansion can also be used to design a PID controller or to find easily a solution to some pde (for example for the cross-current or the spiral heat exchanger).

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