# FLATNESS–BASED FEEDBACK TRACKING CONTROL OF A DISTRIBUTED PARAMETER TUBULAR REACTOR MODEL

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## Abstract

A novel approach for the design of feedback boundary tracking control for parabolic infinite–dimensional systems is illustrated for two coupled nonlinear partial differential equations modelling a tubular reactor. The method is based on a re–interpretation of the flatness–based open–loop design approach using an infinite power series ansatz in the spatial coordinate. It is shown, that by truncating the power series solution, a finite–dimensional approximation of the PDEs can be derived, which is suitable for flatness–based design of feedback boundary control with observer.

## 1 Introduction

Differential flatness has proven to be a very powerful concept for analysis and design of open–loop and stabilizing feedback tracking control for nonlinear finite–dimensional systems [4]. The flatness approach has been extended to the design of open–loop boundary control for infinite–dimensional or distributed parameter systems (DPSs) described by partial differential equations (PDEs). The parameterization of system states and boundary input by a flat output (inverse system) can be obtained for parabolic DPS by assuming a power series expansion of the solution [8, 10]. Applications concern the linear heat conduction equation [8], rather general nonlinear parabolic PDEs describing diffusion, heat conduction, and tubular reactors [10], including time–dependent and space–dependent coefficients [14]. Nevertheless, the use of open–loop boundary control is rather limited due to disturbances acting on the system or model errors. Hence, a closed–loop strategy is needed, which is able to cope with these effects.

For scalar parabolic DPS, two of the authors (Meurer & Zeitz [12]) extended the open–loop results to the design of stabilizing and robust feedback boundary tracking control. The approach is based on a novel use of the power series ansatz, such that the full scope of differential flatness can be exploited for the feedback control design. In this contribution, the flatness–based procedure is demonstrated for the boundary tracking control of a tubular reactor modelled by two coupled nonlinear PDEs. This approach can be seen as an alternative to work in the functional analytic setting on tracking control for linear DPS [3] and offers more general possibilities for motion planning.

This paper is organized as follows. The control problem is introduced in Section 2 using a model of a non–isothermal tubular reactor. Flatness analysis and motion planning are performed in Section 3, serving as a basis for the feedback control design in Section 4. Simulation results are presented in Section 5, followed by some concluding remarks.

## 2 Nonlinear tubular reactor model

The equations of a pseudohomogeneous axial dispersion model of a tubular reactor consist of two coupled PDEs for the reactant concentration $x_1(z,t)$ and the temperature $x_2(z,t)$ (see e.g. [11]). The irreversible first order reaction rate is assumed to follow Arrhenius' law and its temperature dependency can be linearized around a nominal constant temperature $x_{20}$ to obtain an affine approximation suitable for the presented control design

$$k(x_2) = \exp\left(\frac{x_2}{1+x_2/\sigma}\right) \tag{1}$$

$$\approx k(x_{20})\left(1+\frac{x_2-x_{20}}{(1+x_{20}/\sigma)^2}\right). \tag{2}$$

This approximation is sufficient under the assumption of small temperature variations over the reactor length. Note that higher order polynomial approximations can be used if this is not the case. In addition, the governing PDEs[1] are assumed dimensionless with characteristic numbers[2] from chemical engineering [11] for $t>0$ and $z\in(0,1)$:

$$\frac{\partial x_1}{\partial t} = \frac{1}{Pe_1}\frac{\partial^2 x_1}{\partial z^2} - \frac{\partial x_1}{\partial z} - \nu x_1 - \mu x_1 x_2, \tag{3}$$

$$\frac{\partial x_2}{\partial t} = \frac{1}{Pe_2 Le}\frac{\partial^2 x_2}{\partial z^2} - \frac{1}{Le}\frac{\partial x_2}{\partial z} - \frac{\beta x_2}{Le} + \eta x_1 + \kappa x_1 x_2 \tag{4}$$

[1] Differing from [11] it is assumed, that the outlet cooling temperature $T_w$ is constant such that the normalized state $x_2(z,t)$ is introduced with respect to $T_w$ instead of the inflow temperature, i.e. $x_2(z,t) = (T(z,t)-T_w)/T_w$ where $T(z,t)$ denotes the reactor temperature.

[2] Here $Pe_i,\ i=1,2$ denotes the Peclet number for mass and heat transport, respectively, $Le$ the Lewis number, $Da$ the Damköhler number, $B$ the adiabatic temperature rise. Parameters $\beta$ and $\sigma$ represent characteristic quantities for heat exchange and reaction.

with

$$\nu = Da\, k(x_{20})(1 - \frac{x_{20}}{(1 + x_{20}/\sigma)^2}), \quad \eta = \frac{B}{Le}\nu,$$
$$\mu = Da \frac{k(x_{20})}{(1 + x_{20}/\sigma)^2}, \qquad \kappa = \frac{B}{Le}\mu. \tag{5}$$

Danckwert's boundary conditions (BCs) are used, including the inputs $u_1(t)$ and $u_2(t)$ as inflow concentration and temperature:

$$\frac{1}{Pe_1}\frac{\partial x_1}{\partial z}(0,t) = x_1(0,t) - u_1(t), \quad t > 0, \tag{6}$$

$$\frac{\partial x_1}{\partial z}(1,t) = 0, \tag{7}$$

$$\frac{1}{Pe_2}\frac{\partial x_2}{\partial z}(0,t) = x_2(0,t) - u_2(t), \quad t > 0, \tag{8}$$

$$\frac{\partial x_2}{\partial z}(1,t) = 0. \tag{9}$$

Stationary profiles $x_{1,2}^S(z)$ are assumed as initial conditions

$$x_1(z,0) = x_1^S(z), \quad x_2(z,0) = x_2^S(z), \quad z \in [0,1]. \tag{10}$$

The outflow values of concentration and temperature are the controlled variables

$$y_1(t) = x_1(1,t), \quad y_2(t) = x_2(1,t). \tag{11}$$

The considered boundary control problem concerns the transition from the initial stationary profiles $x_1^S(z)$ and $x_2^S(z)$ to the final stationary profiles $x_1(z, t \geq T_1) = x_1^F(z)$ and $x_2(z, t \geq T_2) \geq x_2^F(z)$ in finite times $T_1$, $T_2$ in the presence of model errors or exogenous disturbances. Hence, inspired by the flatness property of finite–dimensional nonlinear systems, a flat output has to be determined in order to parameterize the system states and the boundary inputs [8, 10].

# 3 Flatness–based open–loop control

The analysis of flatness and hence the design of an open–loop control for the considered DPS is based on a power series approach as proposed in [8, 10] and extends the results of [9].

## 3.1 Flatness of the nonlinear DPS

In order to analyze flatness of PDEs (3), (4) with boundary conditions (6)–(9), a power series ansatz for the states $x_1(z,t)$ and $x_2(z,t)$ with unknown time–varying coefficients $a_n(t)$ and $b_n(t)$ is used, i.e.

$$\begin{bmatrix} x_1(z,t) \\ x_2(z,t) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} a_n(t) \\ b_n(t) \end{bmatrix} (1-z)^n \tag{12}$$

which can be formally differentiated with respect to $t$ and $z$. The formal power series ansatz (12) differs from the one proposed e.g. in [8, 9, 10], where the ansatz is defined in terms of scaled arguments $a_n(t)/n!$, $b_n(t)/n!$. However, the formal approach (12) provides significantly better numerical conditioning for the implementation and simulation of the boundary feedback control design derived in Section 4. Substituting (12) into (3), (4) and sorting terms of equal order in $(1-z)^n$ yields

$$\dot{a}_n = \frac{(n+2)(n+1)}{Pe_1} a_{n+2} + (n+1)a_{n+1} - \nu a_n - \mu \sum_{k=0}^{n} a_{n-k} b_k, \quad n \geq 0, \tag{13}$$

$$\dot{b}_n = \frac{(n+2)(n+1)}{Pe_2 Le} b_{n+2} + \frac{n+1}{Le} b_{n+1} - \frac{\beta}{Le} b_n + \eta a_n + \kappa \sum_{k=0}^{n} a_{n-k} b_k, \quad n \geq 0. \tag{14}$$

Since $a_1 = 0$ and $b_1 = 0$ by evaluating (7) and (9) using (12), it is possible to express any coefficient $a_n(t)$, $b_n(t)$, $n \geq 2$ in terms of $a_0(t)$, $b_0(t)$ and their time–derivatives up to infinite order. This will be briefly illustrated in the following: solving (13), (14) for the coefficients $a_{n+2}(t)$, $b_{n+2}(t)$, $n \geq 0$ allows the interpretation of (13), (14) as recurrence relations for the unknown coefficients of the power series ansatz (see [9] for the case of $\beta = 0$). As a result, any coefficient of the ansatz functions with even index can be expressed as

$$\begin{aligned} a_2 &= \psi_{1,2}(\boldsymbol{c}_0, \dot{a}_0), \quad a_{2n} = \psi_{1,2n}(\boldsymbol{c}_{2n-4}, \dot{a}_{2n-2}), \\ b_2 &= \psi_{2,2}(\boldsymbol{c}_0, \dot{b}_0), \quad b_{2n} = \psi_{2,2n}(\boldsymbol{c}_{2n-4}, \dot{b}_{2n-2}) \end{aligned} \tag{15}$$

where $\boldsymbol{c}_k = [a_0, b_0, a_2, b_2, \ldots, a_k, b_k]$. The coefficients with odd index allow algebraic evaluations in terms of the coefficients with even index:

$$\begin{aligned} a_1 &= \psi_{1,1}(a_0) = 0, \quad a_{2n+1} = \psi_{1,2n+1}(\boldsymbol{c}_{2n-2}, a_{2n}), \\ b_1 &= \psi_{2,1}(b_0) = 0, \quad b_{2n+1} = \psi_{2,2n+1}(\boldsymbol{c}_{2n-2}, b_{2n}). \end{aligned} \tag{16}$$

It is important to mention, that equations (16) depend linearly on $a_{2n}(t)$ and $b_{2n}(t)$, $n \geq 1$. In addition, it follows from (12) that $a_0(t) = x_1(1,t) \equiv y_1(t)$ and $b_0(t) = x_2(1,t) \equiv y_2(t)$. Hence by (15), (16), any coefficient $a_n(t)$, $b_n(t)$ can be expressed in terms of $y_1(t)$, $y_2(t)$ and their time–derivatives up to infinite order. Furthermore, evaluation of boundary conditions (6), (8) with (12) provides

$$u_1(t) = \sum_{n=0}^{\infty} \left( a_n(t) + \frac{n+1}{Pe_1} a_{n+1}(t) \right), \tag{17}$$

$$u_2(t) = \sum_{n=0}^{\infty} \left( b_n(t) + \frac{n+1}{Pe_2} b_{n+1}(t) \right). \tag{18}$$

As a results of this analysis, the series expansion for the states $x_{1,2}(z,t)$ defined in (12), as well as the boundary inputs $u_{1,2}(t)$ defined in (17), (18) can be parameterized by $\boldsymbol{y}(t) = [y_1(t), y_2(t)]^T$ and its time–derivatives. This parameterization allows the interpretation of $\boldsymbol{y}$ as a flat output for the system (3)–(9).

## 3.2 Motion planning and convergence of formal solutions

In order to obtain an open–loop boundary control and to solve the control problem as stated in Section 2, smooth desired trajectories $\boldsymbol{y}_d(t) = [y_{1,d}(t), y_{2,d}(t)]^T \in \mathcal{C}^{\infty}(t \in [0, T_{1,2}])$ have to be specified for the flat output $\boldsymbol{y}$ connecting the stationary profiles and ensuring convergence of the series expansions (12). Note since $z \in [0,1]$, it is sufficient to verify radii of convergence $\rho_{1,2}$ of at least 1.

Similar to the proof of convergence given in [9] for $\beta = 0$ in (4), it can be shown that the series (12) converge with unit radii of convergence, if the two components of $\boldsymbol{y}_d(t)$ are Gevrey functions[3] [5] of class $\gamma \in (1,2]$ and the parameters of the tubular reactor (3), (4) satisfy the inequalities[4]

$$\frac{1}{2} \leq \frac{1}{Pe_1} - \frac{1}{R} - \frac{1}{2}(|\nu| + m_2|\mu|), \tag{19}$$

$$\frac{1}{2} \leq \frac{1}{Pe_2} - \frac{Le}{2}\left(\frac{|\beta|}{Le} + \frac{2}{R} + m_1(\frac{|\eta|}{m_2} + Pe_2|\kappa|)\right). \tag{20}$$

It follows necessarily for the Peclet numbers that $Pe_{1,2} < 2$. An extension to higher–order polynomial approximations of (2) is possible as outlined in [12]. In the following similar to [8, 9, 12, 14], motion planning is based on a so–called smooth 'step function' of Gevrey order $\gamma = 1 + 1/\omega$

$$\Theta_{\omega,T}(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq T \\ \frac{\int_0^t \theta_{\omega,T}(\tau)d\tau}{\int_0^T \theta_{\omega,T}(\tau)d\tau} & t \in (0,T) \end{cases} \tag{21}$$

where $\theta_{\omega,T}$ denotes the 'bump function'

$$\theta_{\omega,T}(t) = \begin{cases} 0 & t \notin (0,T) \\ \exp\left(\frac{-1}{\left[(1-\frac{t}{T})\frac{t}{T}\right]^\omega}\right) & t \in (0,T). \end{cases} \tag{22}$$

Therefore, desired trajectories being consistent with both the initial and final stationary profile can be realized by

$$\boldsymbol{y}_d(t) = \begin{bmatrix} x_1^S(1) + (x_1^F(1) - x_1^S(1))\Theta_{\omega_1,T_1} \\ x_2^S(1) + (x_2^F(1) - x_2^S(1))\Theta_{\omega_2,T_2}, \end{bmatrix} \tag{23}$$

where $\boldsymbol{y}_d(0) = [x_1^S(1), x_2^S(1)]^T$ and $\boldsymbol{y}_d(t \geq \max\{T_1, T_2\}) = [x_1^F(1), x_2^F(1)]^T$ with possibly different finite times $T_1,\ T_2$.

Note however, that for implementation the open–loop controls (17), (18) have to be truncated at a certain integer. Due to this fact and in view of exogenous disturbances, model errors or instability, a closed–loop control strategy is needed as will be explained in the next section.

[3]A function $y_i(t)$ is called Gevrey of order $\gamma$, if the following bounds

$$\sup_{t\in\mathbb{R}} |y_i^{(n)}(t)| \leq \frac{m_i(n!)^\gamma}{R^n}, \quad \forall n \geq 0$$

hold for constants $m_i,\ R \in \mathbb{R}^+$.

[4]The proof is based on the results of [9] but uses the technical modification, that the following bounds on time–derivatives of the series coefficients are shown by induction

$$\sup_{t\in\mathbb{R}} |a_n^{(l)}(t)| \leq \frac{m_1 M^n}{R^l}\frac{(n+l)!^\gamma}{n!^\gamma},\ \sup_{t\in\mathbb{R}} |b_n^{(l)}(t)| \leq \frac{m_2 M^n}{R^l}\frac{(n+l)!^\gamma}{n!^\gamma}.$$

# 4 Flatness–based boundary tracking control

In this section, the flatness–based approach of [12] for the design of feedback boundary tracking control is extended to the two nonlinear PDEs (3), (4) with BCs (6)–(9).

## 4.1 Generalized nonlinear controller normal form

Under the assumption of appropriate motion planning, convergence of the formal solution (12) is ensured as illustrated in the previous section. For the design of feedback control, re–consider (13), (14) and recall, that any coefficient with odd index can be expressed algebraically in terms of coefficients with even index using (16) due to the homogeneous boundary conditions (7), (9). As a result, it suffices to consider only the parts of (13), (14) with even index together with (15), (16), namely

$$\begin{aligned}\dot{a}_{2n} &= \alpha^a_{2n+2}a_{2n+2} - \nu a_{2n} - \mu\sum_{k=0}^n a_{2n-2k}b_{2k} \\ &\quad + (2n+1)\psi_{1,2n+1}(\boldsymbol{c}_{2n-2}, a_{2n}) - \mu\phi_{2n}(\boldsymbol{c}_{2n-4}), \quad (24)\\ \dot{b}_{2n} &= \alpha^b_{2n+2}b_{2n+2} - \frac{\beta}{Le}b_{2n} + \eta a_{2n} + \kappa\sum_{k=0}^n a_{2n-2k}b_{2k} \\ &\quad + \frac{2n+1}{Le}\psi_{2,2n+1}(\boldsymbol{c}_{2n-2}, b_{2n}) + \kappa\phi_{2n}(\boldsymbol{c}_{2n-4}) \quad (25)\end{aligned}$$

where $n \geq 0,\ \alpha^a_{2n+2} = (2n+2)(2n+1)/Pe_1,\ \alpha^b_{2n+2} = (2n+2)(2n+1)/(Pe_2 Le)$ and

$$\phi_{2n}(\boldsymbol{c}_{2n-4}) = \begin{cases} 0, & n = 0,1 \\ \sum_{k=1}^{n-2} \psi_{1,2(n-k)-1}(\boldsymbol{c}_{2(n-k-2)}, a_{2(n-k-1)}) & \\ \times\psi_{2,2k+1}(\boldsymbol{c}_{2k-2}, b_{2n}), & n \geq 2. \end{cases} \tag{26}$$

Making use of the formal assumption of convergence, each series expansion of the open–loop controls (17), (18) can be truncated at some upper limit $2N,\ N \in \mathbb{N}$. This is equivalent to consider only the first $2N-2$ terms of summation in (24) and (25), respectively. The main step of the novel approach introduced in [12] is to interpret this finite set of equations (24), (25) as a system of nonlinear ordinary differential equations (ODEs), instead of solving each of these equations recursively as outlined in Section 3.1. For the determination of the thus remaining unknown coefficients $a_{2N}$ in (24) and $b_{2N}$ in (25), it is essential to consider the inflow boundary conditions (6), (8) together with truncated formal ansatz functions (12) , i.e.

$$u_1 \approx \sum_{n=0}^{2N}\left(1 + \frac{n}{Pe_1}\right)a_n, \quad u_2 \approx \sum_{n=0}^{2N}\left(1 + \frac{n}{Pe_2}\right)b_n.$$

Hence it follows directly utilizing (16), that

$$a_{2N} = \frac{Pe_1}{Pe_1 + 2N}\left(u_1 - \sum_{n=0}^{N-1}\left[1 + \frac{2n}{Pe_1}\right]a_{2n} + \right.$$
$$\left.\left[1 + \frac{2n+1}{Pe_1}\right]\psi_{1,2n+1}(\boldsymbol{c}_{2n-2}, a_{2n})\right), \tag{27}$$
$$b_{2N} = \frac{Pe_2}{Pe_2 + 2N}\left(u_2 - \sum_{n=0}^{N-1}\left[1 + \frac{2n}{Pe_1}\right]b_{2n} + \right.$$
$$\left.\left[1 + \frac{2n+1}{Pe_1}\right]\psi_{2,2n+1}(\boldsymbol{c}_{2n-2}, b_{2n})\right) \tag{28}$$

which allows the introduction of the inputs $u_1,\ u_2$ into the ODEs (24), (25) for $n = 0, 1, \ldots, N-1$. Summarizing these results, the schematic state–space representation depicted in Figure 1 is obtained for state $\boldsymbol{\zeta}(t) = [\zeta_1(t), \ldots, \zeta_{2N}(t)]^T$ with $\zeta_n = a_{2(n-1)},\ \zeta_{N+n} = b_{2(n-1)},\ n = 1, \ldots, N$. This input affine nonlinear multi–input multi–output (MIMO) system of finite dimension allows an interpretation as a generalized nonlinear controller normal form due to the obtained band structure of matrix $A$ with one side diagonal and the double (with respect to the two subsystems) triangular structure of $f(\boldsymbol{\zeta})$. It is hence easy to verify, that the outputs (30) are flat outputs of (29) parameterizing the state $\boldsymbol{\zeta}$ and input $\boldsymbol{u}$ following the argumentation of Section 3.1.

### 4.2 Feedback control with observer

In order to track the flat output $\boldsymbol{y}$ along an appropriately designed desired trajectory $\boldsymbol{y}_d(t)$ as defined in (23), feedback boundary tracking control [4, 13] is designed based on the flat representation (29), (30) of the tubular reactor model.

Since flat systems are exactly feedback linearizable [4], asymptotic tracking control can be designed by methods of linear control theory – for details, the reader is referred to e.g. [4, 13]. Following this approach, a static feedback law is obtained

$$\boldsymbol{u} = \boldsymbol{\Omega}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \boldsymbol{y}_d^{(N)} - \boldsymbol{\Lambda}(\boldsymbol{e})). \tag{31}$$

There $\boldsymbol{\Lambda}(\boldsymbol{e}) = [\Lambda_1(\boldsymbol{e}_1), \Lambda_2(\boldsymbol{e}_2)]^T$, $\boldsymbol{e}_k = [e_k, \dot{e}_k, \ldots, e_k^{(N-1)}]$, $k = 1, 2$ can be any type of control, asymptotically stabilizing the tracking error $e_k(t) = y_k - y_{k,d}(t),\ k = 1, 2$. Following [6], 'extended PID–control' will be considered:

$$\Lambda_k(\boldsymbol{e}_k) = p_{k,0}\int_0^t e_k(\tau)d\tau + \sum_{j=1}^{N} p_{k,j} e_k^{(j-1)},\ k = 1, 2. \tag{32}$$

The parameters $p_{k,j},\ j = 0, 1, \ldots, N,\ k = 1, 2$ are assumed as coefficients of Hurwitz polynomials to obtain asymptotically stable tracking error dynamics and can be determined by eigenvalue assignment $\lambda_{k,j},\ j = 0, 1, \ldots, N,\ k = 1, 2$ [13, 6].

Since full state information is necessary for the implementation of the feedback tracking control (31), (32), an observer is applied to estimate the non–measured states. For the observer design it is assumed, that the flat output $\boldsymbol{y}$ is measured (more general cases can be treated similarly if observability is preserved). Hence, a nonlinear tracking observer can be designed based on model (29), (30)

$$\dot{\hat{\boldsymbol{\zeta}}} = A\hat{\boldsymbol{\zeta}} + \boldsymbol{f}(\hat{\boldsymbol{\zeta}}) + B\hat{\boldsymbol{u}} + L(t)(\boldsymbol{y} - C\hat{\boldsymbol{\zeta}}),\ \ \hat{\boldsymbol{\zeta}}(0) = \hat{\boldsymbol{\zeta}}_0, \tag{33}$$

with suitable initial conditions. The time–variable gain matrix $L(t) \in \mathbb{R}^{2N\times 2}$ can be determined based on a linearization of the observer error dynamics along the desired trajectory $\boldsymbol{\zeta}_d(t)$, which is known due to the flatness of (29), (30) [13]. This allows the application of the time–variable Ackermann–formula for the design of $L(t)$, such that appropriate eigenvalues $\hat{\lambda}_{k,j},\ j = 0, 1, \ldots, N-1,\ k = 1, 2$ can be assigned for the observer error dynamics.

The designed observer can also be used for spatial profile estimation throughout the transition process [12]. An estimate $\hat{\boldsymbol{x}}(z,t) = [\hat{x}_2(z,t), \hat{x}_2(z,t)]^T$ of the spatial profiles $\boldsymbol{x}(z,t)$ can be obtained by evaluating the power series ansatz (12)

$$\hat{\boldsymbol{x}}(z,t) = \sum_{n=0}^{2N}\begin{bmatrix}\hat{a}_n(t)\\ \hat{b}_n(t)\end{bmatrix}(1-z)^n, \tag{34}$$

where the time–varying coefficients are replaced by their estimated counterparts:

$$\hat{a}_{2n} = \zeta_{n+1},\ \hat{b}_{2n} = \zeta_{N+n+1},\ n = 0, 1, \ldots, N-1.$$

These coefficients can be directly determined from the estimated states, such that the estimated coefficients with odd index follow by evaluating (16) which in addition provides $\hat{a}_{2N},\ \hat{b}_{2N}$ from (27), (28).

## 5 Simulation results

In order to illustrate the robust performance of the flatness–based feedback tracking control scheme, simulation results are presented for a scenario involving a model error [2].

For the simulation, system (3)–(10) is discretized using the method–of–lines (MOL) approach with spatial discretization $\Delta z = 0.01$ as implemented in MATLAB. Note, that the original reaction rate (1) is used for the MOL plant model. The desired trajectories (23) are parameterized by $x_1^S(1) = 0.4$, $x_1^F(1) = 0.2$, $x_2^S(1) = x_2^F(1) = 0.5$, $\omega_1 = \omega_2 = 1.1$, $T_1 = T_2 = 10.0$, i.e. it is desired to reduce the outflow concentration $x_1(1,t) : 0.4 \to 0.2$ along the prescribed trajectory at constant outflow temperature $x_2(1,t) = 0.5 = x_{20}$.
Feedback control and observer are designed for $N = 5$ in (29), (30). For comparison purposes, simulation results for applying the open–loop controls (17), (18) each truncated at $N = 10$ are depicted additionally. The eigenvalues for asymptotic tracking control and observer are assigned as $\lambda_k = -[1.5, 2, 2.5, 3, 3.5, 4]$, $\hat{\lambda}_k = -[11, 12, 13, 14, 15]$, $k = 1, 2$. Note that special emphasis has to be placed on the eigenvalue location in order to avoid spillover effects due to neglected dynamics while deriving the control design model [1].
The parameters (5) for the plant (3)–(10) are adopted from [7]:

$$
\begin{bmatrix} \dot\zeta_1 \\ \dot\zeta_2 \\ \vdots \\ \dot\zeta_{N-1} \\ \dot\zeta_N \\ \dot\zeta_{N+1} \\ \dot\zeta_{N+2} \\ \vdots \\ \dot\zeta_{2N-1} \\ \dot\zeta_{2N} \end{bmatrix}
=
\underbrace{\begin{pmatrix}
-\nu & \frac{2}{Pe_1} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & \vdots & & & & \vdots \\
\vdots & & & & \vdots & & & & & \\
0 & 0 & \dots & -\nu & \alpha^a_{2N-2} & 0 & 0 & \dots & 0 & 0 \\
-\rho_1^1 & -\rho_2^1 & \dots & -\rho_{N-1}^1 & -\nu-\rho_N^1 & 0 & 0 & \dots & 0 & 0 \\
\eta & 0 & \dots & 0 & 0 & -\frac{\beta}{Le} & \frac{2}{Pe_2 Le} & \dots & 0 & 0 \\
0 & \ddots & & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots \\
0 & 0 & \dots & \eta & 0 & 0 & 0 & \dots & -\frac{\beta}{Le} & \alpha^b_{2N-2} \\
0 & 0 & \dots & 0 & \eta & -\rho_1^2 & -\rho_2^2 & \dots & -\rho_{N-1}^2 & -\frac{\beta}{Le}-\rho_N^2
\end{pmatrix}}_{A}
\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{N-1} \\ \zeta_N \\ \zeta_{N+1} \\ \zeta_{N+2} \\ \vdots \\ \zeta_{2N-1} \\ \zeta_{2N} \end{bmatrix}
+
\underbrace{\begin{bmatrix} f_1(\boldsymbol{c}_0) \\ f_2(\boldsymbol{c}_2) \\ \vdots \\ f_{N-1}(\boldsymbol{c}_{2N-4}) \\ f_N(\boldsymbol{\zeta}) \\ f_{N+1}(\boldsymbol{c}_0) \\ f_{N+2}(\boldsymbol{c}_2) \\ \vdots \\ f_{2N-1}(\boldsymbol{c}_{2N-4}) \\ f_{2N}(\boldsymbol{\zeta}) \end{bmatrix}}_{f(\boldsymbol{\zeta})}
+
\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \rho_N^1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \rho_N^2 \end{pmatrix}}_{B}
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\tag{29}
$$

$$
\boldsymbol{y} = \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots 0 \end{pmatrix}}_{C} \boldsymbol{\zeta}
\tag{30}
$$

Figure 1: State–space representation of the design model (24), (25) in generalized nonlinear controller normal form with substitution of (27), (28). Here $\rho_n^{1,2} = \frac{2N(2N-1)}{2N+Pe_1}(1+\frac{2(n-1)}{Pe_{1,2}})$, $n = 1, \dots, N$ and $\boldsymbol{c}_k = [\zeta_1, \dots, \zeta_k, \zeta_N, \dots, \zeta_{N+k}]$ by definition.

$Pe_1 = Pe_2 = 4$, $B = 12$, $Da = 0.11003$, $Le = 1$, $\beta = 1.5, \sigma = 20$. In the considered simulation scenario, it is assumed that $Da$ describing the reaction speed differs by $50\%$ from the nominal one, i.e. $Da = 0.11003$ is used for control/observer design, whereby $\overline{Da} = 0.16504$ is used in the plant model. As a result, the reactant is consumed faster, releasing more energy due the exothermic reaction, which has to be tackled by the control. Simulation results for this scenario are depicted in Figures 2–5, showing the concentration distribution $x_1(z,t)$ in the $(z,t)$–domain (Fig. 2), the tracking performance (Fig. 3), and the applied control (Fig. 4). It can be clearly seen, that excellent tracking behavior is obtained with zero steady state error. Finally, the applicability of the proposed profile estimation is illustrated in Fig. 5, showing the deviation between the estimated concentration profile $\hat{x}_1(z,t)$ by evaluating (34) and the exact profiles $x_1(z,t)$ from the MOL–simulation of the feedback controlled DPS. The error varies between $[-6, 2]\%$ and decreases to $\approx 0\%$ in steady state, whereby the initially larger errors stem from the assumed model error and the implemented reaction rate (1) in the plant model.

As mentioned, the proposed approach is limited to certain parameter sets (5) to ensure convergence of the series expansions, which are also the basis for the derivation of the finite–dimensional design model (29), (30). Nevertheless, numerical results verify, that the conditions for convergence (19), (20) are rather restrictive and convergence can be ensured also for parameters outside the given ranges (see the used set of parameters above). Furthermore, simulation studies show, that the feedback control scheme with observer is primarily applicable for small $Pe$–numbers ($Pe \leq 4$), i.e. mainly diffusive problems like bleaching reactors.

## 6 Conclusion

This contribution presents an extension of the flatness–based feedback boundary tracking control approach introduced in [12] for scalar nonlinear parabolic PDEs to a system of two coupled nonlinear parabolic PDEs. The approach is essentially based on a re–interpretation of the power series approach, which allows the derivation of an inherently flat finite–dimensional system in a generalized controller normal form approximating the governing infinite–dimensional DPS. This allows the exploitation of the full scope of differential flatness for the design of feedback boundary tracking control, including motion planning and inspection of possible state and/or input constraints. Simulation studies illustrate the performance and robustness of the tracking control scheme when applied to the governing DPS. The proposed approach is completely model based and directly provides design rules for feedforward and feedback control, establishing an analogy to flatness–based control design for finite–dimensional systems.

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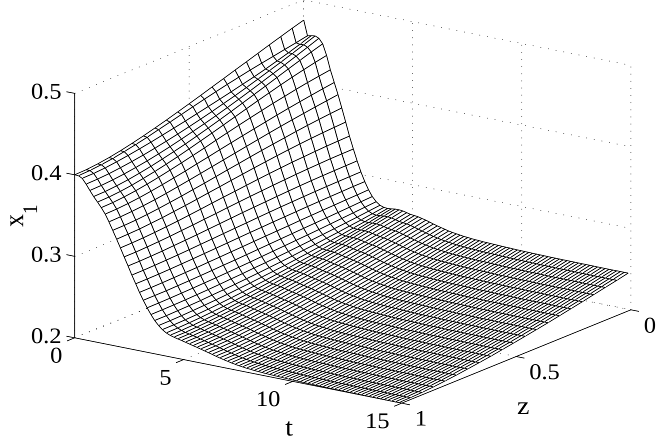


Figure 2: Evolution $x_1(z,t)$ of concentration in the $(z,t)$–domain for tracking control of $\boldsymbol{y}(t)$ along $\boldsymbol{y}_d(t)$ from (23).

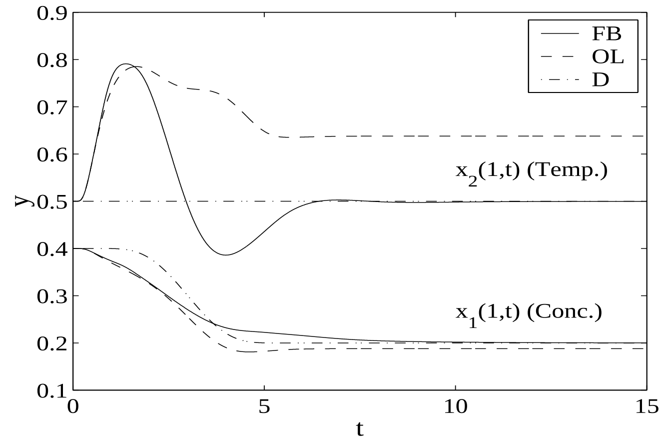


Figure 3: Desired output $\boldsymbol{y}_d(t)$ (D) and real output $\boldsymbol{y}(t) = [x_1(1,t), x_2(1,t)]^T$ for feedback (FB) and open–loop (OL).

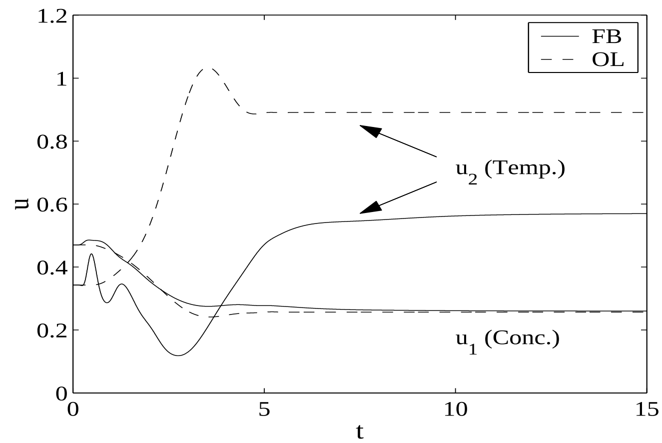


Figure 4: Applied control $\boldsymbol{u}(t) = [u_1(t), u_2(t)]^T$ for feedback (FB) and open–loop (OL).

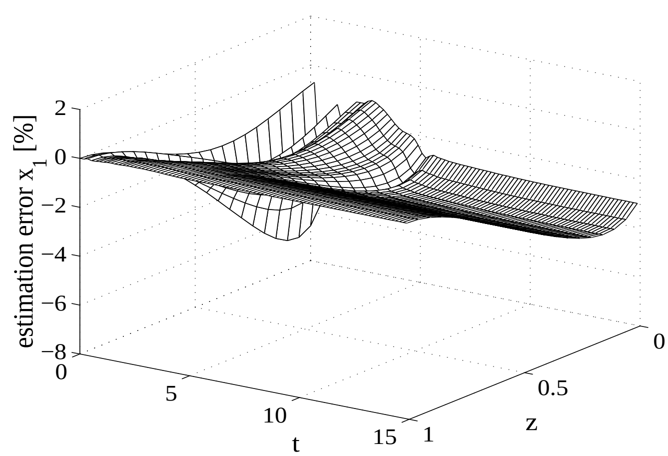


Figure 5: Estimation error $(\hat{x}_1(z,t) - x_1(z,t))/x_1(z,t) \times 100\%$ of observer (33) for profile estimation of $x_1(z,t)$ in Fig. 2.