Relative Gain Array Analysis of Uncertain Multivariable Plants

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Abstract

The input-output pairing of multivariable plants with parametric uncertainty can vary in the face of large plant parameter variations. The Relative Gain Array (RGA) analysis is a powerful tool for the input-output pairing of linear multivariable plants. In the case of parametric uncertainties, RGA elements may vary accordingly. Hence, a test is proposed to identify the change in the input-output pairing in the presence of parametric uncertainties.

1 Introduction

Decentralized controller are widely used in many complex multivariable processes [8], [9], [5]. An appropriate input-output pairing prior to the commencement of the design is vital for desired closed-loop stability and performance. There are different approaches to input-output selection and RGA is the first and the most widely used analytical tool for this problem [3], [6], [1]. However, the proposed approaches are mainky applicable to known multivariable plants and fail in the presence of plant uncertainties. The attempts in [2] and [4] to overcome the uncertainty problem in process models can only partially solve the issue and cannot identify the changes in the input-output selection.

In this paper, the parametric model uncertainty is considered and a graphically based test is presented to identify the possible input-output selection changes resulting from the parameter parameter changes. The test is explicitly stated for two and three input-output multivariable plants. Also, examples are provided to show the effectiveness of the proposed test.

2 RGA analysis in the presence of parametric uncertainties

Consider a linear multivariable plant described by a transfer function matrix $G(s)$ with m inputs and outputs. The RGA matrix has been defined as fallow [3].

$$\Gamma = G(0) \otimes G(0)^{-T} \qquad (1)$$

and each element of Γ (λ_{ij}) can be calculated by equation (2).

$$\lambda_{ij} = \frac{(-1)^{i+j} g_{ij} \det(G^{ij})}{\det(G)} \qquad i,j = 1,2,...,m \qquad (2)$$

G^{ij} is matrix G with eliminate i^{th} row and j^{th} column. A final decision on the input-output pairing is reached through investigating these elements and is given in [3], [7].

2.1 Two input - Two output multivariable plants

In the case of multivariable plants with two inputs and two outputs, equation (2) can be rewritten as

$$\lambda_{11} = \frac{g_{11}g_{22}}{g_{11}g_{22} - g_{12}g_{21}} \qquad (3)$$

and by defining the following variable

$$k = \frac{g_{12}g_{21}}{g_{11}g_{22}} \qquad (4)$$

equation (3) can be written as

$$\lambda_{11} = \frac{1}{1-k} \qquad (5)$$

and , similarly λ_{12} can be written as follows

$$\lambda_{12} = \frac{-k}{1-k} \qquad (6)$$

Since, in such multivariable plants the input-output pairing can be determined from the comparison of the elements of the first row of the RGA, λ_{11} and λ_{12} are compared.

Let $\lambda_{11} > \lambda_{12}$, in this case it follows from equations (5) and (6) that

$$-1 < k < 1 \qquad (7)$$

In the face of parametric uncertainties, k will be an uncertain parameter and can be present

$$k' = k + \Delta k \qquad (8)$$

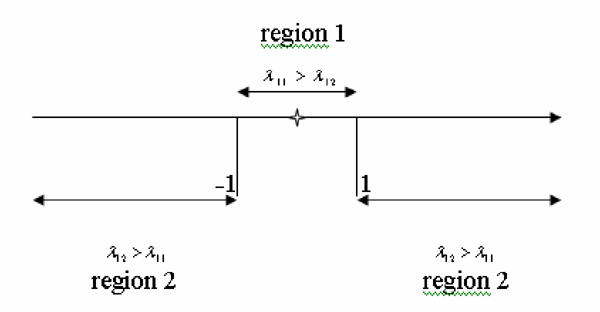
If Δk causes a change in the previous input-output pairing, i.e. $\lambda_{12} > \lambda_{11}$, then

$$\frac{1}{1-k'} < \frac{-k'}{1-k'} \qquad (9)$$

and hence,

$$k' > 1 \text{ or } k' < -1 \qquad (10)$$

Equations (7) and (10) are now presented on an axis, as is shown in fig1. A transfer from region 1 to 2 or vice versa shows a change in the input-output pairing due to the parametric uncertainties.



Two input-Two output plants
Fig 1

Example 1

Consider the following transfer function matrix [2]:

$$G(s) = \begin{bmatrix} \frac{(12.8+\delta_{11})e^{-s}}{16.7s+1} & \frac{-(18.9+\delta_{12})e^{-3s}}{21s+1} \\ \frac{(6.6+\delta_{21})e^{-7s}}{10.9s+1} & \frac{-(19.4+\delta_{22})e^{-3s}}{14.4s+1} \end{bmatrix}$$

where $\delta_{11} \in [-6,2]$, $\delta_{12} \in [-7,3]$, $\delta_{21} \in [-1,3]$, $\delta_{22} \in [-2,2]$, and its corresponding RGA for the nominal case $\delta_{ij} = 0$ ($i,j = 1,2$) is

$$\Gamma_1 = \begin{bmatrix} 2.01 & -1.01 \\ -1.01 & 2.01 \end{bmatrix}$$

and $k = 0.5023$. The RGA matrix shows that $(u_1 - y_1, u_2 - y_2)$ is an appropriate input-output pairing. For $\delta_{11} = -4$, $\delta_{12} = -5$, $\delta_{21} = 1$, $\delta_{22} = 0$, k is changed to

$$k' = \frac{(6.6+1)(-18.9-5)}{(12.8-4)(-19.4)} = 1.064 > 1$$

Which clearly indicates as is shown in Fig1 that a change in input-output pairing has occurred. This Result is also verified by calculating the RGA matrix for the new parameters, i.e.

$$\Gamma_2 = \begin{bmatrix} -15.63 & 16.63 \\ 16.63 & -15.63 \end{bmatrix}$$

which also shows a change in the input-output pairing.

2.2 Three input - Three output multivariable plants

In the case of multivariable plants with three input and output, equations (2) can be rewritten as

$$\lambda_{11} = \frac{g_{11}\det(G^{11})}{g_{11}\det(G^{11}) - g_{12}\det(G^{12}) + g_{13}\det(G^{13})}$$

$$\lambda_{12} = \frac{-g_{12}\det(G^{12})}{g_{11}\det(G^{11}) - g_{12}\det(G^{12}) + g_{13}\det(G^{13})} \quad (11)$$

$$\lambda_{13} = \frac{g_{13}\det(G^{13})}{g_{11}\det(G^{11}) - g_{12}\det(G^{12}) + g_{13}\det(G^{13})}$$

and by defining the following variables

$$k_1 = \frac{g_{12}\det(G^{12})}{g_{11}\det(G^{11})}, k_2 = \frac{g_{13}\det(G^{13})}{g_{11}\det(G^{11})} \quad (12)$$

the element of the RGA matrix can be written as

$$\lambda_{11} = \frac{1}{1+k_2-k_1}$$

$$\lambda_{12} = \frac{-k_1}{1+k_2-k_1} \quad (13)$$

$$\lambda_{13} = \frac{k_2}{1+k_2-k_1}$$

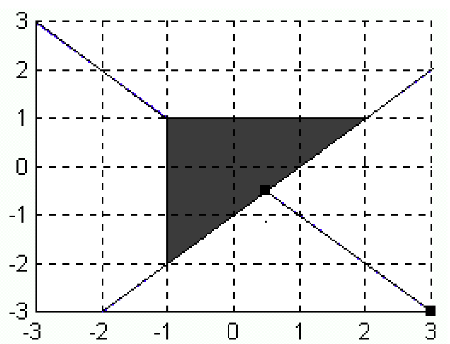
The regions in the (k_1, k_2) coordinates which indicate a change in the input-output pairings are now determined.

Case I: ($\lambda_{11} > \lambda_{12}, \lambda_{11} > \lambda_{13}$)

In this case, equations (13) give a closed region characterized by

$$\begin{aligned} &1+k_2-k_1 > 0 \\ &k_1 > -1 \\ &k_2 < 1 \end{aligned} \quad (14)$$

and is shown in fig 2. The shaded region in fig 2 represents the inequalities in (13) and shows that $(u1-y1, u2-y2)$ are an appropriative pairing, but a final decision must be made after considering the other cases.



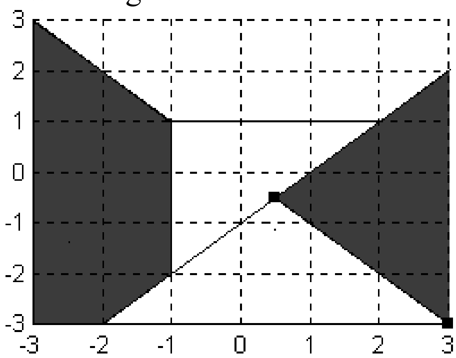
Three input-Three output multivariable plants
Case I: ($\lambda_{11} > \lambda_{12}, \lambda_{11} > \lambda_{13}$)
Fig 2

Case II: ($\lambda_{12} > \lambda_{11}, \lambda_{12} > \lambda_{13}$)

In this case, equations (13) give a closed region characterized by

$$\begin{aligned} &1+k_2-k_1 > 0 \qquad &1+k_2-k_1 < 0 \\ &k_1 < -1 \qquad , &k_1 > -1 \\ &k_2+k_1 < 0 \qquad &k_2+k_1 > 0 \end{aligned} \quad (15)$$

and is shown in fig 3.



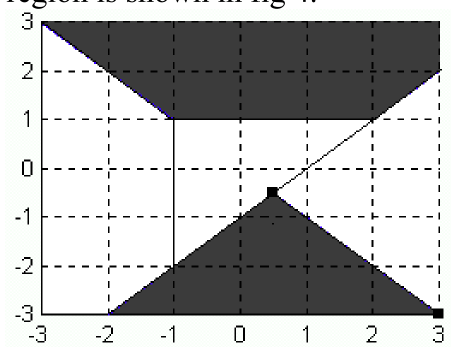
Three input-Three output multivariable plants
Case II: ($\lambda_{12} > \lambda_{11}, \lambda_{12} > \lambda_{13}$)
Fig 3

Case III: ($\lambda_{13} > \lambda_{11}, \lambda_{13} > \lambda_{12}$)

Similarly, equations (13) give

$$\begin{aligned} &1+k_2-k_1 > 0 \qquad &1+k_2-k_1 < 0 \\ &k_2 > 1 \qquad , &k_2 < 1 \\ &k_2+k_1 > 0 \qquad &k_2+k_1 < 0 \end{aligned} \quad (16)$$

and this region is shown in fig 4.



Three input-Three output multivariable plants
Case III: ($\lambda_{13} > \lambda_{11}, \lambda_{13} > \lambda_{12}$)
Fig 4

Test procedure

1) Determine the $\hat{k}_i = (k_1, k_2)$ variables using the following equation.

$$\left[\hat{k}_i\right] = [\Gamma(:,2), \Gamma(:,3)]./[-\Gamma(:,1), \Gamma(:,1)] \quad (17)$$

2) Identify the points $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$ in the $k_2 - k_1$ diagram.

3) If each of the $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$ lie in one of the three regions, then we can use decentralize control.
4) A shift of the indices $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$ from one region to another, indicates a change in the input-outputs pairing.

Example 2

Consider the following transfer function [2]

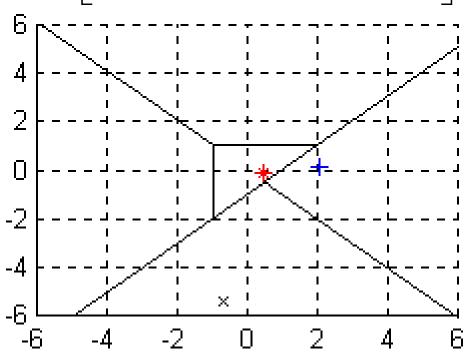
$$G(s)=\begin{bmatrix} \dfrac{0.66e^{-2.6s}}{6.7s+1} & \dfrac{-0.61e^{-3.5s}}{8.64s+1} & \dfrac{-0.0049e^{-s}}{9.06s+1} \\ \dfrac{1.11e^{-0.65s}}{3.25s+1} & \dfrac{-2.36e^{-3s}}{5s+1} & \dfrac{-0.012e^{-1.2s}}{7.09s+1} \\ \dfrac{-33.68e^{-9.2s}}{8.15s+1} & \dfrac{46.2e^{-9.4s}}{10.9s+1} & \dfrac{0.87(11.61s+1)e^{-s}}{(3.89s+1)(18.8s+1)} \end{bmatrix}$$

Case I:

$$G_1 = G(0) + 0.00\times\left(\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \otimes G(0)\right)$$

Fig 5 represents the nominal case and its corresponding RGA is

$$\hat{\Gamma}_1 = \begin{bmatrix} 1.945 & -0.673 & -0.272 \\ -0.664 & 1.899 & -0.235 \\ -0.281 & -0.225 & 1.506 \end{bmatrix}$$



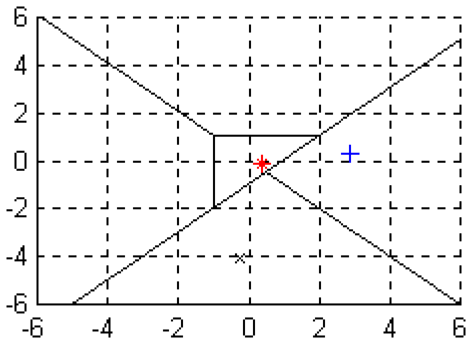
Example 2 : Case I
Fig 5

Case II: Let the nominal plant be changed as follows.

$$G_2 = G(0) + 0.05\times\left(\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \otimes G(0)\right)$$

Fig 6 shows the position of $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$ which indicates that a change in the input-output pairing has not occured. This is verified by the corresponding RGA matrix given by

$$\Gamma_2 = \begin{bmatrix} 1.9913 & -0.7306 & -0.2607 \\ -0.6371 & 1.8213 & -0.1842 \\ -0.3542 & -0.0907 & 1.4449 \end{bmatrix}$$



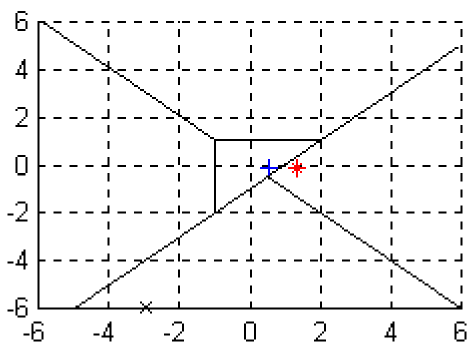
Example 2 : Case II
Fig 6

Case III:

$$G_3 = G(0) + 0.3\times\left(\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \otimes G(0)\right)$$

Fig 7 shows the position of $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$ which indicates that a change has occurred in the input-output pairing. This is also verified by the corresponding RGA matrix given by

$$\Gamma_3 = \begin{bmatrix} -2.0905 & 2.8411 & -0.2495 \\ 3.2140 & -1.7431 & -0.4708 \\ -0.1234 & -0.0979 & 1.2214 \end{bmatrix}$$



Example 2 : Case 3
Fig 7

Acknowledgments

The present input-output pairing methods can easily fail in the case of parametric uncertainties. In this paper, a test procedure has been proposed to nominate the appropriate input-output pairings in the face of parametric plant uncertainties. The regions indicating these pairings are shown graphically to further assist the designer in an input-output selection process. Examples have been provided to show the effectiveness of the proposed.

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