A CHANCE-CONSTRAINED STOCHASTIC INVENTORY PROBLEM UNDER IMPERFECT INFORMATION OF STATE

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Abstract

This paper deals with a chance-constrained stochastic production-planning problem under hypotheses of imperfect information of inventory variables. Assuming a linear-Gaussian nature to the inventory balance system, the mean and covariance, statistical variables, are estimated from the Kalman filter equations. As a consequence, an approach – originally developed for stochastic problems under perfect information of state [8] – is adapted to provide sub-optimal solution for this problem. A simple example illustrates the basic ideas presented.

1 Introduction

The production-planning problem requires a set of decisions that is used to adjust the industrial resources of the company in order to satisfy the exogenous demand for its products. Such decisions are taken over different planning horizon at various levels in the planning decision hierarchy [4]. The fluctuation of demand affects strongly such decisions [8].

At a higher planning level of a hierarchical process, decisions are usually taken over a long-term horizon, and the production-planning problem can only be elaborated at the aggregate pattern of product families [4]. A common problem in this level is to identify the quantity of inventory related to material resources that will be used by company at future periods. Surely, this is not an easy task even for an expert manager. The reason is that there are a variety of uncertainties associated with the process of identifying the quantity of these materials to be used in the shop floor. These uncertainties are due to exogenous and endogenous factors. For example, exogenous factors are to know a priori if a given supplied material will be considered appropriate by consume during the quality test; or to know precisely how will be the behavior of demand in the next month. On the other hands, endogenous factors are related to the quantity of material lost or robed during its handle in the storeroom. As a direct consequence of the exogenous and endogenous randomness, the inventory variable can not be measured precisely. Thus, in order to deal with the lack of accuracy about inventory, a stochastic optimal control problem under hypothesis of imperfect information of state must be formulated.

This paper considers a single stochastic inventory balance system. The idea is to minimize the expected production and holding costs over a finite time horizon [0,T]. It is assumed that demand at each period of time is a Gaussian process. Besides, as said above, the inventory level is not accurately measured. Additionally, probabilistic constraints on inventory and production levels are included. Thus, this kind of problem can be formulated as a linear-stochastic optimal control problem under imperfect information of state and with chance-constraints on production and inventory variables.

A control structure for the above problem is possible by using Kalman Filter [2] to identify the estimates of the inventory level. This structure allows not only determining the optimal mean decision policy but also accommodating the effects of the evolution of the variances of production and inventory variables. Based on Separation Theorem [1-2], the stochastic problem can be split into two other sub-problems: the first one determines a linear optimal gain from the solution of a minimum variance problem [1]. The second solves a deterministic equivalent problem that allows obtaining a mean optimal policy for the stochastic problem. As a result, a sub-optimal decision policy for the stochastic production-planning problem is obtained from the merge of the solution of these two sub-problems. This result is denoted as a linear feedback decision rule [8].

2 The Stochastic Inventory Problem

Let us introduce the notation to be used henceforth:

I_k = inventory level at period k,
P_k = production level at period k,
Y_k = level of inventory effectively measured at period k,
D_k = demand level at period k,
v_k = noise level during inventory measured at period k,
$h \cdot (I_k)^2$ = holding cost of the inventory level at period k,
$c \cdot (P_k)^2$ = production cost at period k.

Consider a linear, discrete-time, stochastic inventory balance equation:

$$I_{k+1} = I_k + P_k - D_k \tag{1}$$

where the demand D_k is a Gaussian variable with mean $\hat{D}_k$ and time-invariant variance $\sigma_D^2 \geq 0$. It's assumed that at $k = 0$ the initial inventory level I_o is estimated. Associated with (1), there is an output device described as follows:

$$Y_k = I_k + v_k \tag{2}$$

where the variable Y_k denotes the output variable (i.e., inventory level plus an error) and v_k denotes an error of measuring I_k from the output device. This error is defined as a normal distributed random variable, with mean $\hat{v}_k = 0$ and finite variance $\sigma_v^2 > 0$.

Since I_k is not directly accessible for control purpose, it is important to introduce the vector Γ_k, which contains all current, and past information related to output (Y_k) and control (P_k) variables.

$$\Gamma_k = \{P_o, P_1, ..., P_{k-1}, Y_0, Y_1, ..., Y_k\} \supset \Gamma_{k-1} \tag{3}$$

Based on the available information given by (3), an optimal non-negative production policy $\{P_o, P_1, ..., P_{T-1}\}$, for some fixed $T \geq 1$, can be find by mean of a control strategy that minimizes the following expected production and holding costs:

$$J(u) = h \cdot E\{I_T^2 \mid \Gamma_T\} + \sum_{k=0}^{T-1} E\{(h \cdot I_k^2 + c \cdot P_k^2) \mid \Gamma_k\} \tag{4}$$

besides, (4) is subject to (1)-(2) and the probabilistic constraints given by:

$$\begin{aligned} &\text{Prob.}(\underline{I} \leq I_{k+1} \leq \overline{I}) \geq 2 \cdot \alpha - 1 \\ &\text{Prob.}(\underline{P} \leq P_{k+1} \leq \overline{P}) \geq 2 \cdot \beta - 1 \end{aligned} \tag{5}$$

where $\alpha, \beta \in [1/2, 1]$ denotes probabilistic indexes fixed *a priori* by the manager.

Some important comments about these formulations:

(a) the Quadratic Criterion model in (2), as justified by [6], represents a realistic structure for production planning process. For instance, the holding costs are incurred for both negative (backlogged sales) and positive inventory (customer satisfaction). Another positive aspect, is that, it allows uncertainties to be handled directly by computing the expected value of the cost;
(b) in long-term planning horizons, the decisions are strongly influenced by stochastic fluctuation of demands. It does not make sense, therefore, to consider deterministic problems in order to favor the application of some mathematical programming procedure, because the result deterministic (i.e. open-loop) production plan has a big chance of becoming a "disaster" when applied for planning purpose in the company; and
(c) the probabilistic constraints (5) are included into the model to ensure that both inventory and production levels won't violate their physical upper ($\overline{I}$ and $\overline{P}$) and lower ($\underline{I}$ and $\underline{P}$) boundaries. Note that the lower boundaries have the following physical interpretations: if we consider $\underline{I} > 0$, it means a hedge against uncertainty, otherwise if we choose $\underline{I} = 0$, for all k, it means that we eliminate all backlogging. In the same way, if we consider a minimal production level $(\underline{P} > 0)$, it allows studying the effect of different production capacity policies [7].

3 Kalman Filter

Since the structure of the system (1-2) is linear and the uncertainty (i.e. demand variables) involved is Gaussian, then the density probability function of the state (I_k) "conditioned" to (3) will be also Gaussian and can be therefore parameterized by the conditional mean and covariance that are generated over the time from the Kalman filter, as follows [2]:

$$\begin{aligned} I_{k+1|k+1} = I_{k|k} + P_k + \hat{D}_k + \\ V_I^{k+1|k} \cdot (V_I^{k+1|k} + \sigma_v^2)^{-1} \cdot [Y_{k+1} - I_{k+1|k}] \end{aligned} \tag{6}$$

and,

$$\begin{cases} V_I^{k+1|k+1} = V_I^{k+1|k} - V_I^{k+1|k} \cdot (V_I^{k+1|k} + \sigma_V^2)^{-1} \cdot \overline{V}_I^{k+1} \\ V_I^{k+1|k} = V_I^{k|k} + \sigma_D^2 \end{cases} \tag{7}$$

where $I_{k|k} = E\{I_k | \Gamma_k\}$ and $V_I^{k|k} = E\{(I_k - I_{k|k})^2\}$.

denotes respectively the mean and covariance estimates of inventory variable. Note that the operator $E\{.\}$ denotes the expected value of a random variable with the distribution function *F*(.).

The initial conditions for (6) and (7) are given by:

$$\begin{cases} I_{0|0} = \hat{I}_0 + V_I^{0|0} \cdot (\sigma_v^2)^{-1} \cdot (Y_0 - \hat{I}_0) \\ V_I^{0|0} = V_I^0 - V_I^0 \cdot (V_I^0 - \sigma_v^2)^{-1} \cdot V_I^0 \end{cases} \tag{8}$$

where $\hat{I}_0 = E\{I_0\}$ and $V_I^0 = E\left\{\left(I_0 - \hat{I}_0\right)^2\right\}$.

denotes respectively the mean and covariance of the initial state I_0.

Since the equations (6) and (7) concentrate all available information about the current state of the inventory in the balance system (1), the approach presented in [3] to deal with the stochastic problem under perfect information of state can be adapted here to deal with the problem described in section 2. Next, this approach will be briefly discussed.

4 The Linear Feedback Rule

As said before, the imperfect information of the state relate to the stochastic inventory equation (1) can be completely represented by its conditional mean and covariance equations given by (6) and (7) respectively.

From the above facts, a linear decision rule can be formulated in order to help to solve the stochastic problem. Such rule is given by:

$$P_k = \hat{P}_k + G_\lambda \cdot \left(I_{k|k} - \hat{I}_k\right) \tag{9}$$

where G_λ is the linear gain and $\hat{I}_k = E\left\{E\left(I_k \mid k\right)\right\} = E\left\{I_{k|k}\right\}$. Note that the control policy (9) is analogous to the one obtained to the perfect information of state [8], the difference is that the statistic information are related to the estimate values of the state given by the Kalman filter (6-7).

The inventory balance equation (1) is represented by $I_{k|k}$ and $V_I^{k|k}$ given by (6) and (7). The means $\hat{P}_k$ and G_λ must be determined by the procedure described next. Note that the inventory level $I_{k|k}$ is a quantity to be estimated from (6) at each period k, and the term $G_\lambda \cdot \left(I_{k|k} - \hat{I}_k\right)$ means a release adjustment according to whether the estimated inventory ($I_{k|k}$) is greater or less than expected inventory ($\hat{I}_k$). For determination of G, the minimum variance problem can be stated [1]:

$$\begin{aligned} &\text{Min } E\{\delta Y_{k+1}^2 + \lambda \cdot \delta P_k^2 \mid \Gamma_k\} \\ &\quad \text{s.t.} \\ &\qquad \delta I_{k+1|k+1} = \delta I_{k|k} + \delta P_k - \delta D_k \\ &\qquad \delta Y_k = \delta I_{k|k} + v_k \\ &\qquad\qquad \chi \\ &\text{Min } \left\{V_I^{k+1|k+1} + \lambda \cdot V_P^k\right\} \\ &\quad \text{s.t. } (7) \end{aligned} \tag{10}$$

where $\delta I_{k|k} = I_{k|k} - \hat{I}_k$ denotes the error of estimation. The parameter $\lambda \in \Re^+$ denotes the trade-off between the production and inventory variances. The closed-loop optimal solution of (10) is given by

$$V_P^k = G_\lambda \cdot V_I^{k|k} \tag{11}$$

where the linear gain is defined as $G_\lambda = \frac{1}{1+\lambda}$. Thus, the close-loop variance of the inventory system (1) can be written as follows:

$$\begin{aligned} V_I^{k+1} &= V_I^k + G_\lambda^2 \cdot V_I^{k|k} + \sigma_D^2 \\ V_I^k &= 0 \end{aligned} \tag{12}$$

Note from (11) and (12) that the variance of inventory V_I^k increases over the time periods and, as an immediate consequence, the production variance V_P^k increases proportionally over the same periods.

The maximum values reached by the inventory and production variances occur at T and T-1 periods respectively. Thus, the idea is to find a value to λ that reduces simultaneously the growing of the both variances. Two alternatives was presenting by [3]. The most simple is the one that uses the Tchebycheff inequality to show that approximately, the trade-off value of λ can be determined as follows:

$$\lambda = \frac{\Delta I^2}{\Delta P^2} \tag{13}$$

where $\Delta I = \bar{I} - \underline{I}$ and $\Delta P = \bar{P} - \underline{P}$ that denote respectively the space between the upper and lower levels of the inventory and production constraints.

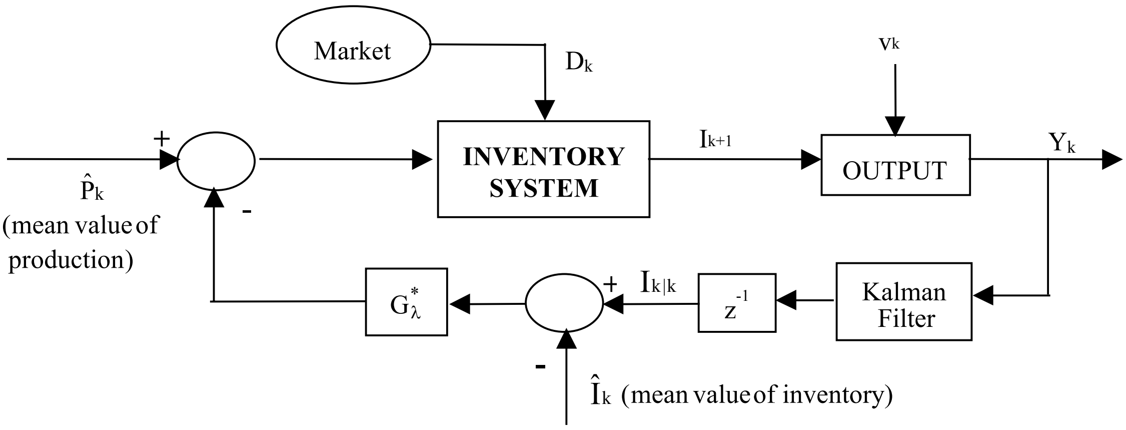

Fig. 1. Schematic representation of the system

Production capacity:					$\underline{P}=0$			$\overline{P}=12$				
Inventory Level:					$\underline{I}=5$			$\overline{I}=15$				
Holding and Production costs:					$h=2$			$c=1$				
Initial and final inventory:					$I_0=I_T=10$							
Probabilistic measures:					$\alpha=0.85$			β=0.5				
Demand features:					$\sigma_D=1.42$							
Month (k)	Jan	Feb	Mar	Apr	May	Jun	Jul	Ago	Sep	Oct	Nov	Dec
$\hat{D}_k$	**7**	**8**	**7**	**7**	**6**	**4**	**5**	**4**	**5**	**5**	**6**	**8**

Table 1 Company's data

Finally, we can introduce the sub-optimal strategy to solve the stochastic production-planning problem, stated in the previous section, as follows:

Let the production policy given by:

$$P_k = \hat{P}_k - G^*_\lambda \cdot \left(I_{k|k} - \hat{I}_k\right) \quad (14)$$

where $\hat{I}_k, \hat{P}_k$ and G^*_λ are determined by the following procedure:

Step 1) *Minimum Variance Problem*: the parameter λ^* is computed from (13) and used to determine the linear gain $G^*_\lambda=1/(1+\lambda^*)$. The optimal gain complete the definition of the linear production rule (14) that is employed to adjust the estimated inventory levels $(I_{k|k})$ to desired levels $(\hat{I}_k)$.

Step 2) *Mean Problem*: knowing G^*_λ and the estimates given by (5) and (6), it is possible to calculate the close-loop inventory and production variances (11) and (12) and their respective probabilistic distributions functions $F_{I,k+1}(\cdot)$ and $F_{P,k}(\cdot)$, for each $k\in[0,T-1]$. As a consequence, the stochastic problem, described by (1-4) can be reduced to a deterministic equivalent whose solution can be provided by any applicable method of mathematical programming:

$$\underset{\hat{P}_k}{\mathrm{Min}}\; h\cdot\hat{I}_T^2 + \sum_{k=0}^{T-1}\left(h\cdot\hat{I}_k^2 + c\cdot\hat{P}_k^2\right)$$

s.t.

$$\begin{aligned}
\hat{I}_{k+1} &= \hat{I}_k + \hat{P}_k - \hat{D}_k \\
\hat{I}_k &= \left\{\underline{I} + F^{-1}_{I,k}(\alpha) \le \hat{I}_k \le \overline{I} - F^{-1}_{I,k}(\alpha)\right\} \\
\hat{P}_k &= \left\{\underline{P} + F^{-1}_{P,k}(\beta) \le \hat{P}_k \le \overline{P} - F^{-1}_{P,k}(\beta)\right\}
\end{aligned} \quad (15)$$

where $F^{-1}(.)$ denote the inverse of the probability distribution function, and α and β are probabilistic index provided by the manager.

Note that (15) provides the mean value of inventory ($\hat{I}_k$) and production ($\hat{P}_k$) that are used as set points in this strategy, see in figure 1 a schematic representation of this situation.

5 Example

A company, whose sales are subject to the effects of seasonal fluctuating demand, tries to develop an aggregated production plan which minimizes total costs over a finite planning horizon (T=12 months). The sub-optimal strategy discussed in previous section has been employed here to provide a decision policy for managers. Table 1 shows the numerical data of the company.

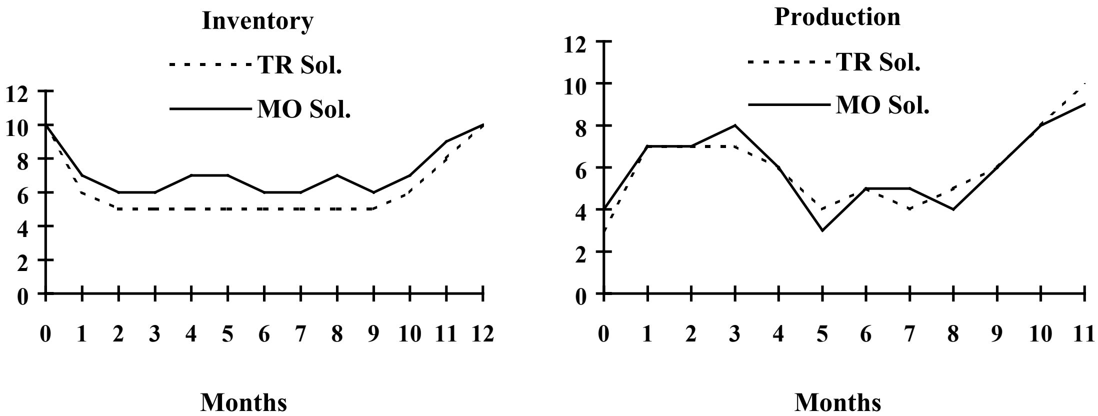

Figure 2. Optimal mean solutions

The optimal solution for the *variance problem* (step 1) is: $\lambda^* = 0.69$ and $G(\lambda^*) = 0.59$. From this result, the *mean problem* (15) can be formulated (step 2) and solved by dynamic programming algorithm. Figure 2 compares the mean optimal (MO) feedback solution of (15) – obtained by using a sub-optimal approach named Open-Loop Feedback Controller (OLFC) [5] – with the true optimal (TR) solution – obtained from the application of stochastic dynamic programming algorithm. It is worth mentioning that the true optimal solution was determined from the same stochastic problem, described in the section 2, but under hypothesis of perfect information of state [5]-[8]. The analysis of the two trajectories shows that the MO solution tries to follow the true solution as close as possible. This interesting characteristic can be explained through the fact that during the solution of the deterministic problem (15), the evolution of variances of inventory and production variables are controlled to the linear gain and, as a consequence, the variances can not grow freely over the time what implies in a smoothing optimal solution [8] .

6 Conclusion

In this paper, a sub-optimal solution for a chance-constrained stochastic production problem under imperfect information was discussed. Considering the linear-Gaussian nature of the inventory balance system, the entire information gathering for control purpose (statistic sufficiency) can be concentrated in the first and second statistic moments, which are estimated from Kalman filter. As a result, the Separation Theorem can be applied to obtain a feedback rule that is based on variances control. This decision rule consists in a linear feedback gain used to reduce the effect of uncertainty due to fluctuating demand over the inventory and production levels. From a simple example, the true optimal solution (TR) derived of the stochastic problem under perfect information of states was compared with mean optimal (MO) feedback solution provided from the application of OLF procedure to the equivalent deterministic problem (15). The results obtained from this sub-optimal approach allows concluding that the linear decision rule, discussed in section 4, can be a good strategies to deal with aggregated production planning problems with imperfect information of decisions variables.

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