COMPATIBILITY OF BEHAVIOR INTERCONNECTIONS

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Abstract

In this paper we discuss the concept of compatible behavior interconnections. We distinguish the case where all variables are available for interconnection (*total interconnection*) from the case where only a part of them are (*partial interconnection*). The main idea behind the notion of compatibility for total interconnection is causality of (the effect of) the interconnection. We also show that state variables are closely related with the concept of compatibility.

The concept of compatibility has been introduced for linear behaviors. In this paper, we propose an extension for the existing concepts. Relations between the proposed concept and the existing ones are also explained.

1 Introduction

Control of dynamical systems can be viewed as interconnection between the system and another system (the controller). This point of view is advocated, for example in the behavioral approach to control theory[2, 5, 6]. A question that arises naturally when considering interconnection of systems is whether every system can be interconnected with every other system. If this question gets a negative answer, then the next logical question will be how to characterize the compatible interconnections. Of course, it should also be made formal what is meant when an interconnection is said to be compatible.

In this paper we discuss the concept of compatibility for behavioral interconnections. Such concept was developed, for example in [6], when the notions of *regular* and *regular feedback* interconnection were introduced for linear behaviors. However, the extension to more general cases is still lacking.

As mentioned above, the behavioral approach to control theory sees solving control problems as finding a system (controller) whose interconnection with the plant results in a desirable system. Formulation of control problems in general behaviors has been done in [5]. There, a construction for the so called *canonical controller* is given, together with some sufficient conditions under which the controller solves the problem. The concept of compatibility adds another dimension to the problem, since it is desirable to obtain a controller that can really be interconnected to the plant. Such problem for linear behaviors has been posed and solved in [1].

The discussion in this paper is organized as follows. In Section 2 some mathematical preliminaries will be given. In Section 3, we discuss the concept of strong and weak compatibility for total interconnections. Finally in Section 4, we present remarks on the compatibility of partial interconnection and the direction for future research.

2 Mathematical preliminaries

In this paper we discuss interconnections of general behaviors. We start with giving a concise introduction to the matters.

A behavior $\mathfrak{B}$ is defined as a collection of trajectories pertaining to the evolution of the variables in a set W over a time axis $\mathcal{T}$. The variables in W take value in a $\mathbf{W}$. Hence, $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$. Examples of commonly used $\mathbf{W}$ are linear spaces (for linear systems) and finite sets (for discrete event systems). The only restriction we impose on the time axis $\mathcal{T}$ is that it admits a total ordering $<$. Given the relation $<$ and the natural identity relation $=$, we define another relation $\leq$.

$$(t_1 \leq t_2) \Leftrightarrow (t_1 < t_2) \vee (t_1 = t_2),\ t_1, t_2 \in T.$$

Let the behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be both subsets of $\mathbf{W}^{\mathcal{T}}$. The *total interconnection* of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is denoted as $\mathfrak{B}_1 \parallel \mathfrak{B}_2$. This is defined as

$$\begin{aligned}\mathfrak{B}_1 \parallel \mathfrak{B}_2 :=& \mathfrak{B}_1 \cap \mathfrak{B}_2, \\ &= \left\{ w \in \mathbf{W}^{\mathcal{T}} \mid w \in \mathfrak{B}_1 \text{ and } w \in \mathfrak{B}_2 \right\}.\end{aligned}$$

It is called total interconnection since all the variables in W are involved in the interconnection. In this paper, we will only discuss the compatibility of total interconnections.

Another important concept that we are going to use in this paper is that of the *dynamic map*. Given a behavior $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$, a dynamic map ϕ takes an element of $\mathfrak{B} \times \mathcal{T}$ and maps it to an element of its codomain Φ. We use the following notation convention throughout this paper.

Notation 2.1 *Let $\phi : \mathfrak{B} \times \mathcal{T} \rightarrow \Phi$ be a dynamic map, and X and Y be subsets of $\mathfrak{B}$ and Φ respectively. A time-indexed dynamic map, notated as $\phi_t(\cdot)$ or $\phi(\cdot;t)$, $t \in \mathcal{T}$, is defined as*

$$\phi_t(w) := \phi(w;t) := \phi(w,t), w \in \mathfrak{B}.$$

Furthermore the following notations also apply in this paper.

$$\phi_t(X) := \{y \in \Phi \mid \exists x \in X, \phi_t(x) = y\},$$
$$\phi_t^{-1}(Y) := \{x \in \mathfrak{B} \mid \exists y \in Y, \phi_t(x) = y\}.$$

The *state similarity operator* generated by the time-indexed dynamic map ϕ_t on $\mathfrak{B}$ is defined to be

$$\bar{\phi}_t : 2^{\mathfrak{B}} \to 2^{\mathfrak{B}},$$
$$\bar{\phi}_t(\cdot) = \phi_t^{-1}(\phi_t(\cdot)).$$

For any subset $X \subset \mathfrak{B}$, $\bar{\phi}_t(X)$ gives the largest subset of $\mathfrak{B}$ whose image under ϕ_t is $\phi_t(X)$.

Two dynamic maps on $\mathfrak{B}$, ϕ and γ are said to be orthogonal, or $\phi \perp \gamma$, if for all $t \in \mathcal{T}$,

$$(\bar{\phi}_t \circ \bar{\gamma}_t)(w) = \mathfrak{B}, \forall w \in \mathfrak{B}, \tag{1a}$$
$$(\bar{\gamma}_t \circ \bar{\phi}_t)(w) = \mathfrak{B}, \forall w \in \mathfrak{B}. \tag{1b}$$

Intuitively, orthogonality between two dynamic maps implies impossibility to make any inferences on the value of one map given the value of the other.

It is also possible to define orthogonality between maps defined on two non-disjoint behaviors. Let ϕ_i be a dynamic map on $\mathfrak{B}_i$, $i = 1, 2$. Assume that $\mathfrak{B}_1 \cap \mathfrak{B}_2 \neq \emptyset$. We say that ϕ_1 and ϕ_2 are orthogonal if for all $t \in \mathcal{T}$,

$$\bar{\phi}_2 \left(\bar{\phi}_{1t}(w) \cap \mathfrak{B}_2\right) = \mathfrak{B}_2, \forall w \in \mathfrak{B}_1, \tag{2a}$$
$$\bar{\phi}_2 \left(\bar{\phi}_{1t}(w) \cap \mathfrak{B}_1\right) = \mathfrak{B}_1, \forall w \in \mathfrak{B}_2. \tag{2b}$$

Again, the intuitive interpretation behind this idea is that when the two maps are orthogonal, it is impossible to make any inferences on the value of one map given the value of the other.

A dynamic map ϕ is said to constitute a *state map* of the behavior $\mathfrak{B}$ if and only if it possesses the patching property as follows.

(Patching) For any $w_1, w_2 \in \mathfrak{B}$ and $\tau \in \mathcal{T}$, the following implication holds.

$$(\phi(w_1, \tau) = \phi(w_2, \tau)) \Rightarrow (w_1 \wedge_\tau w_2) \in \mathfrak{B}.$$

The symbol $\wedge_\bullet$ signifies the patching/concatenation operation, where

$$(w_1 \wedge_\tau w_2)(t) := \begin{cases} w_1(t), & t < \tau \\ w_2(t), & t \geq \tau. \end{cases}$$

For a behavior $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$ and $t \in \mathcal{T}$, we can define the *directability* relation $D_{\mathfrak{B}}(t) \subset \mathfrak{B} \times \mathfrak{B}$ such that $(w_1, w_2) \in \mathfrak{B} \times \mathfrak{B}$ is included in $D_{\mathfrak{B}}(t)$ if and only if $w_1 \wedge_t w_2 \in \mathfrak{B}$.

Definition 2.2 *Let $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$. A state map $\phi : \mathfrak{B} \to \Phi$ is said to be* past-induced *if for any $w_1, w_2 \in \mathfrak{B}$ and $\tau \in \mathcal{T}$,*

$$(w_1(t)|_{t \leq \tau} = w_2(t)|_{t \leq \tau}) \Rightarrow (\phi(w_1, \tau) = \phi(w_2, \tau)).$$

Similarly, ϕ is future-induced *if*

$$(w_1(t)|_{t \geq \tau} = w_2(t)|_{t \geq \tau}) \Rightarrow (\phi(w_1, \tau) = \phi(w_2, \tau)).$$

The following proposition relates past and future induced state maps with directability.

Proposition 2.3 *Let the behavior $\mathfrak{B} \subset \mathbb{W}^{T}$. Let α and ω be any past induced and future induced state maps respectively. The following relation holds for all $w_1, w_2 \in \mathfrak{B}$ and $t \in \mathbb{T}$.*

$$(w_1 D_{\mathfrak{B}}(t) w_2) \Leftrightarrow \bar{\alpha}_t(w_1) \cap \bar{\omega}_t(w_2) \neq \emptyset. \tag{3}$$

Proof. ($\Rightarrow$)Let w_{i-} and w_{i+} denote the past and future of w_i, $i = 1, 2$. Hence, $w_{i-} \in \mathfrak{B}|_{[-\infty,t)}$ and $w_{i+} \in \mathfrak{B}|_{[t,\infty]}$. Denote $w_3 := w_1 \wedge_t w_2$. The past of w_3 is w_{1-} and its future is w_{2+}. Consequently we have that

$$w_3 \in \bar{\alpha}_t(w_1), \tag{4a}$$
$$w_3 \in \bar{\omega}_t(w_2). \tag{4b}$$

Hence $\bar{\alpha}_t(w_1) \cap \bar{\omega}_t(w_2) \neq \emptyset$.

($\Leftarrow$) Assume that $\bar{\alpha}_t(w_1) \cap \bar{\omega}_t(w_2) \neq \emptyset$. Take an element from this set and call it w_3. We know from (4a) that

$$w_4 := (w_1 \wedge_t w_3) \in \mathfrak{B}.$$

Since the future of w_4 is the same as that of w_3, necessarily $\omega_t(w_4) = \omega_t(w_3) = \omega_t(w_2)$. Hence we can construct w_5 such that

$$w_5 := (w_4 \wedge_t w_2) \in \mathfrak{B}.$$

Notice that w_5 has the past of w_1 and the future of w_2. Hence

$$w_5 = (w_1 \wedge_t w_2) \in \mathfrak{B}.$$

Notice that we do not require $w_1, \cdots, w_5$ to be distinct. ■

Corollary 2.4 *The following relation also holds*

$$(w_1 D_{\mathfrak{B}}(t) w_2) \Leftrightarrow w_2 \in (\bar{\omega}_t \circ \bar{\alpha}_t)(w_1), \tag{5}$$
$$\Leftrightarrow w_1 \in (\bar{\alpha}_t \circ \bar{\omega}_t)(w_2). \tag{6}$$

Following that, we introduce the set valued maps $D_{\mathfrak{B},t}(\cdot)$ and $D^{-1}_{\mathfrak{B},t}(\cdot)$ as

$$D_{\mathfrak{B},t}(\cdot) := (\bar{\omega}_t \circ \bar{\alpha}_t)(\cdot), \tag{7}$$
$$D^{-1}_{\mathfrak{B},t}(\cdot) := (\bar{\alpha}_t \circ \bar{\omega}_t)(\cdot). \tag{8}$$

We can construct $\approx$, an equivalence relation within the class of all state maps of a behavior $\mathfrak{B}$, by defining that two state maps ϕ and γ are equivalent, or $\phi \approx \gamma$ if and only if for any $w_1, w_2 \in \mathfrak{B}$ and $t \in \mathcal{T}$,

$$(\phi(w_1, t) = \phi(w_2, t)) \Leftrightarrow (\gamma(w_1, t) = \gamma(w_2, t)),$$

or equivalently

$$\bar{\phi}_t = \bar{\gamma}_t.$$

With that, we also introduce the partial ordering $\preccurlyeq$, by defining $\gamma \preccurlyeq \phi$ if and only if for any $w_1, w_2 \in \mathfrak{B}$ and $t \in \mathcal{T}$,

$$(\phi(w_1, t) = \phi(w_2, t)) \Rightarrow (\gamma(w_1, t) = \gamma(w_2, t)),$$

or equivalently

$$\bar{\phi}_t(\cdot) \subseteq \bar{\gamma}_t(\cdot).$$

A dynamic map ϕ is said to be a *minimal state map*, if there exists no other state map $\gamma \not\approx \phi$ such that $\gamma \preccurlyeq \phi$. The concept of minimal state map is closely related to the concept of *irreducible state* introduced in [7]. Generally, for any given behavior, there is no unique minimal state map (modulo $\approx$) (see [4, 7]). We shall characterize the cases where such minimal state map exists. First, we need the following definitions.

Definition 2.5 *[4, 7]Let $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$. A* canonical past-induced *state map $\phi_- : \mathfrak{B} \to \Phi$, is characterized by the equivalence relation $R_-(t)$ on $\mathfrak{B}$, where for any $w_1, w_2 \in \mathfrak{B}$ and $\tau \in \mathcal{T}$,*

$$(w_1 R_-(t) w_2) :\Leftrightarrow ((w_1 D_{\mathfrak{B}}(t) w) \Leftrightarrow (w_2 D_{\mathfrak{B}}(t) w)), \forall w \in \mathfrak{B}.$$

The state map ϕ_- must then satisfy

$$(\phi_-(w_1, t) = \phi_-(w_2, t)) \Leftrightarrow (w_1 R_-(t) w_2).$$

Notice that we do not characterize a single state map but a family of state maps related by $\approx$. Similarly, a canonical future-induced *state map $\phi_+ : \mathfrak{B} \to \Phi$, is characterized by the equivalence relation $R_+(t)$ on $\mathfrak{B}$, where for any $w_1, w_2 \in \mathfrak{B}$ and $\tau \in \mathcal{T}$,*

$$(w_1 R_-(t) w_2) :\Leftrightarrow ((w D_{\mathfrak{B}}(t) w_1) \Leftrightarrow (w D_{\mathfrak{B}}(t) w_2)), \forall w \in \mathfrak{B}.$$

The state map ϕ_+ must then satisfy

$$(\phi_+(w_1, t) = \phi_+(w_2, t)) \Leftrightarrow (w_1 R_+(t) w_2).$$

These two canonical state maps are minimal.

Proposition 2.6 *A behavior $\mathfrak{B} \subset \mathbf{W}^{\mathcal{T}}$ admits a unique minimal state map (modulo $\approx$) if and only if for any $t \in \mathcal{T}$ the directability relation $D_{\mathfrak{B}}(t)$ is an equivalence relation.*

Proof. ($\Leftarrow$)Assume that $D_{\mathfrak{B}}(t)$ is an equivalence relation, for all $t \in \mathcal{T}$. Then, $D_{\mathfrak{B}}(t)$ induces a partitioning in $\mathfrak{B}$, i.e. we can define a mapping $x_t : \mathfrak{B} \to X$ such that for any $w_1, w_2 \in \mathfrak{B}$,

$$(w_1 D_{\mathfrak{B}}(t) w_2) \Leftrightarrow (x_t(w_1) = x_t(w_2)).$$

Define the state map ϕ such that

$$\phi(w, t) := x_t(w),$$

then ϕ is a minimal state map and other minimal state maps are related to ϕ via $\approx$.

($\Rightarrow$)Assume that $\mathfrak{B}$ admits a unique minimal state map (modulo $\approx$), then the canonical past-induced and future induced state maps must coincide (modulo $\approx$). Let us call this canonical state map ϕ_{can}. Since ϕ_{can} is both past and future induced, we can substitute it for α and ω in Proposition 2.3. Rewriting (3), we obtain

$$(w_1 D_{\mathfrak{B}}(t) w_2) \Leftrightarrow (\phi_{\text{can}}(w_1, t) = \phi_{\text{can}}(w_2, t)).$$

Hence the directability relation $D_{\mathfrak{B}}(t)$ is an equivalence relation. ■

If the dynamic map ϕ is a state map of $\mathfrak{B}$, then its image Φ is called a *state space* of $\mathfrak{B}$. Contrary to the common intuition, the minimality of a state map ϕ does not imply the minimality of the size of its state space.

3 Compatibility of total interconnections

3.1 Directability and compatibility

The definition of directability introduced in the previous section gives rise to the first definition of compatibility.

Definition 3.1 ***(compatibility)*** *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be subsets of $\mathbf{W}^{\mathcal{T}}$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is* compatible *if for any $w_i \in \mathfrak{B}_i$, $i = 1, 2$, there exist a $w \in \mathfrak{B}_1 \parallel \mathfrak{B}_2$, and $t \in \mathcal{T}$ such that $w_1 D_{\mathfrak{B}_1}(t) w$ and $w_2 D_{\mathfrak{B}_2}(t) w$.*

Intuitively, this definition means that all possible pasts in $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be accommodated in the future by the interconnection. In other words, the interconnection should only affect the future. In fact, causality is the crux of this concept.

Notice that in the definition of compatibility, the time instant at which the interconnection can be formed depends on the trajectories of each behavior. To require that the interconnection can be formed at any time is to define a stricter notion of compatibility.

Definition 3.2 ***(uniform compatibility)*** *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be subsets of $\mathbf{W}^{\mathcal{T}}$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is* uniformly compatible *if it is compatible for any $t \in \mathcal{T}$.*

The notion of compatibility given in Definition 3.2 for general behaviors is related to the definition of regular feedback interconnection for linear behaviors[1]. This definition was introduced in [6]. The characterization of a regular feedback interconnection is as follows.

Definition 3.3 ***(regular feedback)*** *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be linear behaviors defined by*

$$\mathfrak{B}_i := \left\{ w \in \mathcal{L}^1_{loc}(\mathbb{R}, \mathbb{R}^q) \mid R_i\left(\frac{d}{dt}\right) w = 0 \right\}, i = 1, 2,$$

where R_1 and R_2 are full-row-rank polynomial matrices[2] with g_1 and g_2 rows respectively and q columns. Define n_i as the McMillan degree of R_i and n as that of (the full row rank version of) $\begin{bmatrix} R_1^T & R_2^T \end{bmatrix}^T$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is a regular feedback interconnection if and only if $n = n_1 + n_2$.

[1] by linear behaviors we refer to those represented by (a system of) constant-coeffi cient ODEs.

[2] The fact that R_1 and R_2 are full row rank can be assumed without any lost of generality. Refer to [2] for explanation.

The McMillan degree of a linear system indicates the dimension of its minimum state space realization. In the following, the relation between regular feedback and uniform compatibility will be explained.

Take any two behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$, both subsets of $\mathbf{W}^{\mathcal{T}}$. Let α_i and ω_i be a past-induced state map and future-induced state map of $\mathfrak{B}_i$ respectively, $i = 1, 2$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is uniformly compatible if and only if for any $w_1 \in \mathfrak{B}_1$, $w_2 \in \mathfrak{B}_2$, and $t \in \mathcal{T}$,

$$D^{-1}_{\mathfrak{B}_2,t}\left(D_{\mathfrak{B}_1,t}(w_1) \cap \mathfrak{B}_2\right) = \mathfrak{B}_2, \tag{9a}$$

$$D^{-1}_{\mathfrak{B}_1,t}\left(D_{\mathfrak{B}_2,t}(w_2) \cap \mathfrak{B}_1\right) = \mathfrak{B}_1. \tag{9b}$$

Let us now turn to linear behaviors. It is a well known fact that linear behaviors admit unique minimal state map (modulo $\approx$). Hence, in the case of linear behaviors, the canonical past-induced and future-induced state maps, α and ω, are identical. Referring to (7) - (8), we conclude that this implies

$$D_{\mathfrak{B},t}(\cdot) = D^{-1}_{\mathfrak{B},t}(\cdot) = \bar{\alpha}_t(\cdot). \tag{10}$$

Applying this fact to the necessary and sufficient conditions for uniform compatibility (9), we infer the following. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ of linear behaviors is uniformly compatible if and only if for any $w_1 \in \mathfrak{B}_1$, $w_2 \in \mathfrak{B}_2$, and $t \in \mathcal{T}$,

$$\bar{\alpha}_{2t}\left(\bar{\alpha}_{1t}(w_1) \cap \mathfrak{B}_2\right) = \mathfrak{B}_2, \tag{11a}$$

$$\bar{\alpha}_{1t}\left(\bar{\alpha}_{2t}(w_2) \cap \mathfrak{B}_1\right) = \mathfrak{B}_1. \tag{11b}$$

Here, $\bar{\alpha}_{it}(\cdot) := \alpha^{-1}_{it}(\alpha_{it}(\cdot))$ and α_i is the (unique) minimal state map of $\mathfrak{B}_i$, $i = 1, 2$. We see that the condition for uniform compatibility of $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is that α_1 and α_2 must be orthogonal.

Now, the algebraic condition $n = n_1 + n_2$ in Definition 3.3 relates to the fact that the dimension of the minimum state space realization of the interconnected system must be equal to the sum of those of the individual systems. Although it will not be proven here, this is equivalent to the fact that α_1 and α_2 are orthogonal. Indeed, in [6] it is explained that being regular feedback is equivalent to the fact that the interconnection can be made at any time.

3.2 Weak directability and compatibility

In the previous subsection we have discussed the concept of compatibility. In this subsection, we are going to discuss a more general kind of compatibility. Notice that compatibility requires that the interconnection can be made without any kind of preparation. A more general criterion would allow some preparation stage to take place prior to the interconnection, and thus accommodating more interconnections.[3]

[3] The authors would like to thank Madhu Belur for a valuable discussion on this issue.

Definition 3.4 ***(weak directability)*** *Let $w_1, w_2 \in \mathfrak{B}$ and $\tau \in \mathcal{T}$. We say that w_1 is* weakly directable *to w_2 at time τ if there exists a trajectory $w_3 \in \mathfrak{B}$ and a $\tau' \leq \tau$ such that*

$$w_3(t) = \begin{cases} w_1(t), & t \leq \tau', \\ w_2(t) & t > \tau. \end{cases}$$

Similar to the case of (strong) directability, we shall use a shorthand notation for weak directability. The fact that w_1 is *weakly directable* to w_2 at time τ can be written as $w_1 D^*(\mathfrak{B}, \tau) w_2$. It is interesting to realize that the time indicated in the relation is such that for any $\tau' \geq \tau$,

$$w_1 D^*(\mathfrak{B}, \tau) w_2 \Rightarrow w_1 D^*(\mathfrak{B}, \tau') w_2.$$

The notion of weak directability is closely related to controllability. Indeed, in behavioral systems theory a system is controllable if any trajectory is weakly directable to any other trajectory [2].

The definition of weak directability leads to the definition of weak compatibility.

Definition 3.5 ***(weak compatibility)*** *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be subsets of $\mathbf{W}^{\mathcal{T}}$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is* weakly compatible *at time $t \in \mathcal{T}$ if for any $w_i \in \mathfrak{B}_i$, $i = 1, 2$, there exists a $w \in \mathfrak{B}_1 \parallel \mathfrak{B}_2$, such that $w_1 D^*(\mathfrak{B}_1, t) w$ and $w_2 D^*(\mathfrak{B}_2, t) w$.*

This definition can be interpreted as follows. If the interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is weakly compatible, then for any trajectories in $\mathfrak{B}_1$ and $\mathfrak{B}_2$, i.e. w_1 and w_2, we can always find a pair of behaviors $\mathfrak{D}_1(w_1)$ and $\mathfrak{D}_2(w_2)$, which in general depends on w_1 and w_2, such that the interconnection $\mathfrak{B}_i \parallel \mathfrak{D}_i(w_i)$, $i = 1, 2$, is (strongly) compatible. This is meant as a preparation stage. When the time is right (in Definition 3.5 this time instant is denoted as t), $\mathfrak{D}_i$ is removed from $\mathfrak{B}_i$ and the interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ can be formed as a (strongly) compatible interconnection. The behaviors $\mathfrak{D}_1(w_1)$ and $\mathfrak{D}_2(w_2)$ are called the *directors*, since their function is to direct w_1 and w_2 to another trajectory w that is accommodated in the interconnection. The director $\mathfrak{D}_i$ can be formed as a subset of $\mathfrak{B}_i$ containing the trajectories that bridge w_i to w. This definition of weak compatibility is similar to the concept of mergeable behaviors introduced in [3].

We can also formulate a time-independent version of weak compatibility, as we have done for (strong) compatibility.

(uniform weak compatibility) Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be subsets of $\mathbf{W}^{\mathcal{T}}$. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is *uniformly weakly compatible* if it is weakly compatible for any $t \in \mathcal{T}$.

Lemma 3.6 *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be linear behaviors defined by*

$$\mathfrak{B}_i := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_i\left(\frac{d}{dt}\right) w = 0 \right\}, i = 1, 2,$$

where R_1 and R_2 are full-row-rank polynomial matrices with g_1 and g_2 rows respectively and q columns. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is uniformly weakly compatible if and only if

$$\mathfrak{B}_1 + \mathfrak{B}_2 = \mathfrak{B}_1^{ctr} + \mathfrak{B}_2^{ctr}, \tag{12}$$

where $\mathfrak{B}_i^{ctr}$ is the controllable part of $\mathfrak{B}_i$, for $i = 1, 2$.

We are interested in finding the relation between weak compatibility for general behavior interconnections and regularity for linear behavior interconnections. The concept of regular interconnections was also introduced in [6] and was used, for example in [1] for designing regular controllers.

Definition 3.7 ***(regularity)*** *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be linear behaviors defined by*

$$\mathfrak{B}_i := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_i \left(\frac{d}{dt}\right) w = 0 \right\}, i = 1, 2,$$

where R_1 and R_2 are full-row-rank polynomial matrices with g_1 and g_2 rows respectively and q columns. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is regular if and only if the row rank of $\begin{bmatrix} R_1^T & R_2^T \end{bmatrix}^T$ is $g_1 + g_2$. Equivalently, $\begin{bmatrix} R_1^T & R_2^T \end{bmatrix}^T$ must have full row rank.

In the following we shall discuss the relation between regularity and weak compatibility for linear behaviors.

Theorem 3.8 *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be linear behaviors defined by*

$$\mathfrak{B}_i := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_i \left(\frac{d}{dt}\right) w = 0 \right\}, i = 1, 2,$$

where R_1 and R_2 are full-row-rank polynomial matrices with g_1 and g_2 rows respectively and q columns. The interconnection $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is uniformly weakly compatible if it is regular. The converse is generally not true.

Proof. Assume that the interconnection is regular. We shall use Lemma 3.6 to prove that it is also uniformly weakly compatible. The behavior $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ contains all $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ characterized by

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \left(\frac{d}{dt}\right) w = 0. \tag{13}$$

We assume that at least one of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is uncontrollable. Otherwise the interconnection is uniformly weakly compatible by directly applying Lemma 3.6. Without any lost of generality, assume that $\mathfrak{B}_1$ is uncontrollable. We can always find two unimodular matrices U_1 and V such that $U_1 R_1 V$ is diagonal.

$$U_1 R_1 V =: \tilde{R}_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & D(\xi) & 0 \end{bmatrix},$$

with I the identity matrix and $D(\xi)$ some diagonal polynomial matrix. Denote $\tilde{R}_2 := R_2 V$. The interconnected behavior is given by the following kernel representation.

$$\begin{bmatrix} I & 0 & 0 \\ 0 & D & 0 \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \end{bmatrix} \left(\frac{d}{dt}\right) \tilde{w} = 0, \tag{14}$$

where

$$\tilde{w} := V^{-1} \left(\frac{d}{dt}\right) w.$$

Notice that since the interconnection is regular, $\tilde{R}_{23}$ is full row rank.

In the following, we shall show that $\mathfrak{B}_1 = \mathfrak{B}_1^{ctr} + \mathfrak{B}_2^{ctr}$. Take any trajectory $\tilde{w} := (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \in \mathfrak{B}_1$. Here the trajectory is partitioned according to the columns of the polynomial matrix in (14). Necessarily, $\tilde{w}_1 = 0$ and $\tilde{w}_2 \in \ker D\left(\frac{d}{dt}\right)$. Now, since $\tilde{R}_{23}$ is full row rank, there exists a $\tilde{v}_3$ such that

$$\tilde{R}_{22} \left(\frac{d}{dt}\right) \tilde{w}_2 = \tilde{R}_{23} \left(\frac{d}{dt}\right) \tilde{v}_3,$$

and $\tilde{v} := (0, \tilde{w}_2, \tilde{v}_3) \in \mathfrak{B}_2^{\text{ctr}}$. Also notice that $\tilde{w} - \tilde{v} = (0, 0, \tilde{w}_3 - \tilde{v}_3) \in \mathfrak{B}_1^{\text{ctr}}$. Therefore, we have shown that

$$\mathfrak{B}_1 = \mathfrak{B}_1^{ctr} + \mathfrak{B}_2^{ctr}. \tag{15}$$

If $\mathfrak{B}_2$ is controllable, it is trivially true that

$$\mathfrak{B}_2 \subset \mathfrak{B}_1^{ctr} + \mathfrak{B}_2^{ctr}. \tag{16}$$

Combining (15) and (16), we get

$$\mathfrak{B}_1 + \mathfrak{B}_2 \subset \mathfrak{B}_1^{\text{ctr}} + \mathfrak{B}_2^{\text{ctr}},$$

which when combined with the trivial inclusion $\mathfrak{B}_1 + \mathfrak{B}_2 \supset \mathfrak{B}_1^{\text{ctr}} + \mathfrak{B}_2^{\text{ctr}}$ yields

$$\mathfrak{B}_1 + \mathfrak{B}_2 = \mathfrak{B}_1^{\text{ctr}} + \mathfrak{B}_2^{\text{ctr}}. \tag{17}$$

If $\mathfrak{B}_2$ is uncontrollable, applying the same procedure as we have done to $\mathfrak{B}_1$, we yield (see (15))

$$\mathfrak{B}_2 = \mathfrak{B}_1^{ctr} + \mathfrak{B}_2^{ctr}. \tag{18}$$

Again, combining (15) and (18) yields (17). Hence by Lemma 3.6, the interconnection is uniformly weakly compatible.

To prove that the converse is not true, consider the following counterexample. Take $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Clearly $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are controllable and, by Lemma 3.6, $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is uniformly weakly compatible. However, $\begin{bmatrix} R_1^T & R_2^T \end{bmatrix}^T$ does not have full row rank. Hence $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is not regular. ∎

Although weak compatibility is weaker than regularity, in a certain aspect they are equivalent. Control problems in behavioral approach can be formulated as follows. Given a plant $\mathcal{P}$ and a specification $\mathcal{S}$, both are behaviors. The problem is to find a controller $\mathcal{C}$ such that $\mathcal{P} \parallel \mathcal{C} = \mathcal{S}$. Moreover, it is also desirable that the interconnection is regular or weakly compatible. This kind of problem was discussed and treated in [1].

One important question related to the control problem is to characterize all specifications that can be achieved, given a plant $\mathcal{P}$. In this sense, regularity and weak compatibility are equivalent.

Theorem 3.9 *Let $\mathcal{P}$ and $\mathcal{S}$ be linear behaviors characterized by*

$$\mathcal{P} := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid P\left(\frac{d}{dt}\right) w = 0 \right\},$$
$$\mathcal{S} := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid S\left(\frac{d}{dt}\right) w = 0 \right\},$$

with $P(\xi) \in \mathbb{R}^{g_p \times q}[\xi]$ and $S(\xi) \in \mathbb{R}^{g_s \times q}[\xi]$. There exists a behavior $\mathcal{C}$,

$$\mathcal{C} := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid C\left(\frac{d}{dt}\right) w = 0 \right\},$$

$C(\xi) \in \mathbb{R}^{g_c \times q}[\xi]$, such that $\mathcal{P} \parallel \mathcal{C} = \mathcal{S}$ and the interconnection is weakly compatible if and only if there also exists a $\mathcal{C}'$ such that $\mathcal{P} \parallel \mathcal{C}' = \mathcal{S}$ and the interconnection is regular.

Proof. ($\Leftarrow$) This is trivial since $\mathcal{P} \parallel \mathcal{C}'$ is also weakly compatible.

($\Rightarrow$) If $\mathcal{P} \parallel \mathcal{C} = \mathcal{S}$ and the interconnection is weakly compatible then necessarily for all $p \in \mathcal{P}$, there exists an $s \in \mathcal{S}$ such that $pD^*(\mathcal{P})s$. Equivalently, this means for every $p \in \mathcal{P}$, there exist an $s \in \mathcal{S}$ and a $p^{\text{ctr}} \in \mathcal{P}^{\text{ctr}}$ such that

$$p = s + p^{\text{ctr}}.$$

$\mathcal{P}^{\text{ctr}}$ is the controllable part of $\mathcal{P}$. Therefore we can have the following relations.

$$\mathcal{S} \subset \mathcal{P}, \tag{19a}$$
$$\mathcal{S} + \mathcal{P}^{\text{ctr}} = \mathcal{P}. \tag{19b}$$

In [1], (19) is proven to be necessary and sufficient conditions for the existence of a $\mathcal{C}'$ such that $\mathcal{P} \parallel \mathcal{C}' = \mathcal{S}$ and the interconnection is regular. ■

4 Remarks

In this paper we discussed the concept of compatibility of general behavior interconnections. The kind of interconnections considered in this paper is total interconnection, where it is assumed that all variables are used in the interconnection. The proposed concept is then compared with the existing concept for linear behaviors, namely regular and regular feedback interconnections. It is shown that for linear behaviors, compatibility and regular feedback are equivalent, while weak compatibility is weaker than regularity.

As already mentioned before, we do not treat the case of partial interconnection. Nevertheless, we would like to give some remarks on this issue. Compatibility of partial interconnections involves more aspect than that of total interconnections. As we have seen in Section 3, compatibility of total interconnection is built on the principle of causality of the interconnections. That is, the interconnection should not affect the behaviors prior to its formation. For partial interconnection there is another key ingredient for the concept of compatibility, namely partitioning of information.

Interconnection of two behaviors can be seen as an exchange of information between them. This exchange of information takes place by means of the variables involved in the interconnection. Partial interconnection means restriction on the access to the information. Generally speaking, we can think of two kinds of information splitting. The first kind is splitting based on observation. In this case, the variables of a behavior is classified into those that are observable/measurable from the environment and those that are not. An example of this kind of splitting is the linear control problem where the state is not measurable and therefore feedback must be done using the output variables. The second kind of splitting is based on manipulation. In this case, the variables of a behavior is classified into those that can be affected by the environment and those that are not. An example of this kind of splitting is the control problem involving disturbance/noise. In this case, it is only sensible to assume that a realizable controller should not affect the disturbance, while the disturbance can be assumed to be measurable[4].

We consider putting these ideas into a formal framework and characterizing the compatibility of partial interconnections as possible directions for future research. In addition, it would also be very interesting to use the concepts developed here in the design process of realizable general controller, as a success would yield a general control theory applicable to a wide range of system classes.

References

[1] M.N. Belur and H.L. Trentelman, *Stabilization, pole placement and regular implementability*, IEEE Trans. Automatic Control **47(5)** (2002), 735–744.

[2] J.W. Polderman and J.C. Willems, *Introduction to mathematical systems theory : A behavioral approach*, Springer, New York, 1998.

[3] P. Rocha and J. Wood, *Trajectory control and interconnection of 1D and nD systems*, SIAM J. Control Optim. **40(1)** (2001), 107–134.

[4] A.J. Van der Schaft, *System theoretic description of physical systems*, CWI Tract No.3, CWI, Amsterdam, 1984.

[5] A.J. Van der Schaft and A.A. Julius, *Achievable behavior by composition*, Proceedings 41st IEEE Conf. Decision and Control (Las Vegas), IEEE, 2002, pp. 7–12.

[6] J.C. Willems, *On interconnections, control and feedback*, IEEE Trans. Automatic Control **42(3)** (1997), 326–339.

[7] ———, *State construction in discrete event and continuous systems*, Proceedings 41st IEEE Conf. Decision Contr. (Las Vegas), IEEE, 2002, pp. 13–15.

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