INTEGRAL ACTION - A DISTURBANCE OBSERVER APPROACH

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Abstract

Integral action is required in many controllers in order to solve real world problems. Modeling errors and disturbances are examples of issues that can be dealt with by using controllers with integral action. This paper explores the use of disturbance observers to introduce integral action into an output feedback controller. In particular, the proposed solution is generalized to include the case of plants where the number of outputs exceeds the number of inputs.

1 Introduction

Integral action is often needed to achieve robustness to modeling errors and disturbance attenuation. In addition, error free tracking of constant reference signals may be achieved by integral action. Integral action may be introduced in several ways. A common approach is to extend the state vector to include integrator states, $\dot{x}_i = r - y$. However, alternatives exist. In this paper, we investigate the use of disturbance observers. In particular, this method is commonly suggested in MPC applications, [4].

In this paper we will show how assumptions on the disturbance model may be used to guarantee integral action in output feedback MIMO controllers. The case of $m \times m$ plants is a straight forward generalization of the SISO case. The main contribution of this paper is the generalization to non-square plants, where the number of outputs exceeds the number of inputs.

Without lack of generality, we could assume that the measured output vector is partitioned as $y = \left[y_z^T y_a^T\right]^T$, where y_z represents the controlled outputs and y_a are the additional measured outputs (if any). We will also assume that the number of inputs of the system equals the number of controlled outputs. The case when the number of inputs equals the number of outputs will be treated first. Thereafter, the controller will be generalized to handle the case when there are additional measured outputs.

2 Square Plants

In this section, a controller with integral action for square plants will be developed. The term *square plant* refers to the fact that the transfer function matrix of a system with an equal number of inputs and outputs is square. We will assume that the plant model is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$. In this case, $y = y_z$ and y_a is not present. (A,B) is assumed to be controllable a pair and (A,C) is assumed to be an observable pair. A standard way of introducing integral action for such systems is described in [5]. In this approach, a constant load disturbance, d, acting on the plant input is assumed. Using an augmented system description, a composite model may be written as

$$\begin{aligned} \dot{x}_e &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x_e + \begin{bmatrix} B \\ 0 \end{bmatrix} u = A_e x_e + B_e u \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} x_e = C_e x_e \end{aligned} \tag{2}$$

where $d \in R^m$ represents the input disturbance and $x_e = \begin{bmatrix} x^T & d^T \end{bmatrix}^T$.

The main idea is to use an observer based on this extended system to estimate the input disturbance d, and to use the disturbance estimation in the control law.

2.1 Observability

Before proceeding, it should be verified that the extended system model is observable. The result is straight forward to derive using the PBH test, and is summarized in the following lemma.

Lemma 1 (Observability) *The system (2) is observable if and only if the system (1) is observable and has no zeros at* $s = 0$.
Proof:
Using the PBH test we obtain the rank condition

$$rank \begin{bmatrix} A - sI & B \\ 0 & -sI \\ C & 0 \end{bmatrix} = n + m$$

If $s \neq 0$ it is readily verified that the matrix has full rank if and only if

$$rank \begin{bmatrix} A - sI \\ C \end{bmatrix} = n.$$

Since (A,C) is assumed to be an observable pair this is always the case. For $s = 0$ we have that

$$rank \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m$$

That is, the system may not have any transmission zeros at s=0, in which case this matrix looses rank.

Remark 1 *The condition that the plant may not have zeros at $s = 0$ is equivalent to that $G(0)$ must be invertible, where $G(s)$ is the transfer function of the plant. This condition is identical to that given for SISO systems in [5]. An interesting observation is that this condition appears as a condition for integral stabilizability (see [1]), where the objective is to stabilize a system using a controller containing integral action. The same condition, but for a more general case, is given in [2].*

2.2 Controller Structure

In order analyze the properties of the controller it is necessary to make some assumptions about the controller structure. We will assume that the control law is given by linear feedback, from the state and disturbance estimations. The observer is assumed to be given by

$$\dot{\hat{x}}_e = A_e \hat{x}_e + B_e u + K(y - C_e \hat{x}_e). \tag{3}$$

where A_e, B_e and C_e are defined according to the augmented system (2). Introducing the feedback law

$$u = -L_x \hat{x} - \hat{d} + L_r r \tag{4}$$

we obtain the following equations for the controller:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{bmatrix} &= \begin{bmatrix} A - BL_x - K_x C & 0 \\ -K_d C & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \\ &+ \begin{bmatrix} K_x \\ K_d \end{bmatrix} y + \begin{bmatrix} BL_r \\ 0 \end{bmatrix} r \\ u &= -L_x \hat{x} - \hat{d} + L_r r. \end{aligned} \tag{5}$$

The feedback and observer gains is assumed to be chosen so that $A - BL_x$ and $A_e - KC_e$ are stable matrices.

This choise of control law, u, has a strong intuitive appeal. It is clear that if $\hat{d}$ is an accurate estimation of d, the effect of the disturbance is canceled. Also, the choice of controller structure is not restrictive in the sense that any design method that yields stable $A - BL_x$ and $A_e - KC_e$ matrices may be used.

Before we proceed, we notice that in stationarity, the following identities hold

$$\begin{aligned} &(A - BL_x)\hat{x} + K_x(y - C\hat{x}) + BL_r r = 0, \quad K_d(y - C\hat{x}) = 0 \\ &\Rightarrow y = -C(A - BL_x)^{-1} BL_r r. \end{aligned}$$

If we choose the gain L_r such that

$$L_r = (C(-A + BL_x)^{-1} B)^{-1} \tag{6}$$

the controller will have a unique equilibrium for $y = r$. This follows from the fact that K_d is invertible (since $A_e - KC_e$ is invertible), which implies that $y - \hat{C}\hat{x} = 0$ in stationarity. Notice also that

$$\text{rank} \begin{bmatrix} A - BL_x & B \\ C & 0 \end{bmatrix} = n + m$$

for any stabilizing L_x. This follows from Lemma 1. But this is equivalent to the matrix $C(A - BL_x)^{-1}B$ having full rank. The inverse in expression (6) thus exists.

It is also clear that the controller has m eigenvalues that are zero, which implies that the controller will have integral action. We shall now investigate in detail this property of the controller.

2.3 Integral Action

In this section an input - output representation of the controller is derived. The aim is to show that the controller contains integral action, which was one of the initial objectives of the design.

Using (5), the control law may be written as

$$\begin{aligned} U(s) = &(I - L_x \Phi(s) B) L_r R(s) - L_x \Phi(s) K_x Y(s) + \\ &+ \frac{1}{s} K_d (I - C\Phi(s) K_x)(C\Psi(s) BL_r R(s) - Y(s)) \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Psi(s) &= (sI - A + BL_x)^{-1} \\ \Phi(s) &= (sI - A + BL_x - K_x C)^{-1} \end{aligned} \tag{8}$$

We assume that the matrix $(-A + BL_x + KC) = \Phi(0)^{-1}$ has full rank. The first term of the controller expression represents feedback and feedforward terms with finite gain. From the assumptions above, it follows that the transfer function $\Psi(s)$ is stable.

The second term in the controller expression represents integral action acting on $r_f - y$, where r_f is the filtered reference signal. The stationary gain of the filter used to obtain r_f is $C\Psi(0)BL_r$, which by design is equal to identity. The integral gain is

$$K_i = K_d (I + C\Psi(0) K_x)^{-1} = K_d (I - C\Phi(0) K_x) \tag{9}$$

so K_i is bounded and has full rank.

2.4 Robustness and Sensitivity

The properties of the controller may also be illustrated by using the concepts of robustness and sensitivity. In practice, there is always a mismatch between the true plant and the plant model. It is of course desirable that modeling error does not severely degrade the control performance. In particular, stability and steady state tracking properties should not be affected.

Let us assume that the true plant has the transfer function $P^0(s)$and that the available model is given by $P(s)$. If we assume an output uncertainty model, we have that

$$P^0(s) = (I + \Delta_P(s))P(s).$$

In summary, the following relation holds:

$$\Delta_Y(s) = S^0(s)\Delta_P(s) \tag{10}$$

where $S^0(s)$ is the sensitivity function of the true system and $Y^0(s) = (I + \Delta_Y(s))Y(s)$, [3]. For small $\Delta_P(s)$, $S^0(s)$ could usually be approximated by $S(s)$. The sensitivity function is readily identified as the transfer function from an output disturbance v to the measured output $y = Cx + v$. The main result result is summarized in the following theorem.

Theorem 1 *Let the system (1) be controlled by (5) and the let the sensitivity function of the closed loop system be $S(s)$. Then*

$$S(0) = 0.$$

Proof:
Introduce the auxillary variable v, representing the ouput disturbance:

$$y = Cx + v.$$

The result is proven by showing that y does not depend on v in stationarity. Using the controller expression (2), we obtain the equations

$$\dot{\hat{x}} = A\hat{x} + Bu + K_x(Cx + v - C\hat{x})$$
$$\dot{\hat{d}} = K_d(Cx + v - C\hat{x})$$

In stationarity we have that $\dot{x} = 0$, $\dot{\hat{x}} = 0$ and $\dot{\hat{d}} = 0$, yielding

$$Cx + v - C\hat{x} = 0 \tag{11}$$

since K_d is invertible. Using the expression (4) we obtain that

$$0 = A\hat{x}_s - BL_x\hat{x}_s + BL_r r_s = (A - BL_x)\hat{x}_s + BL_r r_s$$

where subscript s indicates that the relation holds for constant values of $\hat{x}$ and r. Since $A - BL_x$ is assumed to be a stable matrix, we have that

$$\hat{x}_s = (-A + BL_x)^{-1}BL_r r_s. \tag{12}$$

In particular, due to the choice of L_r, the expressions (11) and (12) it is clear that

$$y_s = Cx_s + v_s = C\hat{x}_s = r_s.$$

Obviously, y is not dependent on v in stationary, and thus $S(0) = 0$.

This result implies that the controller is robust in the sense that the steady state tracking properties are not affected by modeling errors.

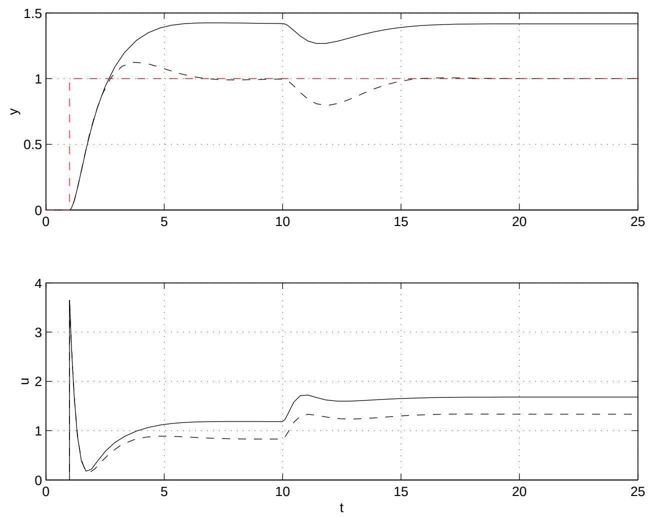

Figure 1: The dashed line represents the case when only the controlled output is measured, whereas the solid line represents the case when both states are measured.

2.5 Example

In Figure 1 the controller described in this section is applied to a simple second order system, given by

$$\dot{x} = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u$$
$$y_z = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$
$$y_a = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

For the true plant, the parameters are $a = -0.2$ and $b = -1.2$, whereas for the model used for control design, the parameters are $a = -0.5$ and $b = -1$. Notice that there is a mismatch between the the real plant and the model used for control design. The plot shows the response for a reference step input at $t = 1$ s and an input step disturbance applied at $t = 10$ s. In the first simulation, represented by the dashed curves, only one measurement is used by the controller, and the scheme works as intended. However, if a controller using both available state measurements is employed, there will be a stationary error. In the first case $S(0) = 0$ as expected, but in the second case we have that

$$S(0) = \begin{bmatrix} 0.28 & 1.39 \\ 0.14 & 0.72 \end{bmatrix}$$

This apparent paradox will be discussed in detail in the next section.

3 Non-Square Plants

Now, suppose that apart from the controlled variables, which are assumed to be measured, there are additional measured signals. We may without lack of generality assume that y is partitioned as $y = [y_z^T y_a^T]^T$, where y_z are the controlled variables and y_a the additional measured outputs. It is reasonable to assume that the number of controlled variables equals the number of inputs. This gives that $y_z \in R^m$ and $y_a \in R^{p_a},\ p_a \geq 1$. Notice that r specifies reference values for y_z, whereas there are no reference values for y_a. Certainly the additional measured variables can be used to estimate the state of the plant. However, it is easily demonstrated that the method presented above for introducing integral action into the controller may fail in such cases. If we assume that both states are available for measurements in the previous example, Figure 1 shows the result. Control performance is significantly degraded - even if more information is available. The controller has integral action, but this is obviously not enough to give the desired steady state tracking properties because of the bias in the additional measurement signals. In the presence of model - plant mismatch, as in the example above, the measured signals are not compatible with the dynamics of the model.

One way to deal with this situation is to determine the confidence associated with the two sets of measured signals y_z and y_a. Since the outputs y_z are controlled, a reasonable assumption is that we are confident in those, i.e. there is no bias in y_z. The fact that we in effect integrate the deviations of y_z from r supports this assumption.

To recover the properties of the controller described in the previous section, the following augmented model is introduced

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{v}_a \\ \dot{d} \end{bmatrix} &= \begin{bmatrix} A & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v_a \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u \\
y_z &= \begin{bmatrix} C_z & 0 & 0 \end{bmatrix} \begin{bmatrix} x^T & v_a^T & d^T \end{bmatrix}^T \\
y_a &= \begin{bmatrix} C_a & I & 0 \end{bmatrix} \begin{bmatrix} x^T & v_a^T & d^T \end{bmatrix}^T
\end{aligned} \tag{13}$$

where $x \in R^n$, $u \in R^m$, $d \in R^m$, $y_z \in R^{p_z}$, $y_a \in R^{p_a}$ and $v_a \in R^{p_a}$. As previously, an input disturbance d is assumed. Also, the disturbance model is augmented to include an output disturbance v_a acting on the additional measured outputs y_a. This assumption reflects the fact that confidence is put in the controlled outputs.

The results derived in the previous section will now be generalized to the modified assumptions stated above.

3.1 Observability

In order to use the model (13) for state and disturbance estimation, it must be verified that the model is indeed observable. The result is summarized in the following lemma.

Lemma 2 (Observability) *The system (13) is observable if the system (1) is observable and has no zeros at $s = 0$.*
Proof:

Using the PBH test we obtain the rank condition

$$rank \begin{bmatrix} A - sI & 0 & 0 \\ 0 & -sI & 0 \\ 0 & 0 & -sI \\ C_z & 0 & 0 \\ C_a & I & 0 \end{bmatrix} = n + m + p_a$$

If $s \neq 0$ it is readily verified that the matrix has full rank if and only if

$$rank \begin{bmatrix} A - sI \\ C_z \\ C_a \end{bmatrix} = n.$$

Since $(A, [C_z^T\ C_a^T]^T)$ is assumed to be an observable pair this is always the case. For $s = 0$ we have that

$$rank \begin{bmatrix} A & 0 & B \\ C_z & 0 & 0 \\ C_a & I & 0 \end{bmatrix} = n + m + p_a$$

But this is equivalent to the matrix

$$rank \begin{bmatrix} A & B \\ C_z & 0 \end{bmatrix} = n + m.$$

having full rank.

Remark 2 *This condition is very similar to the one given in the previous section, the plant may not have zeros at $s = 0$. Notice that the condition applies to the sub plant with input u and output y_z, and does not include the additional outputs y_a. No additional constraints are thus introduced in the presence of extra measurement signals.*

3.2 Controller Structure

We will use the same controller structure as previously. By using the estimator given by equation (3) and the control law (4), the modified controller may now be written as

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{v}}_a \\ \dot{\hat{d}} \end{bmatrix} &= \begin{bmatrix} A - BL_x - K_x C & -K_{xa} & 0 \\ -K_v C & -K_{va} & 0 \\ -K_d C & -K_{da} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{v}_a \\ \hat{d} \end{bmatrix} + \\
&+ \begin{bmatrix} K_{xz} & K_{xa} \\ K_{vz} & K_{va} \\ K_{dz} & K_{da} \end{bmatrix} \begin{bmatrix} y_z \\ y_a \end{bmatrix} + \begin{bmatrix} BL_r \\ 0 \\ 0 \end{bmatrix} r \\
u &= -L_x \hat{x} - \hat{d} + L_r r.
\end{aligned} \tag{14}$$

As previously, we assume that $A - BL_x$ and $A_e - KC_e$ are stable matrices. L_r is chosen as

$$L_r = (C_z(-A + BL_x)^{-1} B)^{-1}$$

The controller then has the unique equilibrium $r = y_z$. The same arguments given previously applies also in this case.

We notice that the order of the controller is now increased compared to the design in the previous section. One extra state for each additional measured signal is introduced. We also notice that the controller has m eigenvalues equal to zero, which implies the presence of integrators.

3.3 Integral Action

As we have seen, the presence of integrators in the controller alone is not sufficient to ensure zero steady state tracking error. In this section we will establish that the controller gives integral action acting on $e_f = r_f - y_{zf}$, where the filters used to obtain r_f and y_{zf} have special properties.

Introducing

$$\Phi_e^{-1}(s) = \begin{bmatrix} sI - A + BL_x + K_xC & K_{xa} \\ K_vC & sI + K_{va} \end{bmatrix}$$
$$C_t = \begin{bmatrix} C_z & 0 \\ C_a & I \end{bmatrix}$$

The following expressions are obtained:

$$\begin{bmatrix} \hat{X}(s) \\ \hat{V}_a(s) \end{bmatrix} = \Phi_e(s)\left(\begin{bmatrix} K_x \\ K_v \end{bmatrix} Y(s) + \begin{bmatrix} BL_r \\ 0 \end{bmatrix} R(s)\right)$$
$$\hat{D}(s) = \frac{1}{s}K_d\left(Y(s) - C_t\begin{bmatrix} \hat{X}(s) \\ \hat{V}_a(s) \end{bmatrix}\right)$$

The controller may then be written as

$$\begin{aligned} U(s) &= -L_x\hat{X}(s) - \hat{D}(s) + L_rR(s) \\ &= \left(I - \begin{bmatrix} L_x & 0 \end{bmatrix}\Phi_e(s)\begin{bmatrix} B \\ 0 \end{bmatrix}\right)L_rR(s) \\ &- \begin{bmatrix} L_x & 0 \end{bmatrix}\Phi_e(s)\begin{bmatrix} K_x \\ K_v \end{bmatrix}Y(s) \\ &+ \frac{1}{s}K_d\left(I - C_t\Phi_e(0)\begin{bmatrix} K_x \\ K_v \end{bmatrix}\right)(C\Psi(s)BL_rR(s) - Y(s)) \\ &= M_0(s)R(s) - \begin{bmatrix} M_1(s) & M_2(s) \end{bmatrix}\begin{bmatrix} Y_z(s) \\ Y_a(s) \end{bmatrix} + \\ &+ \frac{1}{s}K_d\begin{bmatrix} M_3(s) & M_4(s) \end{bmatrix}\begin{bmatrix} C_z\Psi(s)BL_rR(s) - Y_z(s) \\ C_a\Psi(s)BL_rR(s) - Y_a(s) \end{bmatrix} \end{aligned}$$

The definitions of $\Psi(s)$ and $\Phi(s)$ are given by (8). We also make the assumption that the matrix $\Phi_e(0)$ has full rank and that $\Psi(0)$ is stable. It then follows that the matrices $M_0(0)$, $M_1(0)$ and $M_2(0)$ represents finite feedback and feedforward gains.

Now, the aim of the controller is somewhat more elaborate than in the previous case; integral action is desired to act on $r_f - y_{zf}$. In order for the proposed controller to achieve this, it must be verified that $M_4(0) = 0$ and that $M_3(0)$, the integral gain, has full rank. We start by verifying that $M_4(0) = 0$. By direct calculation we obtain:

$$\begin{aligned} M_4(0) &= \left(I - C_t\Phi_e(0)\begin{bmatrix} K_x \\ K_v \end{bmatrix}\right)\begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \left(\begin{bmatrix} 0 \\ I \end{bmatrix} - C_t\begin{bmatrix} -A + BL_x + K_xC & K_{xa} \\ -K_vC & K_{va} \end{bmatrix}^{-1}\begin{bmatrix} K_{xa} \\ K_{va} \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 \\ I \end{bmatrix} - C_t\begin{bmatrix} 0 \\ I \end{bmatrix}\right) = 0 \end{aligned}$$

where we have used the identity

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{-1}\begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

As for $K_i = K_dM_3(0)$ we have

$$\begin{aligned} K_i &= K_d\left(I - C_t\Phi_e(0)\begin{bmatrix} K_x \\ K_v \end{bmatrix}\right)\begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \left(K_{dz} - \begin{bmatrix} K_dC & K_{da} \end{bmatrix}\Phi_e(0)\begin{bmatrix} K_{xz} \\ K_{vz} \end{bmatrix}\right). \end{aligned} \tag{15}$$

But this expression is recognized as the Schur-complement of $\Phi_e^{-1}(0)$ with respect to the matrix

$$\begin{bmatrix} -A + BL_x + K_xC & K_{xa} & K_{xz} \\ K_vC & K_{va} & K_{vz} \\ K_dC & K_{da} & K_{dz} \end{bmatrix}. \tag{16}$$

It is clear that the matrix (15) having full rank is equivalent to (16) having full rank. Now, let us rearrange the elements of this matrix by permuting the rows and columns, an operation that preserves the rank of the matrix

$$\begin{bmatrix} -A + BL_x + K_xC & K_{xz} & K_{xa} \\ K_dC & K_{dz} & K_{da} \\ K_vC & K_{vz} & K_{va} \end{bmatrix}. \tag{17}$$

If we take the Schur-complement of

$$\begin{bmatrix} K_{dz} & K_{da} \\ K_{vz} & K_{va} \end{bmatrix}$$

with respect to (17) we obtain

$$\Phi(0) - \begin{bmatrix} K_{xz} & K_{xa} \end{bmatrix}\begin{bmatrix} K_d \\ K_v \end{bmatrix}^{-1}\begin{bmatrix} K_d \\ K_v \end{bmatrix}C = \Psi(0).$$

The invertability of $[K_d^T \; K_v^T]^T$ follows from the matrix $A_e - KC_e$ having full rank. Further, $\Psi(0)$ is assumed to have full rank by design, and we can conclude that the matrix (15) has indeed full rank.

We have now shown that the integral action property of the controller is recovered. As in the previous case, we have integral action acting on

$$C_z\Psi(s)BL_rR(s) - Y_z(s).$$

Due to the choice of L_r, this gives integral action acting on $r - y_z$ in stationarity. The integral gain, K_i, is given by (15).

3.4 Robustness and Sensitivity

Let us now investigate the robustness properties of the controller. The critical feature of the controller is that $r = y_z$ also in in the presence of modeling errors. To establish this, we use the relation

$$\Delta_Y(s) = S(s)\Delta_P(s).$$

However, now we are concerned with the controlled outputs y_z, and not all of the measured outputs as previously. The additional measured outputs are not controlled, and are likely to be influenced by disturbances as well as modeling errors. For this reason we use

$$\Delta_{Y_z}(s) = S_z(s)\Delta_P(s).$$

That is, only deviations of the controlled outputs from the nominal case $r = y_z$ are considered. In this case $S_z(s)$ consists of the first m rows of the sensitivity function matrix $S(s)$. It follows that $S_z(s)$ is the transfer function form v to y_z. In particular we would like to show that $S_z(0)$ is zero, which would imply that the controller is robust to modeling errors in steady state. We have the following result:

Theorem 2 *Let the system (1) be controlled by (14), and let the sensitivity function of the closed loop system be* $S(s) = \begin{bmatrix} S_z(s)^T & S_a(s)^T \end{bmatrix}^T$. *Then*

$$S_z(0) = 0.$$

Proof:
Introduce the auxillary variables v_a and v_z, representing the output disturbances:

$$y_z = C_z x + v_z, \quad y_a = C_a x + v_a$$

The same technique as in theorem (1) is used; it will be shown that y_z does not depend on v_z or v_a in stationarity. The equations for the controller may be written as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_{xz}(C_z x + v_z - C_z\hat{x}) + \\ &+ K_{xa}(C_a x + v_a - C_a\hat{x} - \hat{v}_a) \\ \dot{\hat{v}}_a &= K_{vz}(C_z x + v_z - C_z\hat{x}) + K_{va}(C_a x + v_a - C_a\hat{x} - \hat{v}_a) \\ \dot{\hat{d}} &= K_{dz}(C_z x + v_z - C_z\hat{x}) + K_{da}(C_a x + v_a - C_a\hat{x} - \hat{v}_a) \end{aligned}$$

In stationarity we have that $\dot{x} = 0$, $\dot{\hat{x}} = 0$ and $\dot{\hat{d}} = 0$ which yields

$$C_z x + v_z - C_z\hat{x} = 0, \quad C_a x + v_a - C_a\hat{x} - \hat{v}_a = 0 \tag{18}$$

since $\left[K_d^T K_v^T\right]^T$ is invertible. Using the expression (4) we obtain that

$$0 = A\hat{x}_s - BL_x\hat{x}_s + BL_r r_s = (A - BL_x)\hat{x}_s + BL_r r_s$$

where subscript s indicates that this relation holds in stationarity, as before. Since $A - BL_x$ is assumed to be a stable matrix, we have that

$$\hat{x}_s = (-A + BL_x)^{-1} BL_r r_s. \tag{19}$$

In particular, due to the choice of L_r, the expressions (18) and (19) it is clear that

$$y_{zs} = C_z x_s + v_{zs} = C_z\hat{x}_s = r_s.$$

Obviously, y_z is not dependent on v in stationary, which implies that $S_z(0) = 0$.

This result proves that the robustness property described in the previous section is recovered.

3.5 Example

In Figure 2, the improved controller structure is simulated, using the same system as in the previous section. The system has one input, and two measured outputs, of which one is controlled. There is also a modeling error present. As we can see, the improved controller preforms well, and achieves zero steady state tracking error. In addition,

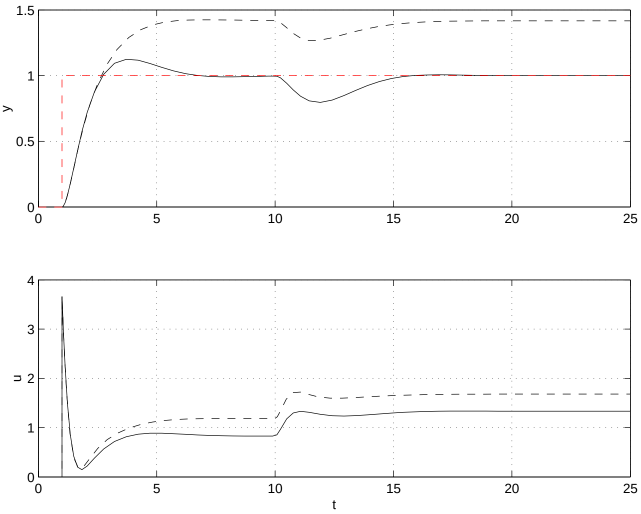

Figure 2: Responses of the controllers derived in Section 2 (dashed) and section 3 (solid).

$$S(0) = \begin{bmatrix} 0.00 & 0.00 \\ 0.20 & 1.00 \end{bmatrix}$$

and in particular $S_z(0) = 0$.

4 Conclusions

In this paper the design of centralized controllers including integral action has been discussed. The proposed solution explores disturbance observers of a certain structure. The case of square plants, including SISO plants, has been generalized to the case of plants with additional measured signals.

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