ROBUST OPTIMAL CONTROL OF ONE-REACH OPEN-CHANNELS

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Abstract

This paper is concerned with the regulation problem of open-channel hydraulic systems. More precisely the water flow and level within the reach are controlled trough the opening rates of two gates localized at each side of the reach. The hydrodynamics of such a system is governed by the non linear Saint-Venant partial differential equations. In the first part of the paper a reduction method which leads to a linear finite dimensional approximation model is shown to be convenient for control purposes. Then an original solution is proposed for the optimal regulation problem. This solution is designed to be robust to the approximation error. Finally, the existence of two very different time-scales in the problem is used to decouple the regulation of the reach and the regulations of the gates. The method is applied to a full non linear partial differential equations model.

1 Introduction

Regulation of irrigation channels has received an increasing interest over the last two or three decades. In Europe and in North-America, a lot of interconnected irrigation networks are already observed and controlled by a distant human operator via communication systems. However, water is becoming a rare and more expensive resource. This has generated the need for fully automatized regulation systems which would be able to minimize the water consumption and supply a time-varying demand "online".

Generally, an irrigation network is a made of a primary open-air canal which deserve secondary canals and/or pressurized network of water distribution which are connected to the primary canal. Canals themselves are made of several long reaches (most of the times, they are several kilometres long) separated by engineering works (like sliding gates for instance). This open-channel hydraulic part is the most complex one. Its dynamical behaviour is characterized by important time lags (due to water transport), wave superposition effects and strong nonlinearity (mainly around the works).
In this paper we will focus on the regulation of a single reach of such an open-channel hydraulic system. Usually the reach dynamics is modelled by a set of non linear hyperbolic partial differential equations : the so-called Saint-Venant equations. These equations are derived from the mass and kinetic momentum balances in an infinitesimal length of the reach.
Assuming a one-dimensional laminar flow of an incompressible and homogeneous fluid (the water) these balances may be written (see [1]) :

$$\begin{aligned}
&\frac{\partial S(x,t)}{\partial t}+\frac{\partial Q(x,t)}{\partial x}=0\\
&\frac{\partial Q(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2(x,t)}{S(x,t)}\right)+gS(x,t)\frac{\partial h(x,t)}{\partial x}\\
&-gS(x,t)\big(I-J(Q(x,t),S(x,t))\big)=0
\end{aligned}\tag{1}$$

where x denotes the reach length coordinate, t the time variable, $S(x,t)$ the wetted cross section, Q the water flow, g the gravity acceleration, h the water level, I the canal slope and J a friction term depending on the water flow and of the wet perimeter (see figure 1). The water cross section depends on the water level (for instance, $S=Bh$ in the case of the rectangular cross section in figure 1). Usually the friction slope is given by the Manning-Strickler formulae :

$$J(Q,S)=\frac{Q|Q|}{D^2}\tag{2}$$

with $D = KS(S/P)^{2/3}$ where P is the wetted perimeter ($P = B+2h$ in the case of the rectangular cross section in figure 1) and K is the Manning-Strickler coefficient which reflects the roughness of the walls and the viscosity of the fluid.

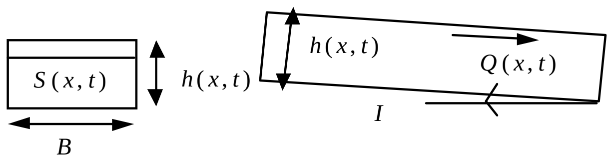

Figure 1 : *parameters and variables for the state equations of a single reach in an open-channel networks.*

Usually, controlled inputs are in the boundary conditions at the two ends of each reach. For instance, with two gates boundary conditions may be written (see [1]) :

$$Q(x=0,t)=C\alpha_1\sqrt{2g(h_{up}(t)-h(x=0,t))}$$
$$Q(x=L,t)=C\alpha_2\sqrt{2g(h(x=L,t)-h_{down}(t))} \tag{3}$$

where L is the reach length, C a coefficient depending on the gate characteristics, α_i the opening rate of the gate i (see figure 2), h_{up} is the upstream water level and h_{down} the downstream one. With such boundary condition, the natural controlled inputs are the two opening rates α_i.

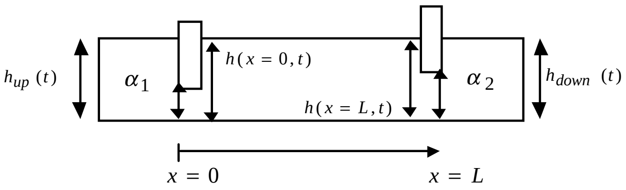

Figure 2 : *classical control of an open-channel reach with the opening rates α_i of two sliding gates.*

These two opening rates are usually realised via an endless screw and an electric engine for which the dynamics is far faster than the dynamics of the water in the reach. Hence these dynamics of the gates are frequently neglected and the controlled inputs may be directly chosen as the values of the flow and/or the water level at each end of the reach. In this paper we will consider that the two controlled variables are :

$$\dot{h}(x=L,t)=u_1(t)$$
$$\dot{Q}(x=0,t)=u_2(t) \tag{5}$$

where the time derivatives are introduced in order to reject constant perturbations. The boundary control problem (1)-(4) has been the object of a wide range of control studies. Some of them used classical finite dimensional techniques (LQ/LQG, GPC, pole placement, PI, etc.) but applied on simplified and at least spatially discretised models. Other works applied to the PDE model (semigroup approach [2]-[3], generalised Lyapunov approach [4] or infinite-dimensional optimisation approach [5]). In this paper, the control law is designed to be robust to the reduction error, that is the additive model error between the PDE model and the discretized finite-dimensional model which is used to design the optimal regulator.

In section two, the linearization of the PDE model, its transformation to an adimensional form and its reduction to a finite dimension state-space realization are performed. In section three, the reduction error on the transfer matrix is computed and robustness constraints are derived from the singular values of this error matrix. In section four a frequency-shaped LQ control approach is applied to compute an optimal regulator which satisfies the robustness constraints. Finally, in section five, this control solution is coupled with a Kalman filter used to estimate the state from the measured upstream water level and downstream water flow. The whole control structure is applied to operate the full nonlinear partial differential equations model.

2 Finite-dimensional linear approximation using orthogonal collocation

In this section, a finite-dimensional linear approximation of the boundary control problem (1)-(4) is derived. Let us first consider the linearization of the Saint-Venant equations. Equations (1) show that there exists a uniform steady-state solution

$$\begin{cases} Q_e(x)=Q_e \\ h_e(x)=h_e \end{cases} \quad \forall x\in[0,L] \tag{4}$$

if and only if the condition $J(Q_e,h_e) = I$ holds. In this paper, we will consider that purpose of the control problem is to reach a water flow Q_e with the uniform water level profile h_e such that this last condition holds. With the dimensionless state variables

$$\hat{h}=\frac{h-h_e}{h_e} \quad , \quad \hat{Q}=\frac{Q-Q_e}{Q_e} \tag{5}$$

the linearized Saint-Venant equations may be written

$$\begin{cases} \dfrac{\partial\hat{h}}{\partial t} = -\dfrac{\partial\hat{Q}}{\partial x} \\ \dfrac{\partial\hat{Q}}{\partial t} = a_1\dfrac{\partial\hat{h}}{\partial x} - a_2\dfrac{\partial\hat{Q}}{\partial x} + a_3\hat{h} - a_4\hat{Q} \end{cases} \tag{6}$$

where

$$a_1=(1-\frac{gB^2h_e^3}{Q_e^2}) \;\; ; \;\; a_2=2 \;\; ; \;\; a_3=g\frac{B^2h^2L}{Q_e^2}(I+J_e+\frac{4J_eR_e}{3h_e}) \;\; ; \tag{7}$$
$$a_4=\frac{2gB^2h_e^5J_eL}{Q_e^2} \;\; ; \;\; R_e=\frac{Bh_e}{B+2h_e}$$

As it has been shown in [6] this linearized partial differential equations model is sufficiently accurate at least for control purposes, even if the water flow and level profiles are far from the steady-states ones. The main non-linearities are localised around the cross-structure but the reach dynamics itself is essentially linear.

An implicit finite difference scheme called the Preismann scheme (see [7]) is usually applied by the hydraulics specialists to perform the numerical integration the Saint-Venant equations. We will use this unconditionally stable algorithm to compute our “reference solution” but the control law will be derived in the sequel from a reduced model obtained from the orthogonal collocation method (see [8]). The aim of this last method is to get an approximation of the PDE solution realized as the solution of a system of ordinary differential equations via a discretization of the space variables only. It has already been successfully applied to the nonlinear model (1) (see [9]) and will be used hereafter to get a linear finite-dimensional approximation of the reach dynamics.

Let us consider N collocation points x_i along the reactor length chosen to be zeros of a $N^{th.}$ order orthogonal polynomials defined on $[0,L]$. The orthogonal collocation method consists in searching an approximated solution by a separation of variables principle in the form of a space interpolation at the collocation points x_i with time-varying coefficients, that is :

$$Q_a(x,t)=\sum_{j=1}^{N} Q_j(t)L_j(x)\ ;\ h_a(x,t)=\sum_{j=1}^{N} h_j(t)L_j(x) \tag{8}$$

where

$$L_j(x) := \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{x-x_k}{x_j-x_k} \tag{9}$$

are the $(N\text{-}1)^{th.}$ order Lagrange interpolation polynomials[1]. Replacing the water flow and level in equations (6) by their approximation defined in (8) gives rise to a linear finite dimensional system of the form :

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{10}$$

where the state vector

$$x(t)=\left(h_1(t),h_2(t),......h_N(t),Q_1(t),Q_2(t),.....Q_N(t)\right)^T \tag{11}$$

is now made of the values of the water level and flow at the collocation points. The boundary conditions (4) define directly the B matrix since

$$\frac{\partial}{\partial t}\begin{bmatrix} \hat{h}(x=L,t) \\ \hat{Q}(x=0,t) \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} h_N(t) \\ Q_1(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{12}$$

and the measured output of the system are chosen to be the complementary variables :

$$y=\begin{bmatrix} h_1(t) \\ Q_N(t) \end{bmatrix} \tag{13}$$

This collocation method has been applied to integrate the model of an experimental micro-canal in which the following values for the coefficients has been used : $L = 8$ [m], $B = 0.1$ [m], $I = 2\cdot10^{-3}$, $Q_e = 4.1\cdot10^{-3}$ [m^3s^{-1}], $h_e = 0.9\cdot10^{-1}$ [m], $K = 100$. In figure 3, a simulation run is performed for the three models : the non-linear Saint-Venant equations, the linearized PDEs and a reduced linear collocation model with $N = 5$ points. The initial water level and flow profile are chosen to be uniform with values $Q(0,x) = 1.8\cdot10^{-3}$ [m^3s^{-1}], $h(0,x) = 0.4\cdot10^{-1}$ [m] for all x in $[0,L]$. As an illustration, the water flow at the three interior collocation points computed with the three models are plotted and shown to be very similar. This is also the case for water levels and flows at any point taken along the reactor length.

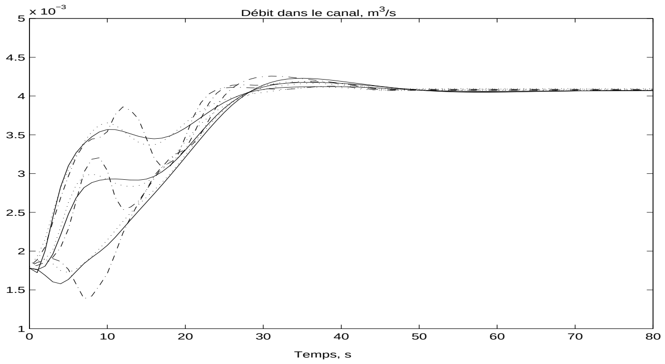

Figure 3 *: the water flow at the three interior collocation points computed with the orthogonal collocation method (continuous line), the Preismann scheme applied to the non-linear Saint-Venant equations (dashed lined) and the Preismann scheme applied to the linearized model (dash-dotted line).*

[1] If the interpolation points x_i are zeros of a $N^{th.}$ order orthogonal polynomials defined on $[0,L]$, the the corresponding Lagrange interpolation polynomials are orthogonal themselves.

3 Reduction error and robustness constraints

Usually, classical control approaches a applied to finite-dimensional approximations of PDEs models without any concern to the numerical errors occurring in the reduction step. However, in this case we point out that the transfer matrix of the linearized Saint-Venant model may be computed as the solution of the following ordinary differential boundary value problem :

$$\begin{aligned} s\hat{h}(x,s) &= -\frac{\partial \hat{Q}(x,s)}{\partial x} \\ s\hat{Q}(x,s) &= a_1\frac{\partial \hat{h}(x,s)}{\partial x} - a_2\frac{\partial \hat{Q}(x,s)}{\partial x} + a_3\hat{h}(x,s) - a_4\hat{Q}(x,s) \\ s\hat{h}(L,s) &= \hat{u}_1(s); s\hat{Q}(0,s)=\hat{u}_1(s); \\ \hat{y}_1(s) &= \hat{h}(0,s); \hat{y}_2(s)=\hat{Q}(L,s) \end{aligned} \tag{14}$$

where s denotes the Laplace variables and the "hat" over a function denotes its Laplace transform. The solution of operational equations (14) may be explicitly computed and written in the form :

$$y(s)=\tilde{G}(s)u(s)\,;\tilde{G}(s)=\begin{bmatrix} \tilde{G}_{11}(s) & \tilde{G}_{12}(s) \\ \tilde{G}_{21}(s) & \tilde{G}_{22}(s) \end{bmatrix} \tag{15}$$

where

$$\begin{aligned} &\tilde{G}_{11}(s)=\frac{\lambda_1-\lambda_2}{s(\lambda_2 e^{\lambda_2}-\lambda_1 e^{\lambda_1})},\tilde{G}_{12}(s)=\frac{\lambda_1\lambda_2(e^{\lambda_1}-e^{\lambda_2})}{s^2(\lambda_2 e^{\lambda_2}-\lambda_1 e^{\lambda_1})}, \\ &\tilde{G}_{21}(s)=\frac{e^{\lambda_1}-e^{\lambda_2}}{(\lambda_2 e^{\lambda_2}-\lambda_1 e^{\lambda_1})},\tilde{G}_{22}(s)=\frac{\lambda_1\lambda_2(e^{\lambda_1}-e^{\lambda_2})^2}{s(\lambda_1-\lambda_2)(\lambda_2 e^{\lambda_2}-\lambda_1 e^{\lambda_1})}+\frac{\lambda_2 e^{\lambda_1}-\lambda_1 e^{\lambda_2}}{s(\lambda_1-\lambda_2)} \end{aligned} \tag{16}$$

with

$$\Lambda=\left(\frac{a_2 s+a_3}{a_1}\right)^2-4\frac{s(s+a_4)}{a_1},\lambda_1=\frac{-(a_2 s+a_3)+a_1\sqrt{\Lambda}}{2a_1},\lambda_2=\frac{-(a_2 s+a_3)-a_1\sqrt{\Lambda}}{2a_1} \tag{17}$$

It is then possible to compute the reduction error between the linearized PDEs model and the finite dimensional linear model obtained by the collocation method expressed in term of frequency response, that is :

$$\begin{aligned} \tilde{G}(j\omega) &= (I+\Delta_s(j\omega))G(j\omega) \\ \tilde{G}(jw) &= (I+\Delta_{is}(jw))^{-1}G(jw) \end{aligned} \tag{18}$$

where

$$G(s)=C(sI-A)^{-1}B \tag{19}$$

is the transfer matrix of the reduced model computed with the collocation method (see equations 10) ; ω denotes the pulsation ; $\Delta_s(j\omega)$ and $\Delta_{is}(j\omega)$ are the output reduction error written respectively in direct and inverse multiplicative forms. It seems then natural to look for a closed loop finite-dimensional controller K, designed on the finite-dimensional model $G(s)$ which is robust to the model errors $\Delta_s(j\omega)$ and $\Delta_{is}(j\omega)$. Defining the sensibility maps

$$S := (I+GK)^{-1}\ ,\ T := (I+GK)^{-1}GK \tag{20}$$

it may be found (see [10], [11]) that robustness conditions in stability and performances may be written in the form

$$\begin{cases} \bar{\sigma}(T(j\omega)) \prec \dfrac{1}{\bar{\sigma}(\Delta_s(j\omega))} & \forall \omega \in IR \\ \bar{\sigma}(S(j\omega)) \prec \dfrac{1}{\bar{\sigma}(\Delta_{is}(j\omega))} & \forall \omega \in IR \end{cases} \quad (21)$$

where $\bar{\sigma}$ denotes the maximum singular value of a given matrix. Hence the actual problem is to find out a controller K which makes the sensitivity matrices S and T as small as possible in order to guarantee stability and performance in spite of the reduction error. In order to do so, we will use a classical loop shaping approach (see [12]). The main idea is to minimize the sensibility matrix S in low frequencies since the system is globally low-pass and to minimize the sensility matrix T in high frequencies since the model reduction error is high essentially in a high frequencies domain (see figure 4 hereafter). Considering the following approximations

$$\begin{aligned} \bar{\sigma}(T(j\omega)) &\cong \bar{\sigma}(GK(j\omega)) & \forall \omega > \omega_h \\ \bar{\sigma}(S(j\omega)) &\cong \underline{\sigma}(GK(jw)) & \forall \omega < \omega_l \end{aligned} \quad (22)$$

where ω_l and ω_h denotes respectively low and high pulsations/frequencies domain boundaries, this leads to the following inequalities

$$\bar{\sigma}(GK(j\omega)) < \frac{1}{\bar{\sigma}(\Delta_s(j\omega))}, \forall \omega > \omega_h \quad \text{and} \quad \bar{\sigma}(\Delta_{is}(j\omega)) < \underline{\sigma}(GK(j\omega)), \forall \omega < \omega_l \quad (23)$$

Let us point out that if these last conditions hold there is still a frequency range corresponding to the pulsations $[\omega_l,\omega_h]$ where robustness conditions in stability or performance are note satisfied. However, the robustness constraints in stability and performance are now expressed in terms of the singular values of the open loop transfer GK ,.and are therefore far easier to handle than the constraints on the sensibility matrices. Practically the range $[\omega_l,\omega_h]$ will be made as small as possible and eventually reduced to a single point.

4 Design of an optimal controller

In order to solve our optimal control problem (stabilisation around a uniform water flow and level profiles) with the robustness constraints (23), we have used an approach based on the linear quadratic optimisation problem with frequency dependant weights on the quadratic cost functional (see [13], [14]). Frequency depending weights are equivalent to the definition of a filter $R(s)$ on the inputs and another filter $P(s)$ on the outputs. These two filters are defined as

$$R(s) = \frac{\tau_R}{1+\tau_R s} I_2 ; P(s) = k_P \frac{1+\tau_P s}{1+10\tau_P s} I_2 \quad (24)$$

where I_2 is the real 2×2 identity matrix. This choice is such that the weight $R(s)$ allows a fast decreasing of the transfer singular values at high frequencies (in order to improve robustness to the error and better noise filtering) and such that the weight $P(s)$ increases the transfer singular values at low frequencies (allowing a better performance at these low frequencies).

It is now possible to find out an optimal regulator K which realize the optimal inputs as a solution of a Riccatti equation where the weights are frequency dependant. These weights are realized as the two filters R and P chosen such that the robustness constraints (23) are satisfied. In figure 4 hereafter are plotted the maximum and minimum singular values of the open loop transfer matrix GK obtained in such a way. They are compared to singular values of the two model multiplicative errors and to the singular values of the two sensibility matrices in order to check that the robustness constraints (23) are indeed satisfied.

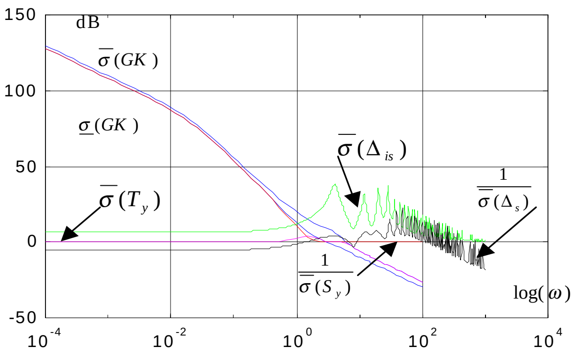

Figure 4 *: robustness constraints express in terms of singular values of the open loop transfer matrix GK, are satisfied. Chosen filters parameters values are* $k_R=1$, $\tau_R=50$, $k_P=300$, $\tau_P=.01$.

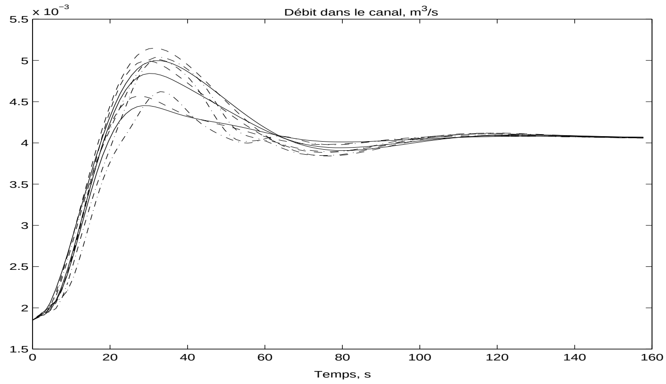

Figure 5 *: the controller computed with the proposed approach is applied to the three different models : the collocation linear model (continuous line), the linearized PDEs model (dash-dot line) and the Saint-Venant nonlinear equations (dashed line). The water flow at the three interior collocation points are plotted.*

This controller has been applied to the finite-dimensional model for which it is optimal. Obviously stabilization around the prescribed steady-state uniform water flow and level occurs. The same controller has been applied to the linearized partial differential equations model in order to check that it is indeed robust to reduction errors. Finally, it has to be pointed out that the controller stabilizes also the non linear Saint-Venant models. This is not so surprising since the computed controller has to be robust to a range of perturbations and not only to perturbations due to the reduction step. Results from the application of the state feedback to the three different models are plotted in figure 5.

Finally it has to be noticed that the state of the reduced model are the water flow and level at the different collocation points. Usually only the water flow and level around the cross-structure are actually measured. In our case the measured outputs are the water level at the entrance of the reach and the water flow at the bottom of the reach. Hence to apply the computed static state feedback, the water levels and flows at the collocation points have to be estimated. The approach proposed in [15] has been applied to design an observer which take into account the robustness constraints. Figure 6 shows the convergence of the estimator and the stabilisation of the water flow around the steady-state value in the case of a closed loop regulator based on the estimated state.

5 Concluding remarks

In this paper a method has been proposed to design classical finite-dimensional controllers to achieve optimal control of linear distributed parameters systems (governed by partial differential state equations). The main idea is to approximate the partial differential equations by a set of ordinary differential equations (reduction of the model) using a pseudo-spectral method (here the orthogonal collocation method) and then to design a controller which is robust to the reduction error. A classical loop-shaping approach may be used to achieve the robustness constraints. It results in the design of two simple first order filters which have to be added to the optimal controller structure.

The method has been applied to the problem of controlling the water flow in an open channel reach modelled by the partial differential Saint-Venant equations. It has been shown to be very efficient even in the presence of a state estimator and non-linear perturbations (due to the linearization step). Generally speaking, authors are convinced that once the reduction procedure has been performed in order to keep the dynamics properties of the original distributed parameters model, and if the controller is designed to be robust to reduction errors, the resulting controller is easier to compute and more efficient than controllers based on an infinite-dimensional approach.

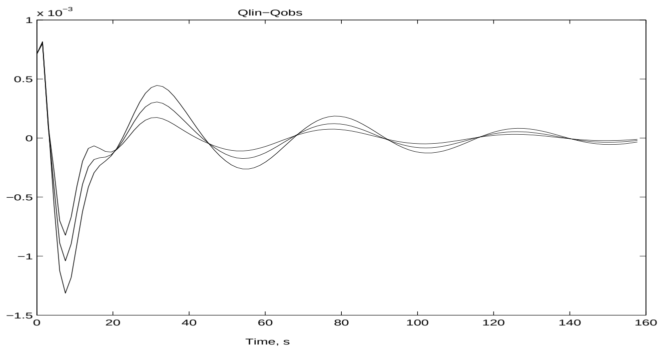

Figure 6 *: estimation errors on the water flows at the three interior collocation points. Both "real" and estimated states are converging to the prescribed uniform steady-state value.*

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