OPTIMAL COST CONVERGENCE WITH RESPECT TO THE TIME HORIZON

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Abstract

The asymptotic behaviour of the optimal cost for problems with increasing time horizon is studied. The dynamics and costs are general nonlinear, possibly with state and control constraints. Apart from basic consistency assumptions, a uniform detectability hypothesis provides the setup for the analysis, which is based on direct evaluations of bounds for costs and trajectories. This investigation is motivated by approximations to infinite horizon problems that would require large amount of computational effort, and by the stability study of receding horizon control problems.

1 Introduction

Consider the sets $\mathcal{Z} \in \mathbb{R}^n$ and $\mathcal{U} \in \mathbb{R}^m$ named state and control spaces respectively, and some measurable functions $f : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}^n$ and $c : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}_+$. Let $\bar{d} : \mathbb{R}^0 \to \mathbb{R}^0$ be a monotonically increasing function with $\bar{d}(0) = 0$, and let $d(z) := \bar{d}(|z|)$, where $|\cdot|$ is a norm in $\mathbb{R}^n$.

Let us introduce the discrete-time nonlinear system

$$\Psi : \quad z_{t+1} = f(z_t, u_t), \quad t \geq 0,\ z_0 = z, \tag{1}$$

where $z_t \in \mathcal{Z}$, $\forall t \geq 0$ is the state trajectory and $u_t \in \mathcal{U}$, $\forall t \geq 0$ is the control sequence. Consider also the cost functional

$$J_u^N(z) = \sum_{t=0}^{N-1} c(z_t, u_t), \tag{2}$$

and the optimal cost

$$J^N(z) = \min_u J_u^N(z), \tag{3}$$

defined whenever $z_0 = z$.

Under general assumptions we derive conditions in this paper for uniform convergence of the optimal cost functional, that is, we find conditions that assure, for each $\sigma > 0$, the existence of $N_0 \geq 0$ such that

$$|J^M(z) - J^N(z)| \leq \sigma d(z), \quad \forall M, N \geq N_0.$$

There are many motivations in the control theory to study the optimal cost convergence as described. For example, one important question is how an infinite horizon problem can be reasonably well approximated by the solution of a finite horizon problem? To solve the nonlinear control problem in (1), (2) and (3) with $N \to \infty$, one has to rely in principle on the dynamic programming or variational techniques, which are feasible for finite horizon problems only. Provided that the above convergence sense is verified, the answer is positive, and the infinite horizon solution can be pursued by means of approximating finite horizon solutions.

A similar motivation arises in problems with large dimensions for which it is desirable to solve the dynamic programming for a certain number of steps only because of costly computation. Nonetheless, one requires certain interesting properties of such a sub-optimal solution, stability being the most sought. This property is however, more akin to infinite horizon problems and in most cases, stability is the initial motivation to the control problem, more important than the search for an optimal control. In this connection, see [8] for an approach in a Markov decision problem.

In a third context, the asymptotic behaviour allow us to handle the stability problem for the class of control problems known as *receding* or *moving horizon* problems that is closely connected with *model predictive control* (MPC) involving a finite horizon cost such as (2). It is possible to employ the optimal cost J^N or a modification of it to act as a Lyapunov function in various interesting control settings. The approach here is based on the following simple evaluation.

Assume for sake of argument that $c(z_k, \cdot) \geq \gamma d(z_k)$; this assumption is not essential but it makes this outline simpler. One can write:

$$J^N(z_k) = c(z_k, u^*(k)) + J^{N-1}(z_{k+1}) \tag{4}$$

which leads to

$$J^N(z_k) - J^N(z_{k+1}) = c(z_k, u^*(k)) + [J^{N-1}(z_{k+1}) - J^N(z_{k+1})]. \tag{5}$$

In order to evaluate the second term on the right hand side of (5), we identify the predicted trajectory $x_{t|k}$, $t \geq k$ with the trajectory z of (1). Then we employ the results of this paper to obtain conditions for the second term to be larger than $-\gamma d(z_k)$, in such a manner that one obtains $J^N(z_k) - J^N(z_{k+1}) > 0$ and the cost functional J serves as a Lyapunov function. The approach just described has been explored in [5] and [9] in different contexts and assumptions. Notice that the control formulation in (1), (2) and (3) is general enough to comprise problems with constrained state and output variables, covering much of the field of recent studies, e.g. see [1, 2, 4, 5, 6, 7, 9].

The main assumptions of the paper are basically general consistency hypothesis related to the finiteness of the cost and cost and drift at the equilibrium (chosen the origin as convention), combined with an uniform detectability assumption.

Section 2 introduces the working assumptions and remarks on them. An outline of the arguments employed to show the asymptotic behaviour of the cost also appears in Section 2. Section 3 concentrates on the technical results leading to the main result, which is presented in Theorem 1.

2 Uniform Convergence of the Cost Functional

Hereafter we consider the following assumptions.

Assumption 1 (equilibrium at origin). $f(0,0)=0$, $c(0,0)=0$, $0\in\mathcal{Z}$ *and* $0\in\mathcal{U}$.

Assumption 2 (existence of optimal controls). *For each* $N\geq 0$ *and* $z\in\mathcal{Z}$, *there exists* $u^*=\arg\min J_u^N(z)$.

Assumption 3 (finiteness of the optimal cost). *For each* $N\geq 0$ *there exists a scalar* $\mathcal{J}^N$ *such that, for each* $z\in\mathcal{Z}$, $J^N(z)\leq \mathcal{J}^N d(z)$.

We shall need the following definitions. Let $t_d\geq 0$ be given. We say that a trajectory z_t, $t\geq 0$, is δ-contractive at time instant $k\geq 0$ when $d(z_{k+t_d})<\delta d(z_k)$.

Definition 1 (cl-detectability). *We say that* (Ψ,J) *is closed loop (cl-)detectable if, for all* u, *there exist integers* N_d, $t_d\geq 0$ *and scalars* $0\leq\delta<1$, $\gamma>0$ *such that* $J_u^{N_d}(z)\geq\gamma d(z)$ *whenever* $d(z_{t_d})\geq\delta d(z)$.

Definition 2 (cl-observability). *We say that* (Ψ,J) *is cl-observable if it is cl-detectable with* $\delta=0$. *Equivalently,* (Ψ,J) *is cl-observable if there exist an integer* N_d *and a scalar* $\gamma>0$ *such that* $J_u^{N_d}(z)\geq\gamma d(z)$, *for all* u.

Remark 1. *Notice that the cl-detectability concept relates the closed loop trajectory* z *and the associated cost* J_u. *Indeed, if the trajectory is not* δ*-contractive at time instant* k, *then we have that there is a contribution of at least* $\gamma d(z_k)$ *to the cost functional in the sense that, for all* u *and* $N\geq k+N_d$,

$$\begin{aligned}J_u^N(z)&=\sum_{t=0}^{N}c(z_t,u_t)\\&=\sum_{t=0}^{N-1}c(z_t,u_t)1_{\{t\notin[k,k+N_d]\}}+\sum_{t=k}^{k+N_d}c(z_t,u_t)\\&\geq\sum_{t=0}^{N-1}c(z_t,u_t)1_{\{t\notin[k,k+N_d]\}}+\gamma d(z_k).\end{aligned}$$

In this section we deal with combinations of the following hypotheses.

H1. (Ψ,J) is cl-detectable.

H2. There exists $\kappa\geq 0$ for which $d(z_{t+1})\leq\kappa d(z_t)$, for all $z_t\in\mathcal{Z}$, with $u_t=\arg\min_u J_u^N(z_t)$.

H3. There exists $\mathcal{J}^+\geq 0$ such that $J^N(z)\leq\mathcal{J}^+ d(z)$, for all $N\geq 0$ and $z\in\mathcal{Z}$.

Remark 2 (H1). *A sufficient condition for hypothesis H1 to hold is that* $c(z,u)\geq\gamma d(z)$, *for all* z,u *and some* $\gamma>0$. *For instance, in the linear quadratic deterministic without state and control constraints, set* $d(z)=z'z$ *and* $c(z,u)=z'Qz+u'Ru$ *with* $Q=Q'\geq 0$. *If one sets* $Q>0$, *then the above relation holds with* $\gamma=\lambda_-(Q)$. *We also have shown that if the system is detectable (in the usual sense for deterministic linear systems), then it is also cl-detectable. Details will be presented elsewhere.*

Remark 3 (H2). *If the system is cl-observable, then hypothesis H2 holds with* $\kappa=\mathcal{J}^N/\gamma$, $N\geq N_d+1$. *Indeed, if we deny this assertion by assuming that* $d(z_{t+1})>\kappa d(z_t)$, *then*

$$\begin{aligned}\mathcal{J}^N d(z_t)\geq J_u^N(z_t)&\geq\sum_{k=1}^{N}c(z_{t+k},u_{t+k})\\&=J_u^{N-1}(z_{t+1})>\gamma d(z_{t+1})\\&>\gamma(\mathcal{J}^N/\gamma)d(z_t)=\mathcal{J}^N d(z_t),\end{aligned}\tag{6}$$

which is an absurd. If the system is linear, and H1 holds, then it can also be shown that H2 holds; the details are omitted here. In the general nonlinear setting an usual assumption is that f *is a bounded function, e.g. [3].*

Remark 4 (H3). *In the linear quadratic context without constraints, i.e.* $\mathcal{Z}=\mathbb{R}^n$, $\mathcal{U}=\mathbb{R}^m$, *and stabilizability (in the usual sense for deterministic linear systems) is a sufficient condition for H3 to hold.*

The basic ideas behind the result of this paper are as follows. First, consider the cl-observable case and assume that H3 holds. Consider also a sequence of controls u and the associated trajectory z_t, $0\leq t\leq N$, and, for the given initial condition z, the ball $\mathcal{B}_r=\{w\in\mathcal{Z}:d(w)\leq rd(z)\}\subset\mathcal{Z}$. From the observability concept one can check that, if the trajectory stays outside $\mathcal{B}_r$ during some interval $0\leq k\leq\eta$, $\eta\geq N_d$, then

$$J^\eta(z)\geq\sum_{\ell=0}^{\hat{k}-1}J^{N_d}(z_{\ell N_d})\geq\sum_{\ell=0}^{\hat{k}-1}\gamma\, d(z_{\ell N_d})\geq\hat{k}\gamma r\, d(z),\tag{7}$$

where $\hat{k}$ is the largest integer for which $\hat{k}\leq(\eta/N_d)$. This allows us to conclude that, for any $r>0$, the trajectory will enter $\mathcal{B}_r$ at some time instant η (otherwise $J^\eta(z)\geq\hat{k}\gamma r\, d(z)>\mathcal{J}^+ d(z)$ for sufficiently large η and $\hat{k}$, which contradicts H3).

Then we evaluate, for a sufficiently large N and $M \geq N$,

$$\begin{aligned} J^N(z) + J^{M-\eta}(z_\eta) & \qquad\qquad (8) \\ &= \sum_{t=0}^{N} c(z_t^*, u_t^*) + \min_u \Big(\sum_{t=\eta}^{M-1} c(z_t, u_t)|_{z_\eta = z_\eta^*} \Big) \\ &\geq \sum_{t=0}^{\eta-1} c(z_t^*, u_t^*) + \min_u \Big(\sum_{t=\eta}^{M-1} c(z_t, u_t)|_{z_\eta = z_\eta^*} \Big) \\ &\geq \min_u J_u^M(z) = J^M(z), \end{aligned}$$

where $u^* = \arg\min_u J^N(z)$ and z_t^*, $0 \leq t \leq N$, is the associated trajectory. This leads to the uniform convergence result

$$0 \leq J^M(z) - J^N(z) \leq J^{M-\eta}(z_\eta) \leq \mathfrak{J}^+ d(z_\eta) \leq \mathfrak{J}^+ r\, d(z). \quad (9)$$

The arguments are more complex in the detectable case. Assume that H2 holds and let us set $N \geq \bar{N} = \max(N_d, t_d)$ and introduce the subsequence t_k of time instants defined recursively as follows:

$$\begin{aligned} & t_0 = 0; \qquad\qquad (10) \\ & \text{if } t_{k-1} + T_d \leq N, \text{ then,} \\ & t_k = \begin{cases} t_{k-1} + \bar{N}, & \text{if} \quad d(z_{t_{k-1}+t_d}) \geq \delta d(z_{t_{k-1}}) \\ & \qquad \text{and } \; t_{k-1} + \bar{N} \leq N; \\ t_{k-1} + t_d, & \text{if} \quad d(z_{t_{k-1}+t_d}) < \delta d(z_{t_{k-1}}). \end{cases} \end{aligned}$$

In connection, we define $\bar{k} \geq 0$ as the largest integer for which $t_{\bar{k}} \leq N$ and, for $0 \leq k \leq \bar{k}$, the following counting function:

$$m_0 = 0, \quad m_k = \sum_{\ell=0}^{k-1} 1_{\{d(z_{t_\ell + t_d}) \geq \delta d(z_{t_\ell})\}}, \qquad (11)$$

in such a manner that m_k is the number of times that the trajectory is not δ-contractive along the past sequence t_ℓ, $0 \leq \ell \leq k-1$. See Figure 1.

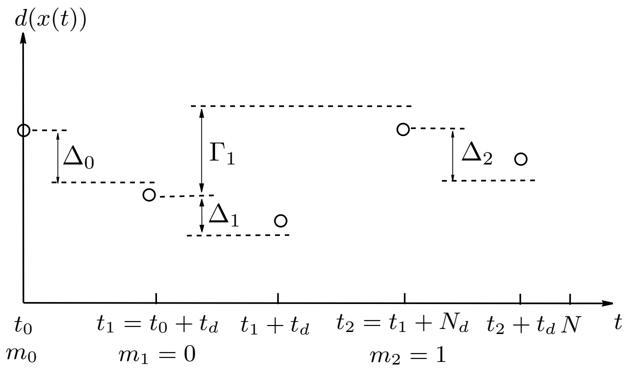

Figure 1: Construction of the subsequence t_k, $0 \leq k \leq \bar{k}$, with $\bar{k} = 2$. We assume that H2 holds and $N_d > t_d$. $\Delta_k = (1-\delta) d(x(t_k))$ and $\Gamma_k = (\kappa^{N_d} - 1) d(x(t_k))$.

Let us return to the explanation of the main arguments. Suppose that z_{t_k} stays outside $\mathcal{B}_r$, for all $k \leq \bar{k}$. Note, from the hypothesis H1, that at each k for which $d(z_{t_{k+1}}) \geq \delta d(z_{t_k})$ holds, a contribution of at least $\gamma d(z_{t_k}) \geq \gamma r d(z)$ is added to the functional J^N (like in (7)).

This leads to $J^N(z) \geq m_{\bar{k}} \gamma r d(z)$, which allows us to conclude that $m_{\bar{k}}$ is bounded, i.e., there exists $m > 0$ for which $m_{\bar{k}} \leq m$, even if $\bar{k}, N \to \infty$ (otherwise, $J^N(z) \geq m_{\bar{k}} \gamma r d(z) > m \gamma r d(z) > \mathfrak{J}^+ d(z)$ for sufficiently large m, N, thus contradicting H3). In this situation it is simple to conclude that, for N sufficiently large, the number of δ-contractions is sufficient to take $z_{t_{\tilde{k}}}$ into $\mathcal{B}_r$ for some $\tilde{k} \leq \bar{k}$. Finally, the uniform convergence result follows from (9) with $\eta = t_{\tilde{k}}$.

In what follows, we formalize these results.

3 Main Result

Lemma 1. *Assume that hypothesis H1 holds and $r > 0$ is such that $d(z_t) > r d(z)$, for each $0 \leq t \leq N$ and some $N \geq \bar{N}$. Consider m_k, $0 \leq k \leq \bar{k}$, defined in (11). Then,*

$$m_k \leq \mathfrak{J}^N / (\gamma r), \quad 0 \leq k \leq \bar{k}.$$

Proof. We evaluate

$$\begin{aligned} J^N(z) &= \sum_{t=0}^{N-1} c(z_t, u_t) \geq \sum_{k=0}^{\bar{k}-1} \sum_{t=t_k}^{t_{k+1}-1} c(z_t, u_t) \qquad (12) \\ &\geq \sum_{k=0}^{\bar{k}-1} J_u^{t_{k+1}-t_k}(z_{t_k}) 1_{\{d(z_{t_k+t_d}) \geq \delta d(z_{t_k})\}} \\ &\geq \sum_{k=0}^{\bar{k}-1} \gamma r d(z) 1_{\{d(z_{t_k+t_d}) \geq \delta d(z_{t_k})\}} \\ &= m_{\bar{k}} \gamma r\, d(z) \end{aligned}$$

where the last inequality follows from hypothesis H1. Then, employing the definition of $\mathfrak{J}^N$, we have that

$$\mathfrak{J}^N d(z) \geq J^N(z) \geq m_{\bar{k}} \gamma r d(z) \Rightarrow m_{\bar{k}} \leq \mathfrak{J}^N / (\gamma r)$$

□

For δ, γ, N_d and t_d as in Definition 1, $\mathfrak{J}^N$ as in Assumption 3, $\bar{N} = \max(N_d, t_d)$, and $r > 0$, we define

$$N_0 = N_0(\mathfrak{J}^N, r) = \bar{N} \Big(\frac{\mathfrak{J}^N}{\gamma r} (1 - \bar{N} \log_\delta(\kappa)) + \log_\delta(r) + 1 \Big) \qquad (13)$$

where we convention that $\log_\delta(\cdot) = 0$ if $\delta = 0$ (observable case).

Lemma 2. *Assume that hypotheses H1 and H2 hold, $r > 0$, and*

$$N \geq N_0(\mathfrak{J}^N, r). \qquad (14)$$

Then $d(z_t) \leq r d(z)$ for some $0 \leq t \leq N$.

Proof. Let us deny the assertion in the lemma and assume that

$$d(z_t) > rd(z) \tag{15}$$

for all t in the prescribed interval. Consider the subsequence t_k, $k \le \bar{k}$, defined in (10). Two cases arise.

Case (i). *Strictly detectable case ($\delta > 0$).* It is simple to check that, for each $k \ge 0$, either (i) $m_k = m_{k-1}$ and $d(z_{t_k}) < \delta d(z_{t_{k-1}})$ or (ii) $m_k = m_{k-1}+1$ and, in this case, from H2 we have that $d(z_{t_k}) < \kappa^{\bar{N}} d(z_{t_{k-1}})$ (see Figure 1 in connection). This leads to

$$d(z_{t_k}) \le \kappa^{\bar{N} m_k} \delta^{k-m_k} d(z)$$

and (15) allows us to employ Lemma 1 to evaluate

$$d(z_{t_k}) \le \kappa^{\bar{N} \mathcal{J}^N/\gamma r} \delta^{k-(\mathcal{J}^N/\gamma r)} d(z)$$

in such a manner that if there exists a $k \ge 0$ for which

$$\begin{cases} \delta^k \le \kappa^{-\bar{N}\mathcal{J}^N/(\gamma r)} \delta^{\mathcal{J}^N/(\gamma r)} r \\ t_k \le N \\ k \le \bar{k} \end{cases} \tag{16}$$

then $d(z_{t_k}) \le rd(z)$ and (15) is contradicted, completing the proof. It remains to show that such a $k \ge 0$ exists. Let $\varphi = \frac{\mathcal{J}^N}{\gamma r}(1 - \bar{N}\log_\delta(\kappa)) + \log_\delta(r)$; from (13) we have that $N_0 = \bar{N}(\varphi+1)$, in such a manner that we can pick ℓ as the integer for which $\bar{N}\varphi < \bar{N}\ell \le N_0$ holds. Now it is a simple matter to check that (16) holds for $k = \ell$: the first inequality in (16) follows immediately from the fact that $\ell > \varphi$; for the second inequality, we recall that $t_k \le k\bar{N}$, for all k, to get that $t_\ell \le \bar{N}\ell \le N_0 \le N$; finally, $t_\ell \le N$ leads to $t_\ell \le t_{\bar{k}}$, recalling the fact that $\bar{k}$ is the largest integer for which $t_{\bar{k}} \le N$.

Case (ii). *Observable case ($\delta = 0$).* The relation in (15) allows us to employ Lemma 1 to conclude that

$$m_{\bar{k}} \le \mathcal{J}^N/(\gamma r) \tag{17}$$

On the other hand, since $\delta = 0$, one has that $d(z_{t_k}) \ge \delta d(z_{t_{k-1}})$ holds at each $k \le \bar{k}$. In this situation, it is simple to check from (10) and (11) that

$$\begin{aligned} t_k &= kN_d \\ m_k &= k. \end{aligned} \tag{18}$$

Recalling that $\bar{k}$ is the largest integer for which $t_{\bar{k}} \le N$, from (18) we conclude that $N - N_d < t_{\bar{k}} \le N$. Dividing by N_d, and employing (18), we obtain

$$N/N_d - 1 < m_{\bar{k}} \le N/N_d. \tag{19}$$

Finally, (17) and (19) lead to

$$N < N_d \left(\frac{\mathcal{J}^N}{\gamma r} + 1 \right) = N_0,$$

which is a contradiction in view of (14), and the result is proven. □

Remark 5. *Notice that Lemma 2 does not ensure that, for each $r > 0$, the inequality (14) holds for some N. This is accomplished only under assumption H3, when the right hand side of (14) is bounded by $N_0(\mathcal{J}^+, r)$.*

At this point we can announce the main result of the paper.

Theorem 1. *Assume that H1 and H2 hold. Then, for $M \ge N \ge N_0$,*

$$0 \le J^M(z) - J^N(z) \le r\mathcal{J}^M d(z) \tag{20}$$

In addition, if H3 also holds, then, for any $r > 0$,

$$|J^M(z) - J^N(z)| \le r\mathcal{J}^+ d(z), \quad \forall M, N \ge N_0(\mathcal{J}^+, r). \tag{21}$$

Proof. The first inequality in (20) is immediate from the definition of the cost functional in (3). For the second inequality, in a similar manner to (8), we write for any t in the interval $0 < t \le N$,

$$\begin{aligned} &J^N(z) + J^{M-t}(z_t^*) \\ &\quad \ge \sum_{k=0}^{t-1} c(z_k^*, u_k^*) + \min_u \left(\sum_{k=t}^{M-1} c(z_k, u_k)|_{z_t = z_k^*} \right) \\ &\quad \ge \min_u J_u^M(z) = J^M(z) \end{aligned}$$

which leads to

$$J^M(z) - J^N(z) \le J^{M-t}(z_t^*) \tag{22}$$

Now we evaluate $J^{M-t}(z_t^*)$. From Lemma 2 we have that there exists $0 \le t \le N$ for which $d(z_t^*) \le rd(z_0^*) = rd(z)$, in such a manner that

$$J^{M-t}(z_t^*) \le \mathcal{J}^M d(z_t^*) \le r\mathcal{J}^M d(z) \tag{23}$$

and the second inequality in (20) follows immediately from (22)–(23).

The uniform convergence result in (21) follows from the first part of this theorem, by setting $\mathcal{J}^M \equiv \mathcal{J}^+, \forall M \ge N_0(\mathcal{J}^+, r)$ in (20). □

4 Conclusions

The asymptotic behaviour of the optimal cost for problems with increasing time horizon is established, in view of the first relation in Theorem 1. When the uniform bound $\mathcal{J}^+$ exists, there exists fixed number N_0 as defined in (13) for any $r > 0$, which provides the uniform result in Theorem 1. General dynamics and costs are considered and the closed-loop detectability hypothesis is introduced and employed together with basic consistency assumptions. For unconstrained linear systems with quadratic costs, it can be shown that the working assumptions used here are all satisfied, including cl-detectability, which holds when the system is detectable in the usual sense. The details will be developed elsewhere.

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