ROBUSTNESS OF MULTIPLE MODEL LQ CONTROL

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Abstract

The optimal control strategy for discrete time multiple model is described. Simulation of control and on-line estimation of model probability is shown. Robustness of stability and comparison of the classical LQ control and LQ control based on multiple models is presented.

1 Introduction

The methodology based on bayesian update of the probability distribution over the set of possible models [4] enables description of a plant by a mixture distribution [7], [1].

LQ (optimal control of Linear system with Quadratic criterion) algorithm based on a mixture of a set of parallel models with common state and different structure only was developed in [3]. Robust stability analysis of such close loop is shown in this paper.

LQG (Gaussian noise) algorithm based on a mixture of a set of parallel models with different parameters and different dimension was developed in [6]. The design of such LQ controller with a set of Kalman filters for on-line estimation of model probability is also presented in this paper.

The outline of the paper is as follows: in section 2 Bayesian approach to state estimation is formulated in a general way. In section 3, Bayesian approach to multiple state development model and Kalman filter in normalized form for estimate the optimal probability distribution over the set of possible models are described. Section 4 solves multiple model approach for quadratic optimal discrete time multiple model. In section 5, simulation results and robustness analysis is presented.

2 Bayesian approach to state estimation

In this section we will analyze general properties of a process model from the bayesian viewpoint. It will be shown that all the concepts used can be generally described in terms of conditional probability density functions (c.p.d.f.).

For the design of manipulated input, the knowledge of the process output based on a finite set of observed input and output data up to time $(t-1)$

$$\mathcal{D}^{t-1} = \{u(1), y(1), \ldots, u(t-1), y(t-1)\}$$

is required. It can be described by a set of c.p.d.f.

$$p\left(y(t)|\mathcal{D}^{t-1}, u(t)\right) \quad \text{for} \quad t = 1, \ldots . \tag{1}$$

If there exists a finite-dimensional vector variable $x(t)$ such that

$$\begin{aligned} p\left(x(t+1), y(t)\left|\mathcal{D}^{t-1}, x(t), u(t)\right.\right) = \\ = p\left(x(t+1), y(t)|x(t), u(t)\right) \end{aligned} \tag{2}$$

i.e. it contains all the relevant information for the prediction of the process output $y(t)$ and state $x(t+1)$, then it is called the *state of the process*.

To obtain the predictive c.p.d.f. (1) given as

$$\begin{aligned} p\left(y(t)\left|\mathcal{D}^{t-1}, u(t)\right.\right) = \\ = \int p\left(y(t)|x(t), u(t)\right) p\left(x(t)\left|\mathcal{D}^{t-1}\right.\right)\ dx(t), \end{aligned} \tag{3}$$

the c.p.d.f.

$$p\left(x(t)\left|\mathcal{D}^{t-1}, u(t)\right.\right) = p\left(x(t)\left|\mathcal{D}^{t-1}\right.\right) \tag{4}$$

representing our knowledge about the state of the process must be also propagated in time. This c.p.d.f. is called the *state estimate*. The condition (4) introduced by [5] is called the *natural condition of control*. It will be used repeatedly in the following text.

The c.p.d.f. $p\left(y(t)|x(t), u(t)\right)$ is defined by the output (measurement) equation of the state-space model of the process

$$y(t) = Cx(t) + Du(t) + e(t), \tag{5}$$

where $e(t)$ is the measurement noise with known distribution with zero mean and covariance $\mathrm{cov}\{e(t)\} = \Gamma_e$, independent of the state and input of the process.

The incorporation of the information contained in a new pair of data $\{u(t), y(t)\}$ (the *data-update* step of the algorithm) can be described as

$$p\left(x(t)\,\middle|\mathcal{D}^t\right) = \frac{p\left(y(t)|x(t), u(t)\right)}{p\left(y(t)\,|\mathcal{D}^{t-1}, u(t)\right)}\, p\left(x(t)\,\middle|\mathcal{D}^{t-1}\right). \tag{6}$$

The *time-update* step of the algorithm, i.e. the predictive c.p.d.f. $p\left(x(t+1)|\mathcal{D}^t\right)$ is given as

$$\begin{aligned} p\left(x(t+1)\,\middle|\mathcal{D}^t\right) &= \\ &= \int p\left(x(t+1)\,\middle|x(t), \mathcal{D}^t\right) p\left(x(t)\,\middle|\mathcal{D}^t\right)\, dx(t). \end{aligned} \tag{7}$$

To complete this step, a state development model defined by the c.p.d.f.

$$p\left(x(t+1)\,\middle|x(t), \mathcal{D}^t\right) = p\left(x(t+1)|x(t), u(t), y(t)\right) \tag{8}$$

which can be obtained from (2) is necessary. The c.p.d.f. $p\left(x(t+1)|x(t), u(t)\right)$ is usually defined by the state transition equation of the state-space model of the process

$$x(t+1) = Ax(t) + Bu(t) + v(t), \tag{9}$$

where $v(t)$ is the process noise with known distribution with zero mean and covariance $\mathrm{cov}\{v(t)\} = \Gamma_v$, independent of the state and input of the process. The role of the term $y(t)$ in the condition of (8) is discussed in [2].

3 Multiple state development models

Suppose a set of h alternative state development models $p\left(x(t+1)|x(t), u(t), y(t)\right)$ parameterized by the active models $m(t) = 1, \ldots, h$

$$\begin{aligned} p_i\left(x(t+1)|x(t), u(t)\right) &= \\ &= p\left(x(t+1)|x(t), u(t), m(t) = i\right) \end{aligned} \tag{10}$$

is given. Then simultaneous filtering of the state and detection of the active model can be developed. It is possible to propagate the state estimates based on the i-th model in parallel and compute the probability distribution over the set of models. In this setting, no changes in the active model are modelled but the (constant) active model is classified from the set of candidates.

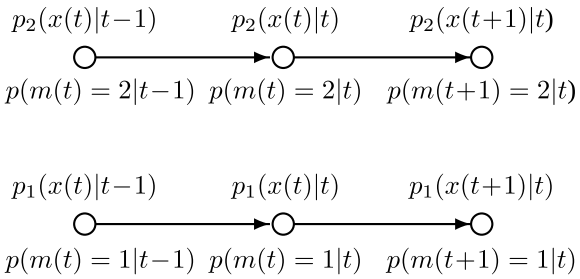

Figure 1: State filtering and model classification with parallel models

Suppose the initial probability distribution (p.d.) over the set of candidate models is given

$$p\left(m(0) = i\,\middle|\mathcal{D}^0\right) = p_i. \tag{11}$$

The probability $\alpha_i(t) = p\left(m(t) = i|\mathcal{D}^t\right)$ can be updated by the data as

$$\begin{aligned} p\left(m(t) = i|\mathcal{D}^t\right) &\propto \\ &\propto p\left(y(t)|\mathcal{D}^{t-1}, u(t), m(t) = i\right) p\left(m(t) = i|\mathcal{D}^{t-1}\right) \end{aligned} \tag{12}$$

where the predictive c.p.d.f. of the output is

$$\begin{aligned} p\left(y(t)\,\middle|\mathcal{D}^{t-1}, u(t), m(t) = i\right) &= \\ &= \int p\left(y(t)|x(t), u(t), m(t){=}i\right) p_i\left(x(t)\,\middle|\mathcal{D}^{t-1}\right) dx(t), \end{aligned} \tag{13}$$

where the c.p.d.f. of the state based on the i-th model

$$\begin{aligned} p_i\left(x(t)\,\middle|\mathcal{D}^{t-1}\right) = p\left(x(t)\,\middle|\mathcal{D}^{t-1}, m(t) = i\right) &\sim \\ \sim \mathcal{N}\left(\widehat{x}_i(t|t-1), \Gamma_{x_i}(t|t-1)\right) \end{aligned} \tag{14}$$

is used (see Figure 1).

The time-update step of the algorithm (for the p.d. over the set of models) is $p\left(m(t+1) = i\,|\mathcal{D}^t\right) = p\left(m(t) = i\,|\mathcal{D}^t\right)$ because there is no "model" of model probability development. But there is the possibility to enable tracking the changes in probability distribution over the set of models to realize forgetting some obsolete information in time update step.

While in its corresponding operation point the i-th output error (or equivalent state space) model provides the "correct" output prediction, for other operating regimes the output prediction is not consistent with the data, resulting in biased probability distribution over the set of models. One of the possibilities how to compensate for this fact is the use of normalized form of models

$$x_i(t+1) = A_i x_i(t) + B_i u(t) + v_i(t) \tag{15}$$

$$y(t) = C_i x_i(t) + D_i u(t) + e_i(t) \tag{16}$$

with (scalar) measurement noise variance and process noise covariance matrix

$$\mathrm{cov}\left\{e_i(t)\right\} = \sigma_{e_i}^2(t), \quad \mathrm{cov}\left\{v_i(t)\right\} = \sigma_{e_i}^2(t) V_i. \tag{17}$$

Also the Kalman filter is implemented in normalized form, operating on the statistics

$$\begin{aligned} p\left(x(t)|\sigma_{e_i}^2, \mathcal{D}^{t-1}, u(t), m(t) = i\right) &\sim \\ \sim \mathcal{N}\left(\hat{x}_i(t|t-1), \sigma_{e_i}^2\, \Gamma_{x_i}(t|t-1)\right), \end{aligned} \tag{18}$$

i.e. the variance of the state estimate is also scaled by the measurement noise variance. Then one-step of the algorithm of the normalized Kalman filter can be written as

$$\begin{aligned} \hat{x}_i(t+1|t) &= A_i \hat{x}_i(t|t-1) + B_i u(t) + \\ &+ L_i(t)\left(y(t) - C_i \hat{x}_i(t|t-1) - D_i u(t)\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \Gamma_{x_i}(t+1|t) &= A_i \Gamma_{x_i}(t|t-1) A_i^T + V_i - \\ &- L_i(t) C_i \Gamma_{x_i}(t|t-1) A_i^T \end{aligned} \tag{20}$$

$$\begin{aligned} L_i(t) &= A_i \Gamma_{x_i}(t|t-1) C_i^T \times \\ &\times \left(C_i \Gamma_{x_i}(t|t-1) C_i^T + 1\right)^{-1}, \end{aligned} \tag{21}$$

i.e. it is independent of the measurement noise variance and the unknown measurement noise variance for each model can be independently estimated from the data.

The covariance of the prediction error of the i-th model $\varepsilon_i(t|t-1) = y(t) - \hat{y}_i(t|t-1)$ is

$$\begin{aligned} \mathrm{cov}\{\varepsilon_i(t|t-1)\} &= \mathrm{cov}\{y(t) - \hat{y}_i(t|t-1)\} = \\ &= \Big(1 + C_i \Gamma_{x_i}(t|t-1) C_i^T\Big)\, \sigma_{e_i}^2. \end{aligned} \tag{22}$$

The c.p.d.f. of the output for known measurement noise variance is

$$\begin{aligned} & p_i\left(y(t)|\sigma_{e_i}^2, \mathcal{D}^{t-1}, u(t)\right) \sim \\ & \sim \mathcal{N}\Big(\hat{y}_i(t|t-1), \big(1 + C_i \Gamma_{x_i}(t|t-1) C_i^T\big)\sigma_{e_i}^2\Big). \end{aligned} \tag{23}$$

The statistics for the bayesian estimate of the variance [5]

$$p\left(\sigma_{e_i}^2(t)\,\middle|\mathcal{D}^{t-1}, u(t)\right) \sim \chi^2_{\nu(t|t-1)}\left(\frac{S_i^2(t|t-1)}{\sigma_{e_i}^2}\right),$$

i.e. variable $S_i^2/\sigma_{e_i}^2$ has a χ^2 distribution with ν degrees of freedom, can be updated (using exponential forgetting with forgetting factor φ) as

$$\begin{aligned} S_i^2(t+1\,|\,t) &= \varphi \cdot \left(S_i^2(t\,|\,t-1) + \frac{\varepsilon_i^2(t\,|\,t-1)}{1 + C_i \Gamma_{x_i}(t|t-1) C_i^T}\right) \\ \nu(t+1\,|\,t) &= \varphi \cdot \Big(\nu(t\,|\,t-1) + 1\Big) \\ \hat{\sigma}_{e_i}^2(t+1\,|\,t) &= \frac{S_i^2(t+1\,|\,t)}{\nu(t+1\,|\,t)}\,. \end{aligned} \tag{24}$$

Using the estimate of measurement noise variance, the p.d.f. (23) should be replaced a Student distribution; however, for sufficiently large $\nu(t+1|t) > 30$, Student distribution can be well approximated by a normal distribution (23) where the estimate of the noise variance (24) is substituted.

4 Multiple model control

In this section, LQ controller for output feedback is design based on a multiple model. First compatible multiple model with common state and different structure only was developed in [3]. In the next paragraph, compatible multiple model with different parameters, structure and different dimension was developed in [6].

4.1 State feedback controller

Suppose a set of h state development particular models is given

$$\begin{aligned} & p_i\big(x(t+1)|x(t), u(t)\big) \sim \\ & \sim \mathcal{N}\big(A_i x(t) + B_i u(t), \Gamma_{v_i}(t)\big), \end{aligned} \tag{25}$$

where $\Gamma_{v_i}(t) = \mathrm{cov}\{v_i(t)\}$. Then the state prediction based on the measured state $x(t)$ is

$$\begin{aligned} & p\left(x(t+1)|x(t), u(t)\right) = \\ & = \sum_{i=1}^{h} \alpha_i p_i\big(x(t+1)|x(t), u(t)\big) \sim \\ & \sim \mathcal{N}\big(\widehat{x}(t+1|t), \Gamma_x(t+1|t)\big), \end{aligned} \tag{26}$$

where the mean $\widehat{x}(t+1|t)$ and covariance $\Gamma_x(t+1|t)$ of c.p.d.f (26) equal

$$\widehat{x}(t+1|t) = \sum_{i=1}^{h} \alpha_i \big(A_i x(t) + B_i u(t)\big) \tag{27}$$

$$\begin{aligned} \Gamma_x(t+1|t) &= \sum_{i=1}^{h} \alpha_i \Big\{\Gamma_{x_i}(t+1|t) + \\ &+ \big(x_i(t+1) - \widehat{x}(t+1|t)\big)\big(x_i(t+1) - \widehat{x}(t+1|t)\big)^T\Big\}. \end{aligned}$$

Consider a loss function

$$\begin{aligned} V\big(x(t), u_t^{N-1}, t\big) &= \mathcal{E}\Big\{x^T(N) Q(N) x(N) + \\ &+ \sum_{k=t}^{N-1} x^T(k) Q(k) x(k) + u^T(k) R(k) u(k)\Big\}, \end{aligned} \tag{28}$$

where

$$u_t^{N-1} = \Big\{u(t),\, u(t+1),\, \ldots,\, u(N-1)\Big\} \tag{29}$$

and its optimal value

$$V^*(x(t), t) = \min_{u_t^{N-1}} V(x(t), u_t^{N-1}, t). \tag{30}$$

The optimal control law after the minimization is

$$\begin{aligned} u^*(t) &= -\left(R(t) + \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) B_i\right)^{-1} \times \\ & \times \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) A_i x(t). \end{aligned} \tag{31}$$

Matrix $P(t)$ is solution of the Riccati equation

$$\begin{aligned} P(t) &= Q(t) + \sum_{i=1}^{h} \alpha_i A_i^T P(t+1) A_i - \\ & - \left(\sum_{i=1}^{h} \alpha_i A_i^T P(t+1) B_i\right) \times \\ & \times \left(R(t) + \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) B_i\right)^{-1} \times \\ & \times \left(\sum_{i=1}^{h} \alpha_i B_i^T P(t+1) A_i\right) \end{aligned} \tag{32}$$

with final condition $P(N) = Q(N)$. The optimal value of the criterion is

$$J^*(t) = V^*\big(x(t),t\big) = x^T(t)P(t)x(t) + + \sum_{k=t}^{N-1} \left\{ \sum_{i=1}^{h} \alpha_i \,\text{trace}\big(P(k)\Gamma_{v_i}(k)\big) \right\}. \quad (33)$$

More details were described in [3].

Note that the optimal feedback gain matrix equals

$$K(t) = \left(R(t) + \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) B_i \right)^{-1} \times \times \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) A_i \quad (34)$$

and finally the optimal feedback control equals

$$u^*(t) = -K(t)x(t). \quad (35)$$

4.2 Output feedback controller

For the output feedback controller, the set of models available reads

$$p_i\left(x(t+1)|\mathcal{D}^t, u(t)\right) \sim \sim \mathcal{N}\big(\widehat{x}_i(t+1|t),\, \Gamma_{x_i}(t+1|t)\big). \quad (36)$$

Then the state prediction $p\left(x(t+1)|\mathcal{D}^t, u(t)\right)$ based on the measured data $\mathcal{D}^t$ is the same as (26), where the mean $\widehat{x}(t+1|t)$ and covariance $\Gamma_x(t+1|t)$ equal

$$\widehat{x}(t+1|t) = \sum_{i=1}^{h} \alpha_i \widehat{x}_i(t+1|t) \quad (37)$$

$$\Gamma_x(t+1|t) = \sum_{i=1}^{h} \alpha_i \Big\{ \Gamma_{x_i}(t+1|t) + +\big(\widehat{x}_i(t+1|t) - \widehat{x}(t+1|t)\big)\big(\widehat{x}_i(t+1|t) - \widehat{x}(t+1|t)\big)^T \Big\}.$$

Consider a loss function

$$V\big(x(t), u_t^{N-1}, t\big) = = \mathcal{E}\left\{ \sum_{i=1}^{h} \alpha_i x_i^T(N) Q(N) x_i(N) + + \sum_{t=1}^{N} \sum_{i=1}^{h} \alpha_i x_i^T(t) Q(t) x_i(t) + u^T(t) R(t) u(t) \right\} \quad (38)$$

and its optimal value

$$V^*(t) = \min_{u_t^{N-1}} V\big(x(t), u_t^{N-1}, t\big). \quad (39)$$

The optimal feedback control is

$$u^*(t) = -\left(R(t) + \sum_{i=1}^{h} \alpha_i B_i^T P_i(t+1) B_i \right)^{-1} \times \times \sum_{i=1}^{h} \alpha_i B_i^T P_i(t+1) A_i \widehat{x}_i(t|t-1). \quad (40)$$

and the special Riccati equation for $P_i(t)$, starting with $P_i(N) = Q_i(N)$ reads

$$P_i(t) = Q_i(t) + A_i^T P_i(t+1) A_i - A_i^T P_i(t+1) B_i \times \times \left(R(t) + \sum_{j=1}^{h} \alpha_j B_j^T P_j(t+1) B_j \right)^{-1} B_i^T P_i(t+1) A_i. \quad (41)$$

The optimal value of the quadratic criterion equals

$$J^*(t) = V^*(t) = \sum_{i=1}^{h} \alpha_i \widehat{x}_i^T(t) P_i(k) \widehat{x}_i(t) + + \sum_{k=t}^{N} \left\{ \sum_{i=1}^{h} \alpha_i \text{tr}\big(Q_i(k)\Gamma_{x_i}(k|k-1)\big) \right\} + + \sum_{k=t}^{N-1} \left\{ \sum_{i=1}^{h} \alpha_i \text{tr}\big(L_i^T(k) P_i(k+1) L_i(k) Q_\varepsilon(N-1|N-2)\big) \right\}.$$

The optimal feedback gain matrices equal

$$K_i(t) = \left(R(t) + \sum_{j=1}^{h} \alpha_j B_j^T P_j(t+1) B_j \right)^{-1} \times \times B_i^T P_i(t+1) A_i \quad (42)$$

and finally the optimal feedback control equals

$$u^*(t) = -\sum_{i=1}^{h} \alpha_i K_i(t) \widehat{x}_i(t|t-1), \quad (43)$$

where $\widehat{x}_i$ is the state estimation by the i-th Kalman filter (19) to (21).

Note that Riccati equations (41) and feedback gain matrices (42) cannot be computed separately for each model.

5 Simulation results

5.1 Example 1

Consider simple SISO system of second order

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1+s\tau)^2} \quad (44)$$

with time constant $\tau \in \langle 20; 50 \rangle\, s$.

The state space description of system (44) is

$$\dot{x}(t) = \begin{bmatrix} -1/\tau & 1/\tau \\ 0 & -1/\tau \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/\tau \end{bmatrix} u(t) \quad (45)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

Note that it is necessary to use the cascade form (see Figure 2) for description of the set of models with time constant $\tau \in \langle 20; 50 \rangle\, s$.

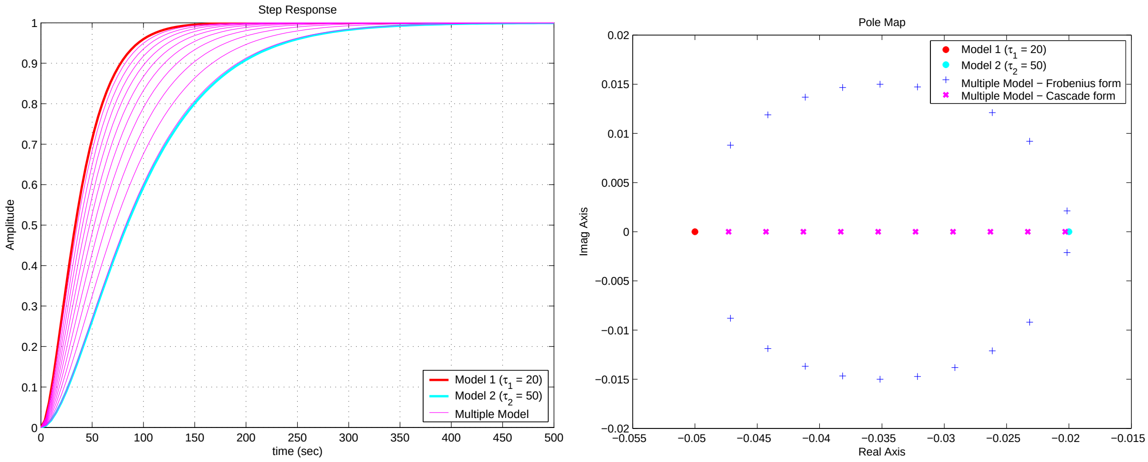

Figure 2: Mixture model step responses and pole map

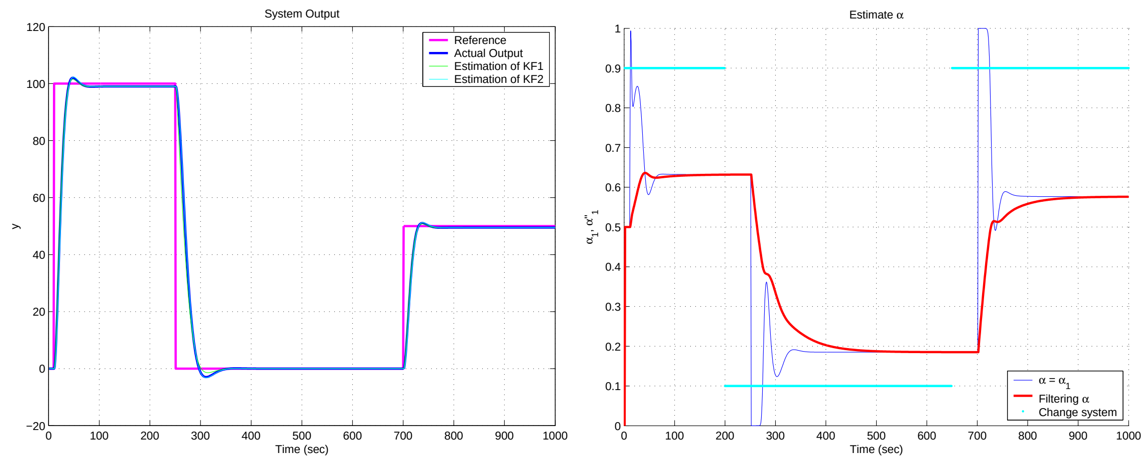

Figure 3: Reference tracking and model probability estimation - deterministic simulation

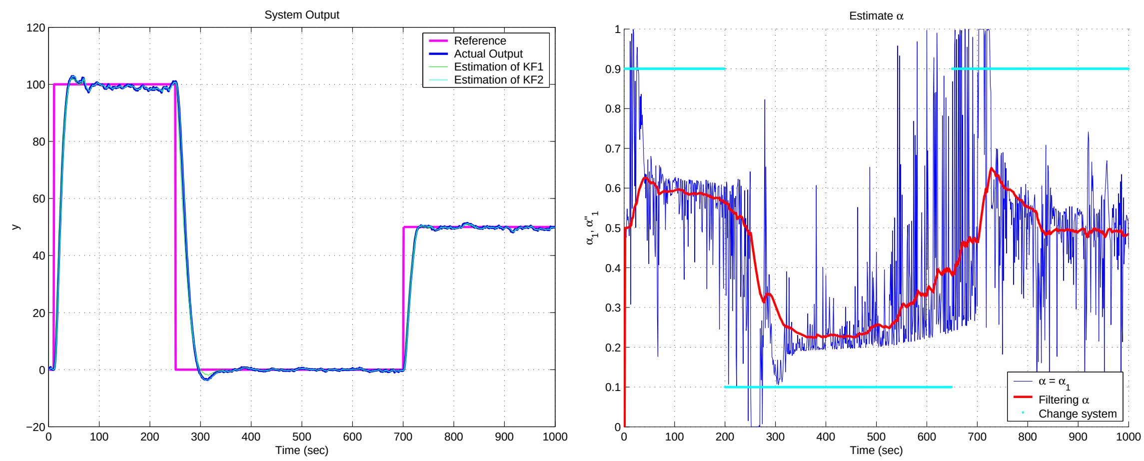

Figure 4: Reference tracking and model probability estimation - stochastic simulation

System (45) is approximated by a set of two models with the matrices

$$A_1 = \begin{bmatrix} -1/\tau_1 & 1/\tau_1 \\ 0 & -1/\tau_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1/\tau_1 \end{bmatrix}, \tag{46}$$

$$A_2 = \begin{bmatrix} -1/\tau_2 & 1/\tau_2 \\ 0 & -1/\tau_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1/\tau_2 \end{bmatrix}, \tag{47}$$

where $\tau_1 = 20s$ and $\tau_2 = 50s$.

For the simulation, the time constant of the system (44) is changed from $\tau = 25s$ to $\tau = 40s$ at time $t = 200s$ and back to $\tau = 25s$ at time $t = 650s$. The probabilities $\alpha_1(t)$ and $\alpha_2(t)$ of models (46) and (47) are estimated by two Kalman filters in normalized form (20), (21). Note that the estimate of model probability $\alpha^*(t)$ together with its filtered value are shown in Figure 3b and Figure 4b. Note that unlike the deterministic case, the process and measurement noise provide sufficient excitation for probability distribution tracking at the time of the change of the time constant.

The LQ control law for reference tracking is designed for a mixture of two models with parameters $\alpha_1 = \alpha^*$, $\alpha_2 = 1-\alpha^*$. The criterion matrices are $Q = 100$, $R = 1$.

Robust stability analysis

For robust stability analysis, the nominal model is chosen from the set of models (44) and the nominal time constant is $\tau_n = 25s$. This nominal model is used for classical LQ controller design.

For the nominal model with the time constant τ_n, the optimal estimate of model probability is $\alpha_n = 0.67$. Such optimal estimate of model probability is used for LQ controller for the multiple model design.

For criterion matrices

$$Q = \begin{bmatrix} 10^5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1 \tag{48}$$

classical LQ controller $K_c(\tau_n)$ and LQ controller for the multiple model $K_m(\alpha_n)$ (31), (32) are designed. For analysis of the robustness, the time constant of the real system (45) is changed from $\tau = 10s$ to $\tau = 60s$.

The eigenvalues of matrices

$$A(\tau) \quad - \quad B(\tau)K_c(\tau_n) \tag{49}$$

$$A(\tau) \quad - \quad B(\tau)K_m(\alpha_n) \tag{50}$$

are shown in Figure 5. The maximum singular values ($\mathcal{H}_2$ norm) of matrices $P_c(\tau)$ and $P_n(\tau)$ are shown in Figure 6. Matrix $P_c(\tau)$ is the solution of discrete Lyapunov equation

$$P_c(\tau) = Q + K_c^T(\tau_n) R K_c(\tau_n) + \tag{51}$$
$$+ \left[A(\tau) - B(\tau)K_c(\tau_n)\right]^T P_c(\tau) \left[A(\tau) - B(\tau)K_c(\tau_n)\right]$$

and matrix $P_n(\tau)$ is the solution of equation

$$P_n(\tau) = Q + K_m^T(\alpha_n) R K_m(\alpha_n) + \tag{52}$$
$$+ \left[A(\tau) - B(\tau)K_m(\alpha_n)\right]^T P_n(\tau) \left[A(\tau) - B(\tau)K_m(\alpha_n)\right].$$

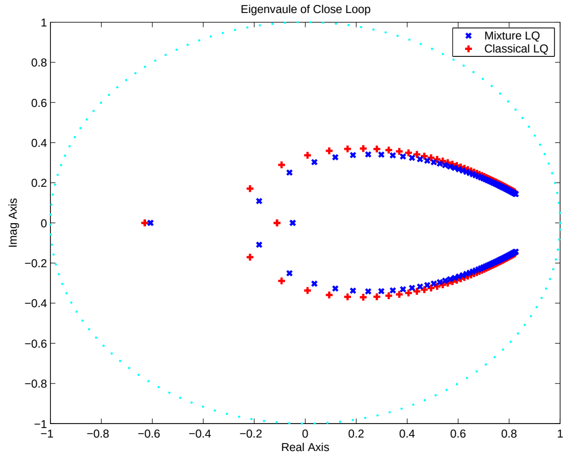

Figure 5: Eigenvalues of close loop

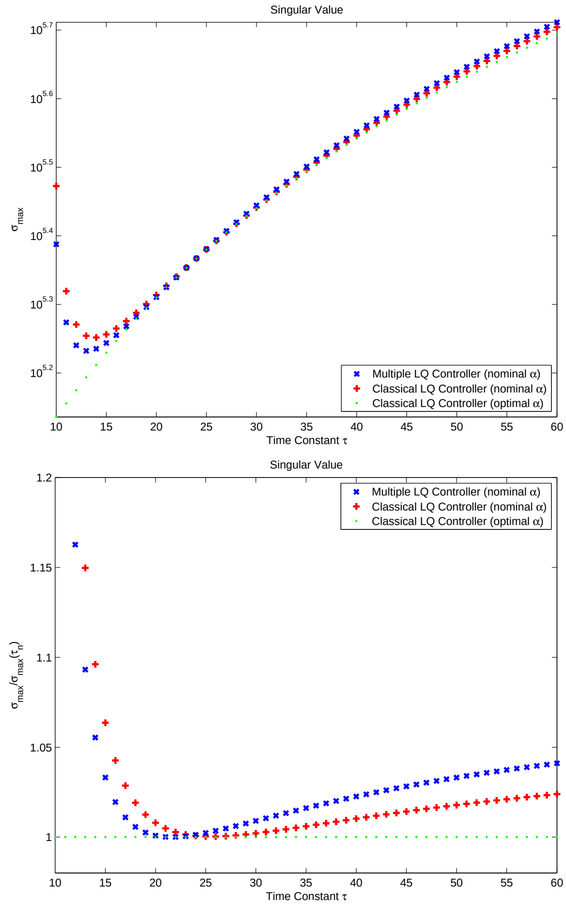

Figure 6: Maximum singular values

Note that the Figure 6b, is just a normalized version Figure 6a.

From Figure 5 follows that LQ strategy based on multiple model is more robust then classical LQ. For nominal model with time constant $\tau_n = 40s$, the eigenvalues of matrices (49) and (50) are almost similar as in Figure 5.

From Figure 6b follows that values of $\mathcal{H}_2$ norm of LQ strategy based on multiple model is less then $\mathcal{H}_2$ norm of LQ strategy based on single model for the time constant $\tau < \tau_n$ and is bigger for the time constant $\tau > \tau_n$. But the differences are not so significant.

5.2 Example 2

SISO system is modelled by mixture of two models with different structure and different dimension.

The nominal model of system is

$$P_0(s) = \frac{1}{0.5s^3 + s^2 + s}\,. \tag{53}$$

It is supposed that system can have complex pole $\sigma \pm j\,\omega$ where $\sigma \in (-\infty, -0.2)$ and $\omega = 10$. The multiplicative perturbation model is

$$P(s) = P_0(s) \cdot \frac{\sigma^2 + \omega^2}{s^2 - 2\sigma s + \sigma^2 + \omega^2}\,. \tag{54}$$

System (54) is approximated by a set of two models with different dimension. The first model $(A_1,\, B_1)$ correspond to the nominal model (53)

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \tag{55}$$

and the second model $(A_2,\, B_2)$ correspond to the worst perturbation model (54) i.e. for $\sigma = -0.2$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & -10 \\ 0 & 0 & 0 & 10 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}. \tag{56}$$

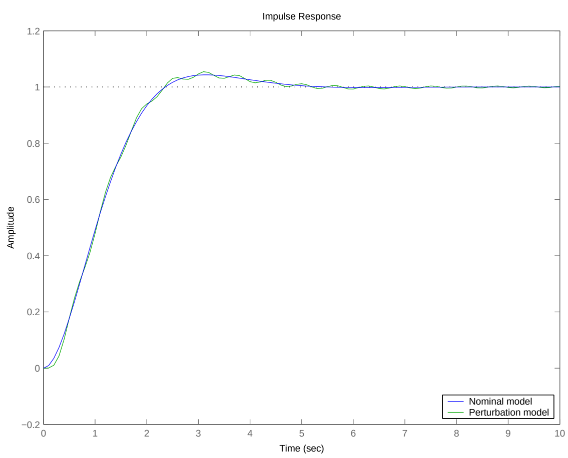

Figure 7: Impulse responses - nominal and perturbation model

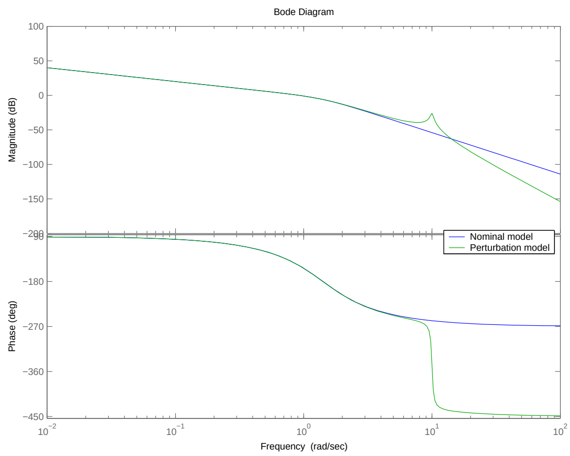

Figure 8: Bode diagrams - nominal and perturbation model

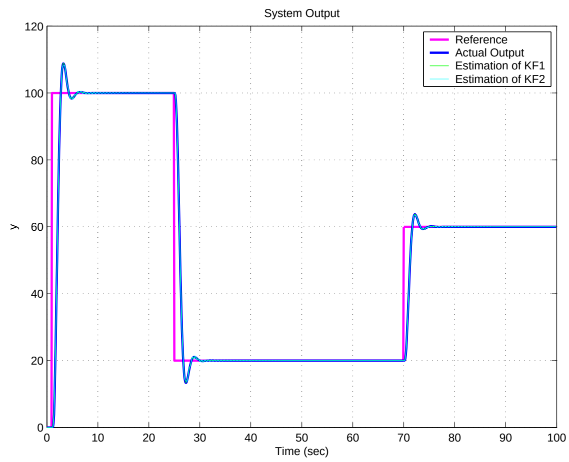

Figure 9: Reference tracking - deterministic simulation

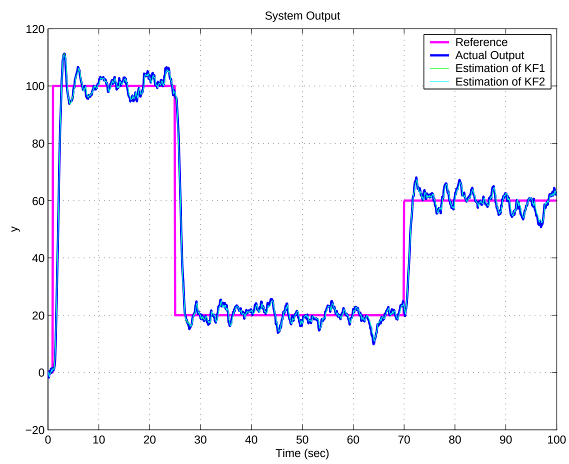

Figure 10: Reference tracking - stochastic simulation

Impulse responses of nominal and perturbation model are in Figure 7. These responses are almost the same and with inexact measurement cannot be distinguished. But the resonance $\omega \approx 10$ can have essential influence on some controllers. The difference of both models is better seen in Bode diagram (see Figure 8).

For simulation, the parameter σ of the system (54) is changed from $\sigma = -1$ to $\sigma = -100$ at time $t = 20s$ and back to $\sigma = -1$ at time $t = 65s$. The probabilities $\alpha_1(t)$ and $\alpha_2(t)$ of models (55) and (56) and the estimate of system states $\widehat{x}_1(t)$ and $\widehat{x}_2(t)$ are provided by two Kalman filters in normalized form (20), (21). The LQ control for reference tracking is designed for a mixture of two models with parameters $\alpha_1 = \alpha^*$, $\alpha_2 = 1 - \alpha^*$. The criterion matrices are $Q_1 = 50$, $Q_2 = 10$ and $R = 1$.

The LQ controller based on the multiple model with different parameters, structure and dimension was designed. The reference tracking is shown in Figure 9 and Figure 10.

6 Conclusion

The design of LQ controller based on the multiple model and analysis of robustness was presented. Simulation results prove the facts which was expected - multiple model approach is more robust then single model approach. But the differences are not so significant.

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