MODELLING AND ROBUST CONTROL OF TRAFFIC SIGNAL SYSTEMS

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Abstract

This paper first proposes a model of traffic signal control systems for avoiding traffic congestion in urban road networks. The model is constructed so that it is controllable, and disturbances of traffic flow and changes of traffic situations can be taken into account. As a result, the proposed model is described as a linear system with polytopic uncertainties. Next, a robust control method is applied to the proposed model so as to achieve robust stability and performance under the uncertainties. The proposed control method gives a control law for real-time and network-wide traffic signal systems. A numerical example is given to show the effectiveness of the proposed method.

1 Introduction

Traffic signal control plays a very important role for road safety and smoothness of traffic flow. Especially, in order to avoid traffic congestion in urban road networks, real-time control of traffic signals have been studied [2, 3, 4, 7, 9, 10].

Recently, Diakaki et al. [3, 4] have proposed a model for such network-wide traffic signal control, and applied optimal linear quadratic (LQ) control to the model. This research is very promising for intelligent transportation systems. However the model and the control method proposed in [3, 4] have the following problems.

- The model to be controlled is usually uncontrollable due to an inappropriate choice of state variables, and, as a result, traffic flows are difficult to control adequately by the proposed method.
- The disturbances which naturally appear in the model are neglected when a feedback gain is designed.
- The control input calculated by the LQ method needs to be modified so that some constraints on the control input are satisfied. Therefore the resulting control input is not optimal for the LQ performance index.

In this paper, we first propose a modified model for traffic signal control that resolves the above issues. The proposed model is a linear system with polytopic uncertainties. We next apply the H^∞ control method based on linear matrix inequalities (LMIs) [1].

This paper is organized as follows: The model for traffic signal control is described in section 2, including discussion of the controllability of the model. In section 3, we apply the LMI-based robust control technique to the model. Then, a numerical example is provided in section 4, which is followed by the conclusion.

2 Model for Traffic Signal Control

The traffic signals are controlled mainly by three parameters called *cycle time*, *split*, and *offset* [4, 6, 8]. Roughly speaking, the cycle time is a period of repeated signal cycles, the split is a ratio of green time to the cycle time, and the offset is the time difference between the beginning times of green signals at multiple intersections. Another element is *lost time* which means time of yellow and all red signal. A simple example of the signal cycle is shown in Figure 1. In general, it is difficult to simultaneously optimize these control parameters, and therefore, many methods for individually optimizing each control parameter have been studied [6, 8]. In this paper, we concentrate on control of the split parameter as well as in [3, 4].

Now we describe a model for traffic signal systems. An urban road network is composed by the sets of links and intersections denoting $L = \{L_1, L_2, \ldots, L_{n_L}\}$ and $J = \{J_1, J_2, \ldots, J_{n_J}\}$, respectively. For each intersection $J_i \in J$, I_i and O_i denote the sets of incoming and outgoing links, respectively. We here make the following assumptions:

- Cycle times at all intersections are equal and fixed;

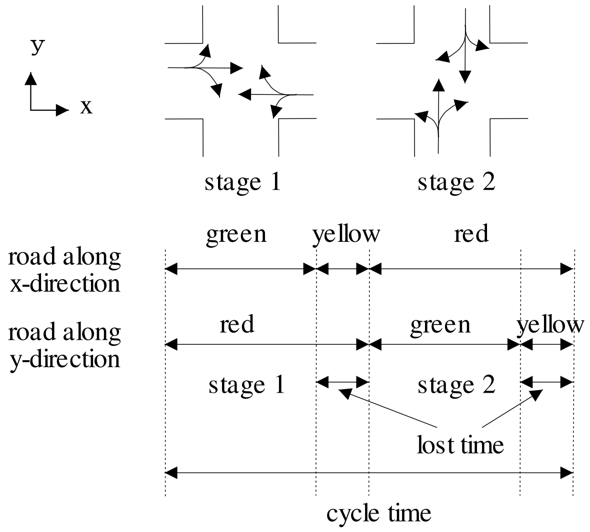

Figure 1: An example of signal cycle.

- Lost time (i.e., time of yellow and all red signal) of each intersection is zero for simplicity;
- The turning movement rates $t_{i,o}$ from $L_i \in I_j$ to $L_o \in O_j$ at intersection J_j are assumed to be known and fixed.

For simplicity, we consider the case where all roads are along x or y direction and all intersections have two signal stages as shown in Figure 1. At intersection J_i, let $g_i \in [0, 1]$ be a split (i.e., a ratio of green time to the cycle time) for the road along x direction. Then, the split for the road along y direction at the intersection is $1 - g_i$.

Remark 1 In [3, 4], each split parameter of roads along x and y directions denoted here by g_x and g_y, respectively is independently calculated by the LQ control. In implementation, however, the resulting parameters no longer satisfy $g_x + g_y = 1$. Therefore, these parameters need to be modified by solving another optimization problem subject to this constraint, and hence are not optimal in a sense of the LQ control. Note that, in order to avoid such modification, we adopt the above settings for the split parameters.

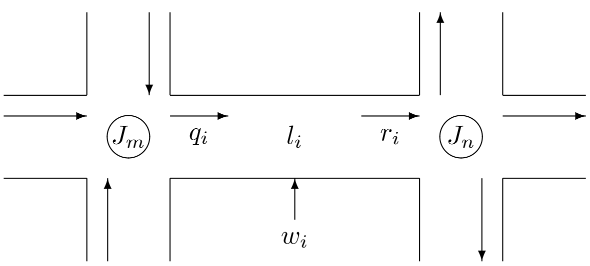

Figure 2: An urban road link.

Consider a single link L_i connecting two intersections J_m, J_n such that $L_i \in O_m$ and $L_i \in I_n$ hold as shown in Figure 2. Then, according to the store-and-forward modelling approach, the dynamics of the link is expressed by

$$l_i(k+1) = l_i(k) + q_i(k) - r_i(k) + w_i(k),$$

where l_i is the volume of traffic (i.e., the amount of vehicles) within link L_i; q_i and r_i are the inflow and outflow, respectively, of link L_i over the period $[kT, (k+1)T]$ with the control interval T which is assumed to be equal to the cycle time; and w_i is a disturbance within link L_i such as demand and exit flows. Moreover we make the following assumptions:

- The inflow to link L_i is given by $q_i(k) = \sum_{L_j \in I_m} t_{j,i} r_j(k)$;
- l_i is sufficiently large, and therefore, the outflow r_i of link L_i is given by

$$r_i = \begin{cases} p_i g_m, & L_i \in L \text{ is a road along } x \text{ direction} \\ p_i(1 - g_m), & L_i \in L \text{ is a road along } y \text{ direction} \end{cases}$$

  where p_i is the saturation flow that is the product of the saturation flow rate and the cycle time T;
- When $w_i = 0$ for all $L_i \in L$, nominal green times g_i^N at all intersections $J_i, i = 1, \ldots, n_J$ that lead to a steady-state system with equilibrium points l_i^N for $L_i \in L$ are available.

To construct a controllable system, we must carefully determine state variables from the links. To this end, we should not choose the origin and destination links as state variables because these links are connected with single intersections and thus are incomplete. Now let $\tilde{L}(\subset L)$ denote a set of links chosen as state variables. Then, for link $L_i \in \tilde{L}$ connecting two intersections as in Figure 2, the dynamics of link $L_i \in \tilde{L}$ along x direction is described as

$$l_i(k+1) = l_i(k) + \sum_{L_j \in I_m} t_{j,i} r_j(k) - p_i g_n(k) + w_i(k).$$

Note that the dynamics of links along y direction are also described in the similar manner. Arranging this equation, we have

$$\begin{aligned} l_i(k+1) &= l_i(k) + \sum_{L_j \in I_m} t_{j,i} s_j p_j g_m(k) \\ &\quad - p_i g_n(k) + w_i(k) + c, \end{aligned} \quad (1)$$

where

$$s_j = \begin{cases} 1, & L_j \text{ is a road along } x \text{ direction} \\ -1, & L_j \text{ is a road along } y \text{ direction} \end{cases}$$

and c is a constant. Since this system has an equilibrium point from the assumption, it follows that

$$l_i^N = l_i^N + \sum_{L_j \in I_m} t_{j,i} s_j p_j g_m^N - p_i g_n^N + c. \quad (2)$$

Subtracting (2) from (1), we have

$$\Delta l_i(k+1) = \Delta l_i(k) + \sum_{L_j \in I_m} t_{j,i} s_j p_j \Delta g_m(k) - p_i \Delta g_n(k) + w_i(k), \quad (3)$$

where $\Delta l_i(k) := l_i(k) - l_i^N$, $\Delta g_m(k) := g_m(k) - g_m^N$, $\Delta g_n(k) := g_n(k) - g_n^N$. Let $n_{\tilde{L}}$ be the number of links chosen as the state variables; $x \in \Re^{n_{\tilde{L}}}$ be the vector of Δl_i for $L_i \in \tilde{L}$; $u \in \Re^{n_J}$ be the vector of Δg_j for $J_j \in J$; and $w \in \Re^{n_{\tilde{L}}}$ be the vector of w_i for $L_i \in \tilde{L}$. Then (3) is summarized as follows:

$$x(k+1) = x(k) + Bu(k) + w(k), \quad (4)$$

where $B \in \Re^{n_{\tilde{L}} \times n_J}$ is a matrix linearly including the saturation flow parameters p_i for $L_i \in \tilde{L}$.

It is known that these saturation flow parameters are changeable due to traffic conditions, and some of them might not be ignorable. To cope with the parameter variation, we introduce uncertainties for p_i. Let $\hat{L}$ be a set of links with such changeable saturation flows, and $n_{\hat{L}}$ be the number of elements of $\hat{L}$, and define p_i for $L_i \in \hat{L}$ as $p_i \in [p_i^{\mathrm{lb}}, p_i^{\mathrm{ub}}]$. Then we obtain

$$B(k) \in \mathbf{Co}\{B_1, B_2, \ldots, B_{2^{n_{\hat{L}}}}\},$$

where $\mathbf{Co}\{B_1, B_2, \ldots, B_{2^{n_{\hat{L}}}}\}$ denotes the convex hull of the constant matrices $B_i, i = 1, \ldots, 2^{n_{\hat{L}}}$ corresponding to the extreme points of the parameter space of the saturation flows with uncertainties. Namely, B belongs to a matrix polytope. To sum up, we obtain the following traffic signal system:

$$\begin{aligned} x(k+1) &= x(k) + B(k)u(k) + w(k), \\ B(k) &\in \mathbf{Co}\{B_1, B_2, \ldots, B_{2^{n_{\hat{L}}}}\}. \end{aligned} \quad (5)$$

Remark 2 In [3, 4], the disturbance w are finally ignored and the uncertainty in the saturation flow is not taken account. We adopt a control method such that these important factors can be appropriately dealt with in the next section.

When we design a control system based on its model, controllability and observability of the model are important properties to be checked. In our case, x is assumed to be measurable, and hence, the observability of system (5) is satisfied. However, the controllability of system (5) is not always satisfied. Related to this, the following proposition holds for system (5).

Proposition 1 *System (5) is controllable if and only if* $\mathbf{rank}\, B(k) = n_{\tilde{L}}$ *for all* $k \geq 0$.

Proof: It is well-known that a linear system given by

$$x(k+1) = Ax(k) + Bu(k), \quad x \in \Re^{n_{\tilde{L}}}, u \in \Re^{n_J}$$

is controllable if and only if the controllability matrix

$$U := [B, AB, \ldots, A^{n-1}B]$$

has full row rank. Since $A = I$ in this case, the proof is obvious. ■

We see from Proposition 1 that $n_{\tilde{L}} \leq n_J$ is a necessary condition for controllability of system (5).

Remark 3 In the conventional approach [4], all links $L_i \in L$ are chosen as state variables, and then, the number of links is usually larger than that of intersections, i.e., $n_{\tilde{L}} > n_J$. Therefore, the system proposed in [4] is usually uncontrollable. As mentioned above, it is important to choose appropriate state variables so that the resulting system is controllable.

3 H^∞ Control Based on the Proposed Model

We assume that the state variables are observable as well as in [3, 4]. Then our aim is to seek a state-feedback $u(k) = Kx(k)$ such that the closed-loop system is stable and satisfies desired properties. Model (5) is a linear system with polytopic uncertainties, and therefore, various control methods are known to be applicable. We here present a control design method via LMIs [1]. From now on, note that "$P > 0$" means that a symmetric matrix P is positive definite.

3.1 Quadratic stabilizability

The closed-loop system with a state-feedback $u(k) = Kx(k)$ is

$$\begin{aligned} x(k+1) &= (I + B(k)K)x(k) + w(k), \\ B(k) &\in \mathbf{Co}\{B_1, B_2, \ldots, B_{2^{n_{\hat{L}}}}\}. \end{aligned} \quad (6)$$

Stabilizability, i.e., a convergence property of all trajectories of system (6), is the most fundamental requirement for constructing the closed-loop system. A sufficient condition for this is quadratic stabilizability defined as follows [1].

Definition 1 System (5) is said to be *quadratically stabilizable* if there exist a state-feedback gain K and a quadratic function $V(\xi) = \xi^T P \xi, P > 0$ that decreases along every nonzero trajectory of (6).

A necessary and sufficient LMI condition for quadratic stabilizability of a continuous-time system is derived in [1]. An LMI condition for the discrete-time system such as (5) can be easily derived as well as in [1].

Proposition 2 *System (5) is quadratically stabilizable if and only if there exist matrices* $Q = Q^T$ *and* Y *satisfying*

$$\begin{bmatrix} Q & Y^T B_i^T + Q \\ Q + B_i Y & Q \end{bmatrix} > 0, \quad i = 1, \ldots, 2^{n_{\hat{L}}}. \quad (7)$$

Then, $K = YQ^{-1}$ *is a quadratically stabilizable state-feedback gain.*

The condition (7) is an LMI of Q and Y. Therefore the feasibility problem to seek Q and Y satisfying (7) is a semidefinite programming problem, and is efficiently solvable by recently developed tools [5, 11, 12].

3.2 H^∞ performance

To evaluate control performance, we introduce the control output shown by

$$\begin{aligned} z(k) &= Cx(k) + Du(k) \\ &= (C + DK)x(k). \end{aligned} \quad (8)$$

We seek a state-feedback gain K such that the H^∞ performance

$$\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2} < \gamma \quad (9)$$

is satisfied, where γ is a specified number and $\|\cdot\|_2$ is the l^2 norm defined by $\|v(k)\|_2^2 := \sum_{k=0}^{\infty} v(k)^T v(k)$. Since the left side in (9) equals the so-called H^∞ norm of the closed-loop system (6), (8) with a constant B, the above property is called the H^∞ performance for a changeable $B(k)$. A sufficient LMI condition for the H^∞ performance is derived as well as in [1].

Proposition 3 *For a given γ, if there exist matrices $Q = Q^T$ and Y satisfying*

$$\begin{bmatrix} Q & Q + Y^T B_{u,i}^T & 0 & QC^T + Y^T D^T \\ Q + B_i Y & Q & I & 0 \\ 0 & I & \gamma^2 I & 0 \\ CQ + DY & 0 & 0 & I \end{bmatrix} > 0, \quad i = 1, \ldots, 2^{n_{\tilde{L}}}, \quad (10)$$

then the closed-loop system (6), (8) with $K = YQ^{-1}$ gives the H^∞ performance.

The condition (10) is also an LMI of Q, Y and γ^2. Therefore, by solving the optimizatoin problem of minimizing γ^2 subject to (10), we can design a state-feedback gain $K = Y_{\mathrm{opt}} Q_{\mathrm{opt}}^{-1}$ where Y_{opt} and Q_{opt} are the optimal solutions. By means of the H^∞ control, we can reduce the effect of the disturbances in the closed-loop system. Here, note that LMI (10) includes LMI (7), and thus, the closed-loop system constructed by solving (10) is guaranteed to be quadratically stable.

Remark 4 We can evaluate the LQ performance by taking C and D with symmetric matrices $\bar{Q} \in \Re^{n_{\tilde{L}} \times n_{\tilde{L}}}$, $\bar{R} \in \Re^{n_J \times n_J}$ as follows.

$$\begin{aligned} C &:= \begin{bmatrix} \bar{Q}^{1/2} \\ 0 \end{bmatrix}, \quad \bar{Q} > 0 \\ D &:= \begin{bmatrix} 0 \\ \bar{R}^{1/2}, \end{bmatrix}, \quad \bar{R} > 0. \end{aligned}$$

Then we see that $\|z(k)\|_2^2 = \sum_{k=0}^{\infty} z(k)^T z(k)$ gives an LQ performance index since

$$z(k)^T z(k) = x(k)^T \bar{Q} x(k) + u(k)^T \bar{R} u(k).$$

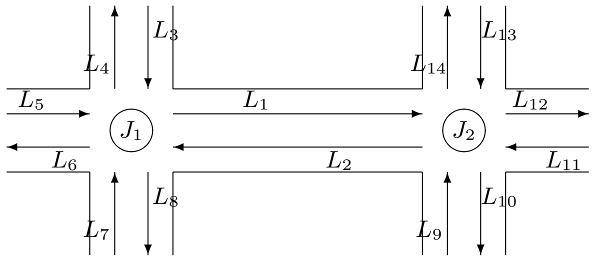

Figure 3: A simple traffic network.

4 Numerical Example

Figure 3 shows a simple traffic network composed by the sets of two intersections and 14 links denoting $J = \{J_1, J_2\}$ and $L = \{L_1, L_2, \ldots, L_{14}\}$, respectively. The incoming links and the outgoing links are as follows:

$$\begin{aligned} I_1 &= \{L_2, L_3, L_5, L_7\} \\ I_2 &= \{L_1, L_9, L_{11}, L_{13}\} \\ O_1 &= \{L_1, L_4, L_6, L_8\} \\ O_2 &= \{L_2, L_{10}, L_{12}, L_{14}\}. \end{aligned}$$

We must to set state variables so that the control system is controllable. Here we choose the numbers of vehicles in links 1 and 2 as the state variables, i.e., $\tilde{L} = \{L_1, L_2\}$. Let the data of the (nominal) saturation flows and the turning movement rates be

$$\begin{aligned} & p_1 = 50, p_2 = 50, p_3 = 40, p_5 = 40, \\ & p_7 = 40, p_9 = 40, p_{11} = 40, p_{13} = 40, \\ & t_{3,1} = 0.25, t_{5,1} = 0.75, t_{7,1} = 0.25, \\ & t_{9,2} = 0.25, t_{11,2} = 0.75, t_{13,2} = 0.25. \end{aligned}$$

We assume that the saturation flows of link L_1 is changeable in this example as follows:

$$p_1 \in [40, 60].$$

Also let $g_i^N (i = 1, 2)$ be the nominal split parameters of the roads along x direction $L_1, L_2, L_5, L_6, L_{11}, L_{12}$.

For the above settings, we summarize the dynamics of links L_1 and L_2 as follows:

$$\begin{aligned} l_1(k+1) &= l_1(k) + 10g_1 - 50g_2 + w_1 + 20 \\ l_2(k+1) &= l_2(k) + 10g_2 - p_1 g_1 + w_2 + 20. \end{aligned} \quad (11)$$

We can see that system (11) has an equilibrium point, i.e., there exist $g_i^N \in [0, 1]$ $(i = 1, 2)$ for $w_i = 0$ $(i = 1, 2)$ and $p_1 \in [40, 60]$. Then, the following equations hold for $p_1 \in [40, 60]$.

$$\begin{aligned} l_1^N &= l_1^N + 10g_1^N - 50g_2^N + 20 \\ l_2^N &= l_2^N + 10g_2^N - p_1 g_1^N + 20. \end{aligned} \quad (12)$$

Subtracting (12) from (11), and denoting $x = [l_1 - l_1^N, l_2 - l_2^N]^T$ and $u = [g_1 - g_1^N, g_2 - g_2^N]^T$, we have the following system:

$$x(k+1) = x(k) + Bu(k) + w(k),$$

$$B \in \mathbf{Co}\left\{ \begin{bmatrix} 10 & -50 \\ -40 & 10 \end{bmatrix}, \begin{bmatrix} 10 & -50 \\ -60 & 10 \end{bmatrix} \right\}. \quad (13)$$

Note that B is full rank for all $p_1 \in [40, 60]$, and thus, this system is controllable.

For system (13), we seek a state-feedback control $u = Kx$ using the proposed method. We set the parameters $\bar{Q} = I$, $\bar{R} = 10000I$. All computation was run on MATLAB.

We first consider the case for reducing the current numbers of vehicles in links L_1 and L_2 by 20 and 5, respectively, under $w(k) \equiv 0$. To this end, we set the initial state $x(0) = [20, 5]^T$. Figures 4–6 show the transient responses of the numbers of vehicles and the split parameters for the saturation flows $p_1 = 40, 50, 60$, respectively. It is seen from these figures that the numbers of vehicles in L_1 and L_2 are reduced by 20 and 5, respectively, within about 5 cycles. Also, the split parameters g_1, g_2 are reasonable because they are within $[0.35, 0.83]$.

To demonstrate control performance against w, we next compare the proposed method with the fixed split parameters, i.e., the case without control. For uniform random disturbances $w_i(k), i = 1, 2$ within the range $[-5, 5]$ shown in Figure 7, the numbers of vehicles and the split parameters by the proposed method are shown in Figure 8. We see from this figure that these values are controlled reasonably. In contrast, Figure 9 shows the numbers of vehicles by the fixed split parameters $g_i = 0.5, i = 1, 2$.

5 Conclusion

In this paper, we have proposed a new model for traffic signal control by modifying the model in [3, 4]. We also have shown that the way of choice state variables from links is important from the viewpoint of the controllability of the model. Then we have presented the H^∞ control design method by means of LMIs, and shown its effectiveness in the simple numerical example. One of topics for further research is to evaluate the proposed control method in more realistic traffic simulation systems.

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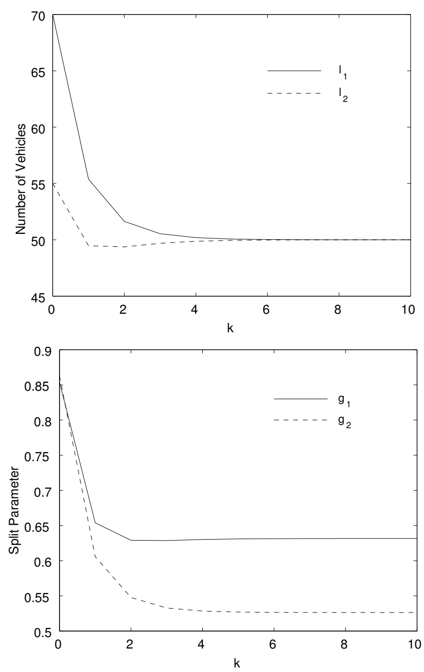

Figure 4: The numbers of vehicles and the split parameters for $p_1 = 40$; l_1 and g_1 (solid); l_2 and g_2 (dashed).

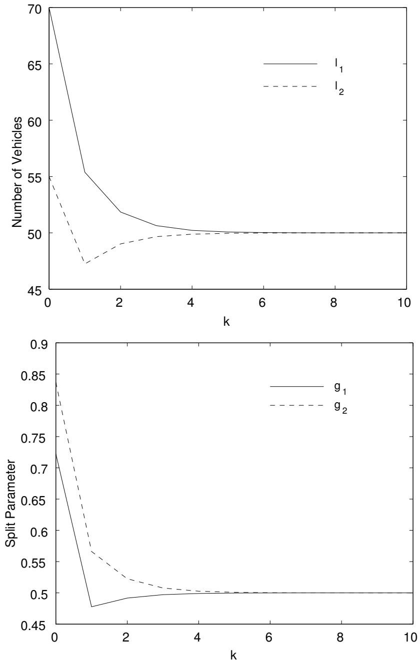

Figure 5: The numbers of vehicles and the split parameters for $p_1 = 50$; l_1 and g_1 (solid); l_2 and g_2 (dashed).

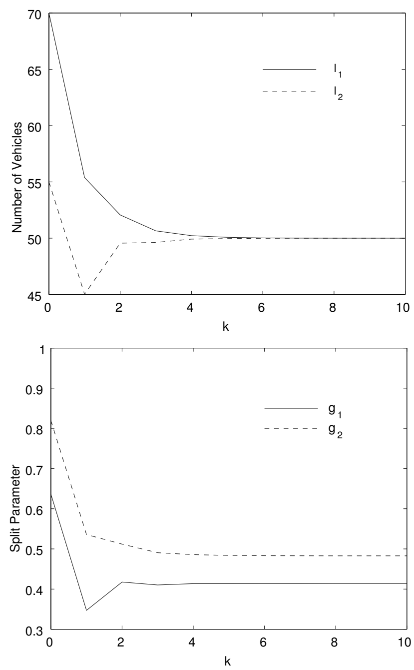

Figure 6: The numbers of vehicles and the split parameters for $p_1 = 60$; l_1 and g_1 (solid); l_2 and g_2 (dashed).

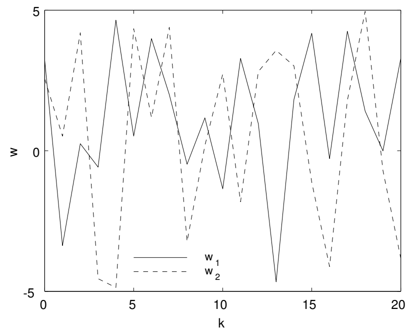

Figure 7: Disturbance w; w_1 (solid); w_2 (dashed).

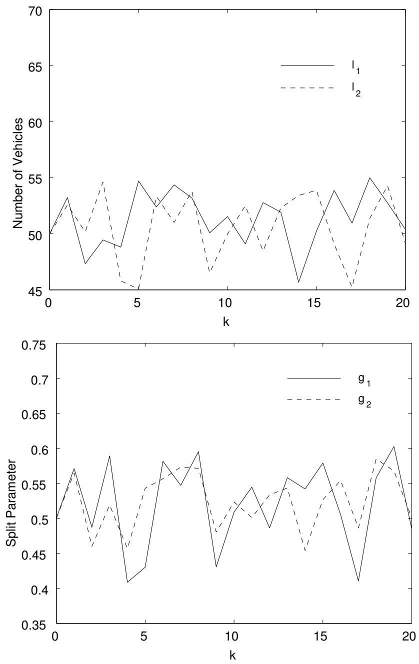

Figure 8: The numbers of vehicles and the split parameters with control; l_1 and g_1 (solid); l_2 and g_2 (dashed).

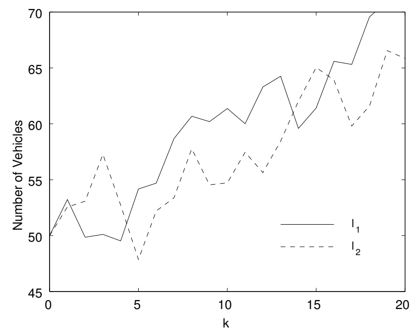

Figure 9: The numbers of vehicles and the split parameters without control; l_1 (solid); l_2 (dashed).