ROBUST CONTROL DESIGN OF LINEAR SYSTEMS WITH POLYTOPIC TIME-VARYING UNCERTAINTY: AN ITERATIVE SDP APPROACH

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Abstract

This paper considers the nonlinear optimization problems arising in robust control synthesis for discrete linear systems with polytopic time-varying uncertainty. Here a linear objective function is minimized under nonlinear matrix-valued function constraints. An iterative semidefinite programming (SDP) method is used to this class of problems. The benefit of the new approach is that the sub-problems in the iterative SDP are all based on linear matrix inequalities (LMIs) and can be solved efficiently. Iterative SDP based algorithms are then proposed for the robust stabilization and optimal control problems, respectively. A numerical example is used to compare the SDP and LMI based design which shows that the SDP one is less conservative.

1 Introduction

Robust control synthesis problems for uncertain linear systems have been widely studied in the last 15 years. Recently a new linear matrix inequality (LMI) [1, 6] based criterion was proposed for discrete linear systems with polytopic uncertainty [3]. Here

$$x(k+1) = A(k)\,x(k) \tag{1}$$

where $A(k)$ is an uncertain matrix and belongs to a convex cone, i.e., at any time instant k, it can be expressed as

$$A(k) = \sum_{i=1}^{l} \mu_i(k)\,A_i$$

The parameter vector $\mu_i(k)^T = [\mu_1(k)\ \mu_2(k) ... \mu_l(k)]$ is a function of uncertainties. Using this criterion results in an LMI problem, the feasibility of which guarantees the stability of the system given in equ. (1) (See Section 2 for detail). Simultaneously a parameter-varying Lyapunov function is obtained from the optimization result. The quadratic Lyapunov function, which is widely used in robust control [5], can be regarded as a special case of parameter-varying one. Thus the design based on the new criterion is more flexible and less conservative.

However, applying this criterion in the controller design will result in a non-LMI problem of the form,

$$\begin{aligned}&\min d^T x\\&\text{subject to } \mathcal{A}(x) < 0\end{aligned}$$

where d is a given vector, x is a vector containing all the decision variables, $\mathcal{A}(x)$ is a nonlinear matrix function and < 0 means negative definite. Note that the constraint in this optimization problem is nonlinear and in general it cannot be converted into LMIs, which means it may not be solved using the current LMI packages.

Iterative semidefinte programming (SDP) is a commonly used approach in the BMI optimization problem. An optimal solution can be found by solving a series of linear optimization sub-problems, which are constructed from linearizations of the objective function and constraints of the original nonlinear problem. In the iterative SDP method, one importance issue is to calculate the appropriate move limit for each sub-problem [2]. If the move limit is too large, the optimization procedure may not converge. If it is too small, the convergence will be very slow.

In this paper, two new iterative SDP algorithms are developed for robust control designs. By solving some linear sub-problems, a state feedback controller is found to robustly stabilize the uncertain system. A controller is then optimized to minimize the given objective function for the specified initial condition. A major difference from the general iterative SDP method is that the move limit is not computed for each sub-problem. Instead, it is incorporated into the objective function so that convergence of the move limit can be achieved.

The paper is organized as follows: the problems are formulated in Section 2. In Section 3 the nonlinear optimization problems are established for the two robust controller design problems of interest. Then two iterative SDP algorithms to provide numerical solution are proposed in Section 4. In Section 5 a numerical example illustrate the efficiency and the advantages of the new approaches. Section 6 concludes the paper.

2 Preliminaries

Consider the discrete uncertain linear dynamic system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{2}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $u(k) \in \mathbb{R}^{n_u}$ the control signal and $y(k) \in \mathbb{R}^{n_y}$ the output. The time-varying uncertain matrix triple $[A(k) \mid B(k)]$ belongs to a polytopic cone,

$$\begin{aligned}&[A(k) \mid B(k)] \in \\ &\mathbf{Co}\{[A_1 \mid B_1], [A_2 \mid B_2], ..., [A_l \mid B_l]\}\end{aligned} \tag{3}$$

At any time instant k, the matrices in (2) can be expressed as,

$$[A(k) \mid B(k)] = \sum_{i=1}^{l} \mu_i(k)[A_i \mid B_i] \tag{4}$$

where the uncertain parameter

$$\mu(k) = [\mu_1(k), \mu_2(k), ...\mu_l(k)]$$

satisfies the condition

$$\sum_{i=1}^{l} \mu_i(k) = 1 \;\; \mu_i(k) > 0 \qquad \forall i = 1, 2, ..., l \tag{5}$$

To stabilize the uncertain linear system in equ. (2), a robust state feedback controller

$$u(k) = Fx(k) \tag{6}$$

is designed, where $F \in \mathbb{R}^{n_x}$ is the feedback gain. The optimal control problem then involves choosing a state feedback controller so that the following objective function is minimized.

$$J_\infty = \sum_{i=0}^{\infty}\left(\|x(k)\|_{Q_1}^2 + \|u(k)\|_R^2\right) \tag{7}$$

where $Q_1 \geq 0$ and $R > 0$ are suitable weight matrices.

In both cases, the closed-loop systems are in the form of (1). To begin, the stability criterion proposed in [3] is included here for completeness.

Theorem 1 *[3] The uncertain linear system (1) is stable if there exist matrices G_i and $Q_i > 0$ such that*

$$\begin{bmatrix} G_i + G_i^T - Q_i & * \\ A_i G_i & Q_j \end{bmatrix} > 0 \qquad \forall i, j = 1, 2, ..., l \tag{8}$$

and the Lyapunov function is given as

$$V(x(k), k) = x^T \sum_{i=1}^{l} \mu_i(k) Q_i^{-1} x \tag{9}$$

Note that the using this criterion gives a parameter-varying Lyapunov function (9) depending on the uncertain time-varying parameter $\mu(k)$. Clearly this is more general than the quadratic Lyapunov function $V(x) = x^T P x$, which is a special case of (9) by letting matrices $Q_i = Q$ be constant and choosing matrix P to be $P = Q^{-1}$.

3 Controller designs

3.1 Robust controller

Suppose F to be the robust state feedback gain matrix for the system (2). The closed-loop system then can be written as

$$x(k+1) = [A(k) + B(k)F]x(k) \tag{10}$$

Clearly, from (4) at any time instant k, the matrix $A(k) + B(k)F$ satisfies,

$$A(k) + B(k)F = \sum_{i=1}^{l} \mu_i(k)(A_i + B_i F) \tag{11}$$

According to Theorem 1, the following lemma is then obtained.

Lemma 2 *The uncertain system (2) is robustly stabilized by the state feedback controller (6) if there exist matrices $G_i \in \mathbb{R}^{n_x \times n_x}$, $Q_i > 0 \in \mathbb{R}^{n_x \times n_x}$ such that the follow inequalities hold.*

$$\begin{bmatrix} G_i + G_i^T - Q_i & * \\ (A_i + B_i F) G_i & Q_j \end{bmatrix} > 0 \qquad \forall i, j = 1, 2, ..., l \tag{12}$$

and the Lyapunov function is given in (9).

Proof. The result can be directly achieved by applying Theorem 1 to the closed-loop system (10). ■

Remark 1 *In the case that $G_i = G, i = 1, 2, ...l$ are constant, letting $FG = Y$, the inequalities (12) are linear, which can be solved by using any LMI packages. Obviously, this is more conservative than solving (12) directly.*

Remark 2 *Solving inequalities (12) gives a state feedback controller for stabilizing the time-varying uncertain system (2). If the uncertain system is time independent, the inequalities (12) simplify to,*

$$\begin{bmatrix} G_i + G_i^T - Q_i & * \\ (A_i + B_i F) G_i & Q_i \end{bmatrix} > 0 \qquad \forall i = 1, 2, ..., l \tag{13}$$

Letting matrices $G_i = G$ be constant, from using the same approach as in Remark 1 these inequalities can be recast as LMIs and used in robust controller design for linear systems with time-invariant uncertainty [4].

3.2 Optimal robust controller

As pointed out in Lemma 2, the controller that satisfies the inequalities (12) robustly stabilizes the uncertain system (2), from which a Lyapunov function is found for the closed-loop system. To find an *optimal* state feedback controller which minimizes the objective function in equ. (7), suppose further that the Lyapunov function satisfies that,

$$V\left(x\left(k+1\right),k+1\right)-V\left(x\left(k\right),k\right)< \\ -\left(\left\|x\left(k\right)\right\|_{Q_1}^2+\left\|u\left(k\right)\right\|_R^2\right) \tag{14}$$

Summing (14) from $k=0$ to ∞, because $x\left(\infty\right)=0$, it follows that

$$V\left(x\left(0\right),0\right)>J_{\infty} \tag{15}$$

Thus the optimal robust control problem is now re-casted to as one of finding an upper bound $V\left(x\left(0\right),0\right)$ for the objective function (7).

Theorem 3 *The optimal robust feedback gain F can be obtained by solving the following optimization problem (if it exists).*

$$\min_{Q_i,G_i,F}\gamma$$

subject to

$$\begin{bmatrix} 1 & * \\ x\left(0\right) & Q_i \end{bmatrix}>0 \tag{16}$$

and

$$\begin{bmatrix} G_i+G_i^T-Q_i & * & * & * \\ \left[A_i+B_iF\right]G_i & Q_j & * & * \\ Q_1^{1/2}G_i & 0 & \gamma I & * \\ R^{1/2}FG_i & 0 & 0 & \gamma I \end{bmatrix}>0 \\ \forall i,j=1,2,...,l \tag{17}$$

Proof. See Appendix A. ■

Similarly, letting $G_i=G$ be constant, optimization problem in Theorem 3 is an LMI problem and can be solved using any LMI packages.

4 Iterative semidefinite programming

Note that in the constraints (12) and (17), the inequalities are not linear for the optimization variables F and G_i. The design problems of Section 3 thus cannot therefore be solved by directly applying the current LMI packages.

Iterative semidefinte programming is one of the most straightforward nonlinear optimization methods. The basic idea is to recursively solve a series of linearly approximated sub-problems, where each intermediate solution is the starting point for the subsequent sub-problem. The linear part of the Taylor series expansion is generally adopted as the approximation to the nonlinear problem. A similar strategy is employed here whereby the nonlinear matrix inequality is decomposed into a sequence LMIs, solving which gives an optimal solution for the original nonlinear problem.

4.1 Robust controller design via SLP

By linearizing the nonlinear inequalities, the kth sub-problem (12) thus can be formulated as,

$$\min_{\triangle G_{i,k},\triangle Q_{i,k},\triangle F_k}\sum_{i=1}^{l}\lambda_k\left(\left\|\triangle G_{i,k}\right\|+\left\|\triangle Q_{i,k}\right\|\right) \tag{18}$$

subject to

$$\begin{bmatrix} G_{i,k}+G_{i,k}^T-Q_{i,k} & * \\ \left(A_i+B_iF_k\right)G_{i,k} & Q_{j,k} \end{bmatrix}+ \\ \begin{bmatrix} \triangle G_{i,k}+\triangle G_{i,k}^T-\triangle Q_{i,k} & * \\ \left(A_i+B_iF_k\right)\triangle G_{i,k}+B_i\triangle F_kG_{i,k} & \triangle Q_{j,k} \end{bmatrix} \\ >0 \\ \forall i,j=1,2,...,l \tag{19}$$

where F_k, G_{ik} and Q_{ik} are the kth starting points. The coefficient λ_k is added here to force the sub-problems to converge to zero quickly. The sequence $\{\lambda_k\}$ can be chosen as $\lim_{k\to\infty}\lambda_k=\infty$. The $(k+1)$th iterate is revised with respect to the solution of the kth sub-problem as follows.

$$\begin{aligned} F_{k+1}&=F_k+\triangle F_k \\ G_{i,k+1}&=G_{i,k}+\triangle G_{i,k} \\ Q_{i,k+1}&=Q_{i,k}+\triangle Q_{i,k} \\ &\forall i=1,2,...,l \end{aligned} \tag{20}$$

From equ. (18) it can be seen that, if the problem is feasible, the differences $\triangle F_k, \triangle G_{i,k}$ and $\triangle Q_{i,k}$ will tend to zero. Consequently a feasible solution is found for the nonlinear problem described by (12).

The kth sub-problem can be re-cast as an LMI one. Note the constraints (19) are already LMIs, and the norm optimization objective (18) is equivalent to the following problem,

$$\min_{\triangle G_{ik},\triangle Q_{ik},\triangle F_k}\sum_{i=1}^{l}\lambda_k\left(g_{i,k}+q_{i,k}\right) \tag{21}$$

subject to

$$\begin{bmatrix} g_{i,k}I & * \\ \triangle G_{i,k} & g_{i,k}I \end{bmatrix}>0 \\ \begin{bmatrix} q_{i,k}I & * \\ \triangle Q_{i,k} & q_{i,k}I \end{bmatrix}>0 \\ \forall i=1,2,...,l \tag{22}$$

In this way each sub-problem can be solved efficiently using any LMI toolbox packages.

Notice that the move of the feedback gain $\triangle F_k$ is not included in the optimization objective (18). According to our numerical experiments, if $\triangle F_k$ is included, the sequence of the differences $\{\triangle F_k\},\{\triangle G_{i,k}\}$ and $\{\triangle Q_{i,k}\}$ may not converge. Actually from the constraints (19), it can be seen that if $\triangle G_{i,k}$ and $\triangle Q_{i,k}$ tend to zero, the norm bound of $\triangle G_{i,k}$ and $\triangle Q_{i,k}$ will force the sequence $\{\triangle F_k\}$ to converge to zero as well.

In addition, different technique from the general linear programming is used to compute the move limit. Due to the requirement that the matrices on the left side of inequality constraints must be positive definite, applying the move limit always results in a non-feasible LMI sub-problem if the starting point is far away from the feasible solution. In the new approach, the norms of difference matrices $\triangle G_{i,k}$ and $\triangle Q_{i,k}$ are added to the objective function, which forces these differences tend to zero. When these norms are small enough, a group of feasible solution must be found.

Summarizing, the algorithm for the design of the robust controller can be described as follows.

Algorithm 1

---

1. Initialize the matrices $F_1, G_{i,1}$ and $Q_{i,1}$ with random values and set $k = 1$.
2. Solve the kth LMI optimization problem (21) subject to the constraints (19) and (22). If the optimization problem is not feasible, then the controller design problem cannot be solved by the SLP algorithm. The iteration is stopped.
3. Update the starting point using (20) and set $k = k + 1$.
4. If the inequalities (12) are satisfied for the new values of $F_k, G_{i,k}$ and $Q_{i,k}$, a feasible solution is found. Otherwise go to step 2 and continue.

4.2 Optimal controller design via iterative SDP

Consider the nonlinear optimization problem in Theorem 3. As before, the kth sub-problem can be formulated as follows.

$$\min_{\triangle G_{ik},\triangle Q_{ik},\triangle F_k,\triangle\gamma_k} \sum_{i=1}^{l} \lambda_k \left(g_{i,k} + q_{i,k}\right) + \gamma_{dk} \qquad (23)$$

subject to

$$\begin{bmatrix} 1 & * \\ x(0) & Q_{i,k} + \triangle Q_{i,k} \end{bmatrix} > 0 \qquad (24)$$

$$\begin{bmatrix} G_{i,k} + G_{i,k}^T - Q_{i,k} + \triangle G_{i,k} + \triangle G_{i,k}^T - \triangle Q_{i,k} & * & * & * \\ \left[A_i + B_i F_k\right]\left(G_i + \triangle G_i\right) + B_i \triangle F_k G_i & Q_{j,k} + \triangle Q_{j,k} & * & * \\ Q_1^{1/2}\left(G_{i,k} + \triangle G_{i,k}\right) & 0 & \left(\gamma_k + \triangle\gamma_k\right) I & * \\ R^{1/2}\left(F_k G_{i,k} + \triangle F_k G_{i,k} + F_k \triangle G_{i,k}\right) & 0 & 0 & \left(\gamma_{k+}\triangle\gamma_k\right) I \end{bmatrix} > 0 \qquad (25)$$

$$\begin{bmatrix} \gamma_{dk} & * \\ \triangle\gamma_k & \gamma_{dk} \end{bmatrix} > 0 \qquad (26)$$

and inequalities (22). The inequalities (24) and (25) are from the linearization of (16) and (17), respectively, and the inequality (26) gives the norm bound on $\triangle\gamma_k$.

A feasible solution is found, by solving this optimization problem. However the value of γ may not optimal. To obtain an optimal γ, the objective function is adapted to.

$$\min_{\triangle G_{ik},\triangle Q_{ik},\triangle F_k,\triangle\gamma_k} \sum_{i=1}^{l} \lambda_k \left(g_{i,k} + q_{i,k}\right) + \gamma_{dk} + \triangle\gamma_k \qquad (27)$$

An optimal γ can then be found by minimizing the value of $\triangle\gamma_k$. The algorithm for the design of the optimal robust controller is then described as follows:

Algorithm 2

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1. Initialize the matrices $\gamma_1, F_1, G_{i,1}$ and $Q_{i,1}$ with random values and set $k = 1$.
2. Set the initial optimization objective to (23).
3. Solve the kth LMI optimization problem subject to the constraints (24,25,26, 22). If the optimization problem is not feasible, then stop the iteration.
4. Update $F_{k+1}, G_{i,k+1}$ and $Q_{i,k+1}$ using (20) and set $\gamma_{k+1} = \gamma_k + \triangle\gamma_k$, the set $k = k + 1$.
5. If the inequalities (16) and (17) are not satisfied for the new values of $F_k, G_{i,k}$ and $Q_{i,k}$, go to step 3 and continue. Otherwise, if the optimization objective is (27), then an optimal solution is found.
6. Change the optimization objective to (27) in kth sub-problem and go to step 3.

Note that this algorithm actually contains two steps. The first step is to find a feasible solution for the objective function (23). If it is found, the objective function is then changed to (27) to optimize the value of γ.

It should be noted that the solution obtained by the iterative SDP may not be globally optimal due to the nonlinearity of the optimization problems. Thus the initial values are quite important since different initial values may give different optimal solutions. An alternative for finding an appropriate group of initial values is to solve the simplified LMI problems. By setting the matrices $G_i = G$ to be constant, both inequality constraints (12) and (17) can be cast as LMIs. If they are feasible, a group of initial values is obtained by solving LMIs and can then subsequently be used in iterative SDP.

5 Numerical example

This example is taken from ([7]) and consists of a two-mass-spring system as shown in Figure (1).

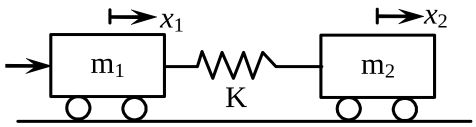

Figure 1: Coupled spring-mass system

Using Euler's first-order approximation for the derivative and a sampling time of $0.1s$, the following discrete-time state-space equations are obtained by discretizing the continuous-time equations of the system.

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -\frac{0.1K}{m_1} & \frac{0.1K}{m_1} & 1 & 0 \\ \frac{0.1K}{m_2} & -\frac{0.1K}{m_2} & 0 & 1 \end{bmatrix} x(k) \\ &+ \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix} u(k) \qquad (28) \\ y(k) &= x_2(k) \end{aligned}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ is system state, x_1, x_2 are the positions of body 1 and 2, and x_3, x_4 are their respective velocities. Also, m_1 and m_2 are the masses of the two bodies and K is the spring constant.

Here it is assumed that the mass m_2 and the spring constant K are uncertain such that

$$m_2 \in [0.1, 1], K \in [0.5, 10]$$

By applying Algorithm 1, a robust state feedback gain can be quickly found. The sequence of norms of the optimization variables is listed in Table 1.

k	**norm**
1	4.5157×10^5
2	1.2803×10^5
3	16.7759
4	0.0928
5	0.0081
6	7.2270×10^{-13}

Table 1: Norms of optimization variables in each sub-problems

From these values it can be seen that the iterative SDP method converges rapidly since the final solution is obtained after solving 6 sub-problems

To design the robust optimal controller, both the LMI based approach, in which the G matrix is set to be constant, and the iterative SDP algorithm are used. The control objective is to regulate the uncertain system from initial condition to origin with the matrices in objective function (7) given as

$$Q_1 = I, R = 1$$

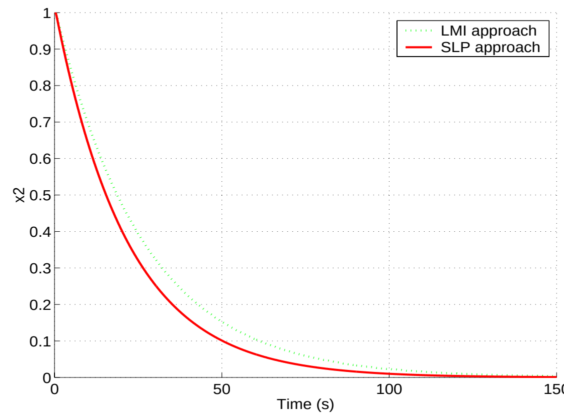

Figure 2: System outputs from the different robust design approaches

With the initial condition to be $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$, the upper bound computed from the LMI based design is

$$\gamma_{lmi} = 855.2675$$

and the one from the iterative SDP design is

$$\gamma_{sdp} = 630.6379$$

The iterative SDP based design thus gives a lower upper bound. Applying the controllers from both designs to the uncertain system, the outputs are given in Fig. 2.

Both designs are based on the same objective function and adopt parameter-varying Lyapunov functions. Using a constant matrix G to approximate the inverse of Q_i gives a LMI based problem. But obviously better optimal solution can be obtained by choosing different G_i for each Q_i. From Fig. 2 it can be seen that the closed-loop system resulting from the iterative SDP based design settles faster. In addition, since the optimal upper bounds depend on the Lyapunov functions, it also shows that two different Lyapunov functions are obtained.

6 Conclusion

In this paper two nonlinear programming problems are formulated for the robust state feedback controller designs for linear systems with polytopic uncertainties: robust stabilization and optimal control problems. Compared to the LMI formulations, the associated nonlinear based design is then less conservative but difficult to be solved. To solve these optimization problems under the nonlinear matrix-valued inequality constraints, new iterative semidefinte programming (SDP) algorithms are developed, in which the sub-problems are LMIs and can be solved efficiently. A numerical example is included which shows the efficiency and the advantage of the new algorithms.

A Proof of Theorem 5

Proof. Suppose that the Lyapunov function $V(x(k), k) = x(k)^T P(k) x(k)$, where the matrix $P(k) > 0$ is time-varying positive definite. First, letting $\gamma = V(x(0), 0)$, the problem of minimization of $V(x(0), 0)$ can be reformulated as

$$\min_{P(0),\gamma} \gamma$$

subject to

$$\gamma = x(0)^T P(0) x(0) \tag{29}$$

Letting that $P(k) = \gamma Q(k)^{-1}$ and using the Schur complement, this optimization problem is equivalent to

$$\min_{Q(0),\gamma} \gamma$$

subject to

$$\begin{bmatrix} 1 & * \\ x(0) & Q(0) \end{bmatrix} > 0 \tag{30}$$

Defining that $Q(k) = \sum_{i=1}^{l} \mu_i(k) Q_i$, it can be seen that the left side of inequality (30) is the convex combination of the left side of inequalities (16), from which the inequalities in (16) are thus established. It is then proved that condition (14) is guaranteed by the inequalities in (17). The condition (14) can be written as

$$\begin{aligned} & x(k+1)^T P(k+1) x(k+1) - \\ & x(k)^T P(k) x(k) + x(k)^T Q_1 x(k) + \\ & u(k)^T R u(k) < 0 \end{aligned}$$

Recalling the system equation (2) and the controller in equ. (6), it follows that

$$\begin{aligned} & x(k)^T \{[A(k) + B(k) F]^T \times \\ & P(k+1)[A(k) + B(k) F] - \\ & P(k) + Q_1 + F^T R F\} x(k) < 0 \end{aligned}$$

This is satisfied for all $x(k)$ if,

$$\begin{aligned} & [A(k) + B(k) F]^T P(k+1) \times \\ & [A(k) + B(k) F] - P(k) + \\ & Q_1 + F^T R F < 0 \end{aligned} \tag{31}$$

Since $P(k) = \gamma Q(k)^{-1}$, from the Schur complement, it follows that (31) is equivalent to

$$\begin{bmatrix} Q(k) & * & * & * \\ [A(k) + B(k) F] Q(k) & Q(k+1) & * & * \\ Q_1^{1/2} Q(k) & 0 & \gamma I & * \\ R^{1/2} F Q(k) & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{32}$$

Since inequalities (17) hold, the matrices G_i are non-singular. Letting $G(k) = \sum_{i=1}^{l} \mu_i(k) G_i$ and multiplying (32) from the left by

$$diag\left(G(k)^T Q(k)^{-1}, I, I, I\right)$$

and from the right by

$$diag\left(Q(k)^{-1} G(k), I, I, I\right)$$

inequality (32) is converted to the form,

$$\begin{bmatrix} G(k)^T Q(k)^{-1} G(k) & * & * & * \\ [A(k) + B(k) F] G(k) & Q(k+1) & * & * \\ Q_1^{1/2} G(k) & 0 & \gamma I & * \\ R^{1/2} F G(k) & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{33}$$

Also, recalling that $Q(k) = \sum_{i=1}^{l} \mu_i(k) Q_i > 0$ and since

$$(Q(k) - G(k))^T Q(k)^{-1} (Q(k) - G(k)) \geq 0$$

gives

$$G(k)^T Q(k)^{-1} G(k) \geq G(k) + G(k)^T - Q(k)$$

Thus inequality (33) holds if

$$\begin{bmatrix} G(k) + G(k)^T - Q(k) & * & * & * \\ [A(k) + B(k) F] G(k) & Q(k+1) & * & * \\ Q_1^{1/2} G(k) & 0 & \gamma I & * \\ R^{1/2} F G(k) & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{34}$$

Observe that the left side of inequality (34) is a convex combination of the left side of (17), i.e.

$$\mathcal{B}(k) = \sum_{i=1}^{l} \sum_{j=1}^{l} \mu_j(k+1) \mu_i(k) \mathcal{B}_{i,j}$$

where $\mathcal{B}_{i,j}$, $\mathcal{B}(k)$ are the left sides of inequality (17) and 34, respectively. The theorem therefore is proved as required. ■

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