ON WEIGHT ADJUSTMENTS IN $\mathscr{H}_\infty$ CONTROL DESIGN

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Abstract

$\mathscr{H}_\infty$ control design is generally performed iteratively. At each iteration, the weights constraining the desired closed-loop transfer functions are adjusted until satisfactory performance margins are obtained. The way in which the weights are adjusted is generally heuristic and based solely on past experience and engineering judgement/intuition. It is consequently important to understand and provide guidelines on how weight adjustments directly affect the synthesised controller, and more importantly, the corresponding closed-loop transfer function matrices. This article presents a thorough study of this problem based on small weight adjustments in $\mathscr{H}_\infty$ control design.

1 Introduction

In recent years, $\mathscr{H}_\infty$ control design has become a well known method to design model based controllers satisfying a number of constraints expressed by amplitude bounds, in the form of weights, on the "to be designed" closed-loop transfer functions. This method whose theoretical basis can be found in the works of [3, 4, 19] has known numerous applications on real life systems (see [1, 18] and references therein).

It is well known that the design of weights in $\mathscr{H}_\infty$ control problems is a non-trivial task. Usually, suitable weights are obtained via a trial and error process based primarily on engineering judgement and intuition [7]. This trial and error process becomes increasingly complicated as the number of weighted channels increase, since it may not be possible to sensibly choose the weights for each channel independently. For problems involving requirements on both the standard closed-loop sensitivity and complementary sensitivity functions, for example, it is not possible to arbitrarily specify weights for each of these channels, since the two are coupled. Such tradeoffs are common in control system design. It is consequently important to understand and provide guidelines on how weight adjustments directly affect the synthesised controller (the central controller in this paper), and more importantly, the corresponding closed-loop transfer function matrices. Related work that addresses this problem from a different (from an optimisation) perspective can be found in [10, 13–15].

Since their inception, $\mathscr{H}_\infty$ control problems have been amenable to a variety of solution techniques. The framework used in this paper to analyse the effect of weight adjustments on the synthesised controller and the corresponding closed-loop transfer function matrices is the chain-scattering approach to $\mathscr{H}_\infty$ control of [8]. This approach is very similar (and in fact equivalent in some sense) to the J-spectral factorisation approach to $\mathscr{H}_\infty$ control of [5, 6, 9]. We use this framework because we envisage adjustments of the weighting functions that may change the McMillan degree of the frequency domain symbol. These adjustments are easily dealt in the frequency domain operator-theoric framework of chain-scattering, but are more cumbersome and not easily cast in state-space descriptions.

This paper is an extension of [2] to Multiple Input Multiple Output (MIMO) systems and to four-block $\mathscr{H}_\infty$ problems. The analysis is based on the assumption of "small" weight adjustments (with smallness exactly qualified in the paper). This assumption is required so as to construct a *linear map* from weight adjustments to controller and closed-loop transfer function modifications. The first contribution of this work is to give a first-order approximation of the modification of the central controller due to a small weight adjustment after a successful $\mathscr{H}_\infty$ control synthesis step. This approximation is a function of the weight adjustment and the variables involved in the initial control design problem, and is therefore computable without requiring another $\mathscr{H}_\infty$ control synthesis step (i.e. a new synthesis step involving the adjusted weights). A second contribution of this paper is that we show that the modification of the central controller persists outside the frequency region where the weight is mainly adjusted. Finally, we analyse the effect of this controller modification on the corresponding closed-loop transfer functions.

2 Background Material

2.1 Chain-scattering representation of generalised plants

Consider a generalised plant Σ with two kinds of inputs (w, u) and two kinds of outputs (z, y) represented as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \tag{1}$$

where z represents the errors to be reduced [$\dim(z) = m$], y denotes the measured outputs [$\dim(y) = q$], w represents the exogenous signals [$\dim(w) = r$], u denotes the control inputs [$\dim(u) = p$], and assume that it satisfies the following assumption:

Assumption (A1): $q \le r$, $p \le m$ and $\mathrm{rank}[\Sigma_{21}(j\omega)] = q$, $\mathrm{rank}[\Sigma_{12}(j\omega)] = p$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

If Σ_{21}^{-1} exists (i.e. if $r = q$), then the generalised plant Σ can be alternatively represented by

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

where

$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{12} - \Sigma_{11}\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{11}\Sigma_{21}^{-1} \\ -\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{21}^{-1} \end{bmatrix}. \quad (2)$$

This type of representation is usually referred to as a *chain-scattering representation* of Σ.

Now, let the plant Σ or its chain-scattering equivalent G be controlled by a controller $u = Ky$. Then the closed-loop transfer function matrix T_{zw} mapping exogenous inputs w to errors z is given by

$$\begin{aligned} T_{zw} &= \mathcal{F}_l(\Sigma, K) := \Sigma_{11} + \Sigma_{12}K(I - \Sigma_{22}K)^{-1}\Sigma_{21} \\ &= \mathrm{HM}(G, K) := (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}, \end{aligned}$$

where $\mathcal{F}_l(\cdot,\cdot)$ denotes the "lower Linear Fractional Transformation" frequently used in control theory and $\mathrm{HM}(\cdot,\cdot)$ denotes the "Homographic Transformation" frequently used in classical circuit theory.

2.2 Connections to $\mathscr{H}_\infty$ control

A *normalised $\mathscr{H}_\infty$ control problem* can be stated as follows: "Find a controller K such that the closed-loop system $T_{zw} = \mathcal{F}_l(\Sigma, K) = \mathrm{HM}(G, K)$ is internally stable and the closed-loop transfer function T_{zw} satisfies $\|T_{zw}\|_\infty < 1$". A controller K is said to be *admissible* if it solves the normalised $\mathscr{H}_\infty$ control problem. We also usually seek to characterise the set of all admissible controllers. For such a set, a controller K_c is said to be a *central controller* if it is achieved by setting a certain free parameter characterising this set to zero.

The following lemma characterises the set of all admissible controllers in terms of a solution of a J-cospectral factorisation problem when the normalised $\mathscr{H}_\infty$ control problem is solvable. Here, J_{mr} denote the signature matrix, defined by $J_{mr} := \mathrm{diag}(I_m, -I_r)$.

Lemma 1 ([8]) *Suppose that the normalised $\mathscr{H}_\infty$ control problem is solvable for a generalised plant $\Sigma \in \mathscr{RL}_\infty$ given by equation* (1) *that satisfies assumption (A1). Then there exists a unimodular Ξ in $\mathscr{RH}_\infty$ satisfying*

$$\Xi\, J_{pq}\, \Xi^\sim = \begin{bmatrix} I_p & 0 \\ \Sigma_{22} & \Sigma_{21} \end{bmatrix} \begin{bmatrix} \Sigma_{12}^\sim\Sigma_{12} & \Sigma_{12}^\sim\Sigma_{11} \\ \Sigma_{11}^\sim\Sigma_{12} & \Sigma_{11}^\sim\Sigma_{11} - I_r \end{bmatrix}^{-1} \begin{bmatrix} I_p & \Sigma_{22}^\sim \\ 0 & \Sigma_{21}^\sim \end{bmatrix}. \quad (3)$$

In this case, all admissible controllers are given by

$$K = \mathrm{HM}(\Xi, S)$$

for some $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$.

The unimodular matrix Ξ in $\mathscr{RH}_\infty$ satisfying equation (3) is unique up to right multiplication by a constant nonsingular real matrix $\hat{\Psi}$ which satisfies $\hat{\Psi} J_{pr} \hat{\Psi}^T = J_{pr}$ (i.e. there are several different possible unimodular matrices Ξ solving equation (3)). References [11,12] describe in detail how to appropriately select a particular unimodular matrix Ξ by fixing the choice of $\Xi(j\infty)$. It is very important for the analysis of Section 4 to pin down one particular unimodular matrix Ξ that solves equation (3) because first order approximations only make sense when considering the effect of small changes on the same unimodular matrix Ξ.

3 Considered Problem

As stated in the introduction, the aim of this paper is to analyse the effect of small adjustments in the weighting functions of an $\mathscr{H}_\infty$ design on the central controller K_c (uniquely defined after Ξ has been pinned down at infinite frequency) and the resulting closed-loop transfer function matrices T_{zw}. Here, the central controller is defined by

$$K_c := \mathrm{HM}(\Xi, 0) \quad (4)$$

for a particular choice of unimodular matrix Ξ that satisfies equation (3) and the resulting closed-loop transfer function matrix T_{zw} is defined by

$$T_{zw} := \begin{bmatrix} P \\ I \end{bmatrix} (I - K_c P)^{-1} \begin{bmatrix} -K_c & I \end{bmatrix} \quad (5)$$

where P is the nominal plant model (not the generalised plant Σ). Note that T_{zw} contains all four important transfer function matrices that ought to be considered in any sensible control system design. Weighting functions are assigned to entries in T_{zw} in accordance with the design paradigm adopted (e.g. some design paradigms weight each closed-loop transfer function matrix individually and others weight the plant P directly).

We suppose that the originally posed $\mathscr{H}_\infty$ control problem with a specified weighting function W is solvable and that we have the unimodular matrix Ξ of interest that satisfies equation (3), the corresponding central controller K_c and the resulting closed-loop transfer function matrix T_{zw}. Then we adjust the weighting function by a small amount ΔW to give $W_{new} := W + \Delta W$. After solving the new $\mathscr{H}_\infty$ control problem that results from this change in weight, the unimodular matrix Ξ changes to $\Xi_{new} := \Xi + \Delta\Xi$, the corresponding central controller K_c changes to $K_{c,new} := K_c + \Delta K_c$ and the resulting closed-loop transfer function matrix T_{zw} changes to $T_{zw,new} := T_{zw} + \Delta T_{zw}$. In order to describe what is meant by a small change in weight, we assume the following:

Assumption (A2): The change in weight ΔW is chosen small enough to ensure that:

(a) the mapping $\Delta W \mapsto \Delta\Xi \mapsto \Delta K_c \mapsto \Delta T_{zw}$ is linear,

(b) the $\mathscr{H}_\infty$ control problem remains solvable after adjusting the weight.

In this work, we will effectively construct a mapping $\Delta W \mapsto \Delta\Xi \mapsto \Delta K_c \mapsto \Delta T_{zw}$ based on first order approximations. This mapping allows us to understand quantitatively the effects of changing a weight by a small amount on the closed-loop transfer function matrices (and it is also a guide to the effects of a larger change in weight). While the smallness requirements of assumption (A2) ensures this mapping is linear, we will show that this mapping is in general not memoryless since $\Delta K_c(j\omega_1)$ does not only depend on $\Delta W(j\omega_1)$ but on $\Delta W(j\omega)$ for, in principle, all $\omega \in [0, \infty)$.

3.1 Application to the $\mathscr{H}_\infty$ loop-shaping design procedure

In order to explicitly study how changes in weights map to changes in the central controller and the closed-loop transfer function matrices, we need to choose some $\mathscr{H}_\infty$ control design paradigm for the sake of the discussion. In this paper, we use the $\mathscr{H}_\infty$ loop-shaping design procedure proposed by [16] to illustrate the concepts, as it is an effective method for designing robust controllers and has been successfully used in a variety of applications. A detailed tutorial on how to design robust controllers using this design procedure can be found in [18].

In this paradigm, the $\mathscr{H}_\infty$-norm objective is to synthesise an internally stabilising controller K_c such that

$$\left\| \begin{bmatrix} W_2 & 0 \\ 0 & W_1^{-1} \end{bmatrix} T_{zw} \begin{bmatrix} W_2^{-1} & 0 \\ 0 & W_1 \end{bmatrix} \right\|_\infty < \gamma.$$

It can be shown that this $\|\cdot\|_\infty$ is always greater than or equal to unity and hence for a feasible problem $\gamma > 1$.

Some algebraic manipulations should convince the reader that the above $\mathscr{H}_\infty$-norm objective can be restated as: "Synthesise an internally stabilising controller K such that

$$\left\| \mathcal{F}_l \left(\left[\begin{array}{cc|c} 0 & \frac{1}{\gamma} W_2 P W_1 & \frac{1}{\gamma} W_2 P \\ 0 & \frac{1}{\gamma} I & \frac{1}{\gamma} W_1^{-1} \\ \hline -W_2^{-1} & P W_1 & P \end{array} \right], K \right) \right\|_\infty < 1 \quad (6)$$

for some $\gamma > 1$".

Consequently, the generalised plant Σ for the $\mathscr{H}_\infty$ loop-shaping problem is given by the term in square brackets in equation (6). From this representation of Σ, it is clear that this problem is a four-block problem since neither Σ_{21} nor Σ_{12} are square. Furthermore, since both W_1 and W_2 are typically chosen to be units in $\mathscr{RH}_\infty$, assumption (A1) is trivially satisfied in this case.

Using Lemma 1 and rewriting equation (3) for the $\mathscr{H}_\infty$ loop-shaping case, we get

$$\Xi\ J_{pq}\ \Xi^{\sim} = -\begin{bmatrix} (W_1 W_1^{\sim}) & 0 \\ 0 & (W_2^{\sim} W_2)^{-1} \end{bmatrix}$$
$$+\ \gamma^2 \begin{bmatrix} I \\ P \end{bmatrix} \left[P^{\sim}(W_2^{\sim} W_2) P + (W_1 W_1^{\sim})^{-1} \right]^{-1} \begin{bmatrix} I & P^{\sim} \end{bmatrix}. \quad (7)$$

Since the post-compensator W_2 is usually held fixed as a low-pass filter, we will analyse how a change in the pre-compensator W_1 maps to changes in the central controller K_c and the closed-loop transfer function matrices T_{zw}. In this sense, we will derive linearisations of equations (7), (4) and (5), since equation (7) relates W_1 to Ξ, equation (4) relates Ξ to K_c and equation (5) relates K_c to T_{zw}. Note that a similar analysis could have been performed on W_2 instead of W_1, if it is so required.

4 The Effects of Small Weight Adjustments

Proofs are omitted for the sake of brevity and they will be published elsewhere.

4.1 Effect on the transfer function matrix Ξ

We will first need to find an approximation for $\Delta\Xi$ since Ξ defines the central controller K_c through equation (4). This approximation for $\Delta\Xi$ is given in Theorem 2 below. Recall that it is very important to pin down one particular unimodular matrix Ξ that solves equation (7) because first order approximations only make sense when considering the effect of small changes on the same unimodular matrix Ξ.

Theorem 2 *Suppose a number $\gamma > 1$, a nominal plant $P \in \mathscr{RL}_\infty$ and some weights W_1 and W_2 (units in $\mathscr{RH}_\infty$) are given for which the normalised $\mathscr{H}_\infty$ control problem stated in equation* (6) *is solvable. Let Ξ (unimodular in $\mathscr{RH}_\infty$) denote the solution of equation* (7) *and force uniqueness on Ξ by pinning down $\Xi(j\infty)$ as described in [11,12].*

Then consider the adjustment of weight W_1 by some small[1] *amount ΔW_1 to give a new weight $W_{1,new} := W_1 + \Delta W_1$. As a result of this weight change, the selected Ξ changes to Ξ_{new} and a first order approximation of the change $\Delta\Xi := \Xi_{new} - \Xi$ is given by*

$$\Delta\Xi \ \approx\ \Xi\,\Phi J_{pq}, \quad (8)$$

where Φ is a stable transfer function matrix solving

$$\Phi + \Phi^{\sim} \ =\ \Xi^{-1}\Gamma(\Delta W_1)\,\Xi^{-\sim} \quad (9)$$

and $\Gamma(\Delta W_1)$ is defined as follows

$$\Gamma(\Delta W_1) := -\begin{bmatrix} W_1\,\Delta W_1{}^{\sim} + \Delta W_1\, W_1^{\sim} & 0 \\ 0 & 0 \end{bmatrix}$$
$$+\,\gamma^2 \begin{bmatrix} I \\ P \end{bmatrix} \left[(W_1 W_1^{\sim}) P^{\sim} (W_2^{\sim} W_2) P + I \right]^{-1} \left[W_1\,\Delta W_1{}^{\sim} + \Delta W_1\, W_1^{\sim} \right]$$
$$\times \left[P^{\sim}(W_2^{\sim} W_2) P (W_1 W_1^{\sim}) + I \right]^{-1} \begin{bmatrix} I & P^{\sim} \end{bmatrix}. \quad (10)$$

If we were to consider a different $\mathscr{H}_\infty$ design paradigm from the $\mathscr{H}_\infty$ loop-shaping design procedure, the above theorem would remain essentially the same with only the definition of $\Gamma(\Delta W_1)$ being different. Similarly, if we consider changes in W_2 instead of W_1, only the definition of $\Gamma(\cdot)$ would be different.

The matrix Φ resulting from equation (9) is unique up to an additive constant "skew-symmetric" matrix. It will be shown in Theorem 3 that the only part of the stable transfer function matrix Φ that is important in analysing how the central controller changes for a small weight adjustment is Φ_{12} (of dimension $p \times q$). We will thus choose the corresponding (12)-block of the above-mentioned additive skew-symmetric matrix such that

$$\Phi_{12}(j\infty) = 0.$$

This is done so that $K_{c,new}(j\infty) \approx K_c(j\infty)$, as should be the case since ΔW_1 is typically selected to be strictly proper.

[1] "Small" in the sense of assumption (A2).

4.2 Effect on the central controller K_c

Theorem 2 gives us an approximation of the difference between Ξ_{new} and Ξ. This approximation is a function of ΔW_1 and the variables involved in the original (i.e. the one with weights W_1 and W_2) problem. This result now allows us to derive an approximation for the difference between the new central controller $K_{c,new}$ and the original central controller K_c using the relation (4) between the object Ξ and the central controller K_c. This expression is given in the following theorem.

Theorem 3 *Let the suppositions of Theorem 2 hold with Ξ partitioned as follows*

$$\Xi \; = \; \begin{array}{cc} \left[\begin{array}{cc} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{array}\right] & \begin{array}{l} \}p \\ \}q \end{array} \\ \begin{array}{cc} \underbrace{}_{p} & \underbrace{}_{q} \end{array} & \end{array}$$

and define the central controller as in equation (4)*.*

Then consider the adjustment of weight W_1 by some small amount ΔW_1 to give a new weight $W_{1,new} := W_1 + \Delta W_1$. As a result of this weight change, the selected central controller K_c changes to $K_{c,new}$ and a first order approximation of the change $\Delta K_c := K_{c,new} - K_c$ is given by

$$\Delta K_c \; \approx \; -(\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{21})\,\Phi_{12}\,\Xi_{22}^{-1}, \qquad (11)$$

where Φ_{12} and Φ_{21} are stable transfer function sub-matrices of Φ in (9) *that solve*

$$\Phi_{12} + \Phi_{21}^{\sim} \; = \; \begin{bmatrix} I_p & 0 \end{bmatrix} \Xi^{-1}\Gamma(\Delta W_1)\,\Xi^{-\sim}\begin{bmatrix} 0 \\ I_q \end{bmatrix} \qquad (12)$$

and $\Gamma(\Delta W_1)$ is defined as in (10)*. To force uniqueness in the decomposition of equation* (12) *require Φ_{12} to be also strictly proper.*

It should be clear that the approximation of ΔK_c given in (11) has zero gain at infinite frequency since Φ_{12} is selected to be strictly proper. Furthermore, it can be easily shown that if the original central controller K_c is (open-loop) stable, then the approximation to the resulting difference between $K_{c,new}$ and K_c is also stable.

4.3 Effect on the closed-loop transfer functions T_{zw}

Theorem 3 gives us an approximation of the difference between $K_{c,new}$ and K_c. This approximation is again simply a function of ΔW_1 and the variables involved in the original (i.e. the one with weights W_1 and W_2) problem. This result (which is computable prior to solving the modified $\mathscr{H}_\infty$ control problem) now allows us to derive an approximation for the difference between the new closed-loop transfer function matrices $T_{zw,new}$ and the original ones T_{zw} using the relation (5) between K_c and T_{zw}. This expression is given in the following theorem.

Theorem 4 *Let the suppositions of Theorem 3 hold and define the closed-loop transfer function matrices of interest T_{zw} as in equation* (5)*.*

Then consider the adjustment of weight W_1 by some small amount ΔW_1 to give a new weight $W_{1,new} := W_1 + \Delta W_1$. As a result of this weight change, the closed-loop transfer function matrices of interest T_{zw} change to $T_{zw,new}$ and a first order approximation of the change $\Delta T_{zw} := T_{zw,new} - T_{zw}$ is given by

$$\Delta T_{zw} \approx \begin{bmatrix} P \\ I \end{bmatrix}(I - K_c P)^{-1}\Delta K_c\,(I - PK_c)^{-1}\begin{bmatrix} -I & P \end{bmatrix} \quad (13)$$

where ΔK_c denotes the difference between the new central controller after weight change and the original central controller K_c, or an approximation of this difference as given in equation (11)*.*

Equation (13) suggests how the controller K_c should be modified in order to obtain the desired effects on the closed-loop transfer functions.

5 Controller Changes outside the Adjusted Frequency Band

In this section, we analyse the relation between the weight change ΔW_1 and the strictly proper stable transfer function matrix Φ_{12} in equation (12), as this will give us the effect on ΔK_c through equation (11) and in turn the resulting effect on $K_{c,new}$.

Typically, the adjustment of weight W_1 by ΔW_1 will mainly be restricted to the frequency band of interest $[\omega_1, \omega_2]$. Consequently, ΔW_1 will in general be a pass-band stable filter that is strictly proper, has a blocking zero at zero frequency and its maximum singular value is non-negligible only in the frequency band $[\omega_1, \omega_2]$. For illustration, in a SISO setting, ΔW_1 will have the form

$$\Delta W_1 = \frac{Ks}{(s+\omega_1)(s+\omega_2)}$$

with $\omega_1, \omega_2 > 0$ and $K \in \mathbb{R}$. The blocking zero and strict properness requirements ensure that $\Delta W_1(j\omega) \to 0$ as $\omega \to 0$ and ∞.

The transfer function matrix Φ_{12} will generally have a different frequency plot from the weight change ΔW_1 because of the particular decomposition in equation (12). Indeed, $\Phi_{12}(j\omega)$ converges to 0 at high frequencies (see Theorem 3) just like $\Delta W_1(j\omega)$ converges to 0 at high frequencies, but $\Phi_{12}(j\omega)$ generally converges to a *non-zero* real constant matrix at low frequencies unlike ΔW_1. This phenomenon is captured by the following theorem.

Theorem 5 *Given a stable strictly proper weight adjustment ΔW_1 that has a blocking zero at $s = 0$, define $\Gamma(\Delta W_1)$ as in equation* (10)*.*

Then, the stable strictly proper transfer function matrix Φ_{12} that solves equation (12) *does not in general have a blocking zero at $s = 0$.*

Understanding that Φ_{12} generally has a different low frequency behaviour from ΔW_1 is important because the low frequency behaviour of Φ_{12} determines the low frequency modification ΔK_c of the central controller through equation (11) and in turn it also determines the corresponding low frequency modification ΔT_{zw} of the closed-loop transfer function matrices. Note that $(\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{21})$ and Ξ_{22} in equation (11) are bi-proper transfer function matrices because Ξ is a unit in $\mathscr{RH}_\infty$. Thus, the low frequency behaviour of ΔK_c will be directly determined by the low frequency behaviour of Φ_{12}.

In view of Theorem 5, we can assert that the modification of the central controller K_c (and hence the closed-loop transfer functions T_{zw}) due to the weight change ΔW_1 will not only be restricted to the frequency band of interest $[\omega_1, \omega_2]$ where the weight change occurs, but will also persist at lower frequencies than ω_1. In this sense, we say that the changes in the central controller (and hence the closed-loop transfer functions) occur in the frequency range $[0, \omega_2]$, even though the weight change ΔW_1 is only significant in the frequency band $[\omega_1, \omega_2]$. At frequencies much higher than ω_2, there will be no significant changes in K_c (and correspondingly T_{zw}) because $\Phi_{12}(j\omega)$ converges to zero as $\omega \to \infty$.

6 Numerical Example

Consider the following scaled-down model of the High Incidence Research Model constructed at the University of Cambridge in order to study problems associated with the control of air-vehicles at high angles of attack [17]:

$$P(s) = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -23.8 & -3.36 & 4.60 & -0.239 & 1.67 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -16.8 & -0.0248 & 22.8 & -0.916 & 0 & 1.39 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

The plant P has two inputs (roll and yaw thrusters) and two outputs (roll and yaw angles).

We want to control this MIMO plant using the $\mathscr{H}_\infty$ loop-shaping design procedure. Throughout this example, the weight W_2 will be set to the identity matrix I_2 in order to simplify the expressions. Furthermore, we initially select W_1 to be of the form $W_1 = kI_2$ where k is some constant. The constant $k = 800$ is chosen so that the cross-over frequency of the shaped plant PW_1 is approximately 30 rad/s.

The $\mathscr{H}_\infty$ control design procedure with these weights W_1 and W_2 and $\gamma = 5$ delivers the central controller K_c whose maximum singular value is represented in Figure 1. The maximum singular values of the corresponding closed-loop transfer function matrices are depicted in Figure 2. Figure 2 shows that the resulting closed-loop behaviour is satisfactory except for T_{21} which has a too large resonance peak around $\omega = 50$ rad/s.

A new weight $W_{1,new}$ is now chosen by making a small weight adjustment ΔW_1 to the initial weight W_1. This adjustment must decrease the resonance peak of T_{21} without modifying too much the other satisfactory features of the initial design (e.g. the cross-over frequency of $PW_{1,new}$ must be reasonably the same as for PW_1). Following conventional wisdom, we decrease the weight W_1 around the frequency of the resonance peak. That is, we

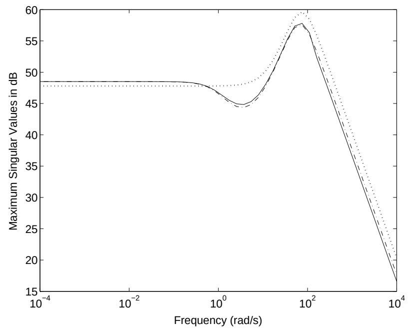

Figure 1: $\overline{\sigma}(K_c(j\omega))$ (dotted), $\overline{\sigma}(K_{c,new}(j\omega))$ (dashdot), $\overline{\sigma}(K_{c,app}(j\omega))$ (solid)

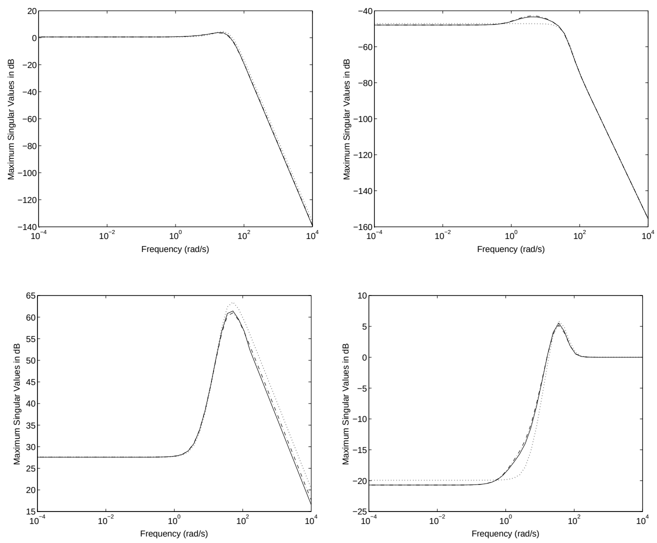

Figure 2: $\overline{\sigma}(T_{ij}(j\omega))$ (dotted), $\overline{\sigma}(T_{ij,new}(j\omega))$ (dashdot), $\overline{\sigma}(T_{ij,app}(j\omega))$ (solid)

choose the new weight $W_{1,new}$ as follows:

$$W_{1,new} = W_1 + \Delta W_1 = W_1 + \begin{bmatrix} \frac{-7566s}{(s+1)(s+20)} & 0 \\ 0 & \frac{-7566s}{(s+1)(s+20)} \end{bmatrix}$$

Figure 3 shows both $W_{1,new}$ and ΔW_1. From this figure, it

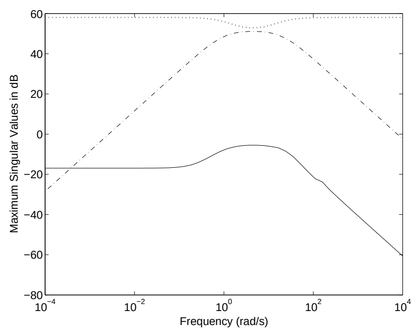

Figure 3: $\overline{\sigma}(\Phi_{12}(j\omega))$ (solid), $\overline{\sigma}(\Delta W_1(j\omega))$ (dashdot), $\overline{\sigma}(W_{1,new})$ (dotted)

is clear that ΔW_1 can be considered a small deviation (in the sense of Assumption A2) of W_1 since it causes simply a 7 dB change in the magnitude of W_1. Furthermore, the new weight decreases the gain of the shaped plant in the frequency band $[1, 20]$ rad/s.

We then use the results of this paper in order to get a quantitative idea of the behaviour that we anticipate as a consequence of the adjustment ΔW_1. For this purpose, we first compute Φ_{12} using the decomposition in equation (12). The maximum singular value of Φ_{12} is represented in Figure 3. This figure shows that $\Phi_{12}(j\omega)$ does not converge to 0 when $\omega \to 0$, unlike ΔW_1. The quantity Φ_{12} is then used to compute an approximation of the new central controller $K_{c,new}$ that would be obtained if we were to perform an $\mathscr{H}_\infty$ control design with weight $W_{1,new}$. The maximum singular value of this approximation of the new central controller is represented in Figure 1. In this figure, we note that the change in the controller is not limited to the frequency band $[1, 20]$ rad/s, but persists in the frequency band $[0, 20]$ rad/s. This is a non-obvious change resulting from the low frequency behaviour of $\Phi_{12}(j\omega)$ (see Figure 3). The approximation of $K_{c,new}$ can now be used to analyse the consequences of ΔW_1 on the closed-transfer function matrices. The maximum singular values of these approximations $T_{ij,app}$ are represented in Figure 2. We observe that the chosen adjustment ΔW_1 in the weight W_1 delivers the desired effect, which is quantifiable with this theory: the resonance peak of T_{21} is indeed reduced by about 5 dB.

For the sake of comparison, a new $\mathscr{H}_\infty$ control synthesis is also performed using $W_{1,new}$ (and W_2, γ). This delivers a new central controller $K_{c,new}$ and the corresponding new closed-loop transfer function matrices whose maximum singular values are also represented in Figures 1 and 2 respectively. In these figures, we observe that the approximations $K_{c,app}$ and $T_{ij,app}$ were reliable enough to accurately predict quantitative changes in the the frequency responses of the new transfer function matrices $K_{c,new}$ and $T_{ij,new}$ when the weight W_1 was replaced by $W_{1,new}$. This is because the curves representing the approximations and the actual frequency responses are almost overlapping.

7 Conclusions

In this paper, we have analysed the effects of *small* weight adjustments on the synthesised central controller and the corresponding closed-loop transfer function matrices in an $\mathscr{H}_\infty$ control design setting. Only small changes were considered because this allowed us to construct a linear map from weight adjustments to controller and closed-loop transfer function modifications. Approximations to these modifications can be very easily computed using simple formulae, and without having to solve a second $\mathscr{H}_\infty$ synthesis problem for the new problem with adjusted weights. Thus, in this sense, these approximations allow us to determine quantitatively the effects of weight adjustments “a priori”. We show also that the modification in the central controller persists outside the frequency band where the weight was is mainly adjusted.

Specific details have been illustrated on the $\mathscr{H}_\infty$ loop-shaping design procedure, but different $\mathscr{H}_\infty$ design paradigms may be used because the arguments of this paper easily follow through. This is because the underlying principle is independent of the design method considered. The results of this paper are also independent of the structure of the weighting functions. Thus, adjustments of diagonal and non-diagonal weights are treated in the same framework with similar ease.

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