SOLUTION TO THE GENERAL ROBUST STRICTLY POSITIVE REAL SYNTHESIS PROBLEM FOR POLYNOMIAL SEGMENTS

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Abstract

For any two n-th order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real.

1 Introduction

The strict positive realness (SPR) of transfer functions is an important performance specification, and plays a critical role in various fields such as absolute stability/hyperstability theory [15, 20], passivity analysis [11], quadratic optimal control [3] and adaptive system theory [16]. In recent years, stimulated by the parametrization method in robust stability analysis [1, 4, 6], the study of robust strictly positive real systems has received much attention, and great progress has been made [2, 5] [7]-[10] [12]-[14] [17]-[19] [21]-[35]. It was proved in [9, 21] that the strict positive realness of an entire interval transfer function family can be guaranteed by the same property of only eight prespecified vertex transfer functions in this family. However, most available results in the literature belong to the category of robust SPR analysis. Valuable results in robust SPR synthesis are rare. The following fundamental problem is still open [14, 18, 22, 23] [25]-[28] [30] [31]-[34]:

Suppose $a(s)$ and $b(s)$ are two n-th order Hurwitz polynomials, does there exist, and how to find a (fixed) polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR?

By the definition of SPR, it is easy to know that the Hurwitz stability of the convex combination of $a(s)$ and $b(s)$ is necessary for the existence of polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. In [13, 14, 19], it was proved that, if $a(s)$ and $b(s)$ have the same even (or odd) parts, such a polynomial $c(s)$ always exists; In [2, 13, 14, 17, 22, 23, 25, 27, 30, 31], it was proved that, if $n \leq 4$ and $a(s), b(s) \in K$ (K is a stable interval polynomial set), such a polynomial $c(s)$ always exists; Recent results show that [22, 23, 25, 27, 28, 30] [32]-[34], if $n \leq 6$ and $a(s)$ and $b(s)$ are the two endpoints of the convex combination of stable polynomials, such a polynomial $c(s)$ always exists. Some sufficient condition for robust SPR synthesis are presented in [2, 5, 10, 17, 22, 23, 27], especially, the design method in [22, 23] is numerically efficient for high-order polynomial segments or interval polynomials, and the derived conditions are necessary and sufficient for low-order polynomial segments or interval polynomials.

This paper presents a constructive proof of the following statement: for any two n-th order polynomials $a(s)$ and $b(s)$, the Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. This also shows that the conditions given in [22, 23] are also necessary and sufficient and the above open problem admits a positive answer. Some previously obtained SPR synthesis results for low-order polynomial segments [22, 23, 25, 27, 28, 30] [32]-[34] become special cases of our main result in this paper.

2 Main Results

In this paper, P^n stands for the set of n-th order polynomials of s with real coefficients, R stands for the field of real numbers, $\partial(p)$ stands for the order of polynomial $p(\cdot)$, and $H^n \subset P^n$ stands for the set of n-th order Hurwitz stable polynomials with real coefficients.

In the following definition, $p(\cdot) \in P^m, q(\cdot) \in P^n, f(s) = p(s)/q(s)$ is a rational function.

Definition 1 [3, 16, 29] $f(s)$ is said to be strictly positive real(SPR), if

(i) $\partial(p) = \partial(q)$;

(ii) $f(s)$ is analytic in $\mathrm{Re}[s] \geq 0$, (namely, $q(\cdot) \in H^n$);

(iii) $\mathrm{Re}[f(j\omega)] > 0, \quad \forall \omega \in R$.

If $f(s) = p(s)/q(s)$ is proper, it is easy to get the following property:

Property 1 [3, 9, 16, 20] If $f(s) = p(s)/q(s)$ is a proper rational function, $q(s) \in H^n$, and $\forall \omega \in R, \mathrm{Re}[f(j\omega)] > 0$, then $p(s) \in H^n \cup H^{n-1}$.

The following theorem is the main result of this paper:

Theorem 1 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, the necessary and sufficient condition for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both Strict Positive Real, is

$$\lambda b(s) + (1 - \lambda)a(s) \in H^n, \lambda \in [0, 1].$$

The statement is obviously true for the cases when $n = 1$ or $n = 2$. We will prove it for the case when $n \geq 3$.

Since SPR transfer functions enjoy convexity property, by Property 1, we get the necessary part of the theorem.

In what follows, the notation $[x]$ stands for the largest integer that is smaller than or equal to the real number x and $\langle y \rangle_z$ stands for the remainder of the nonnegative integer y divided by the positive integer z [1].

To prove sufficiency, we must first introduce some lemmas.

Lemma 1 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n$, then $\alpha(s) = s^{n-1} + \alpha_1 s^{n-2} + \cdots + \alpha_{n-1} \in H^{n-1}$, where [2]
$\alpha_i = a_i^{\langle n-i \rangle_2} \left(\dfrac{a_i a_{n-1} - a_{i-1} a_n}{a_{n-1}} \right)^{\langle n-1-i \rangle_2}, i = 1, 2, \cdots, n-1,$
$a_0 = 1, a_l = 0$ if $l < 0$ or $l > n$.

Proof it can be directly proved by using Hurwitz criterion.

Lemma 2 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n$, then $\beta(s) = s^{n-1} + \beta_1 s^{n-2} + \cdots + \beta_{n-1} \in H^{n-1}$, where [3] $\beta_i = (a_1 a_{i+1} - a_{i+2})^{\langle i \rangle_2} \left(\dfrac{a_{i+1}}{a_1} \right)^{\langle i+1 \rangle_2}, i = 1, 2, \cdots, n-1,$
$a_0 = 1, a_l = 0$ if $l < 0$ or $l > n$.

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[1] For example, $[1.5] = 1, [0.5] = 0, [-1.5] = -2$, etc., and $\langle 0 \rangle_2 = 0, \langle 1 \rangle_2 = 1, \langle 11 \rangle_3 = 2$, etc..

[2] For example, when $n = 3$, we have:
$$\alpha_1 = \frac{a_1 a_2 - a_3}{a_2}, \alpha_2 = a_2.$$
when $n = 4$, we have:
$$\alpha_1 = a_1, \alpha_2 = \frac{a_2 a_3 - a_1 a_4}{a_3}, \alpha_3 = a_3.$$
when $n = 5$, we have:
$$\alpha_1 = \frac{a_1 a_4 - a_5}{a_4}, \alpha_2 = a_2, \alpha_3 = \frac{a_3 a_4 - a_2 a_5}{a_4}, \alpha_4 = a_4.$$
when $n = 6$, we have:
$$\alpha_1 = a_1, \alpha_2 = \frac{a_2 a_5 - a_1 a_6}{a_5}, \alpha_3 = a_3, \alpha_4 = \frac{a_4 a_5 - a_3 a_6}{a_5}, \alpha_5 = a_5.$$

[3] For example, when $n = 3$, we have:
$$\beta_1 = a_1 a_2 - a_3, \beta_2 = \frac{a_3}{a_1}.$$
when $n = 4$, we have:
$$\beta_1 = a_1 a_2 - a_3, \beta_2 = \frac{a_3}{a_1}, \beta_3 = a_1 a_4.$$
when $n = 5$, we have:
$$\beta_1 = a_1 a_2 - a_3, \beta_2 = \frac{a_3}{a_1}, \beta_3 = a_1 a_4 - a_5, \beta_4 = \frac{a_5}{a_1}.$$
when $n = 6$, we have:
$$\beta_1 = a_1 a_2 - a_3, \beta_2 = \frac{a_3}{a_1}, \beta_3 = a_1 a_4 - a_5, \beta_4 = \frac{a_5}{a_1}, \beta_5 = a_1 a_6.$$

Proof it can be directly proved by using Hurwitz criterion.

Lemma 3 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n$, then, for every $k \in \{1, 2, \cdots, n-2\}$, the following quadratic curve is an ellipse in the first quadrant (i.e., $x_i \geq 0, i = 1, 2, \cdots, n-1$) of the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$ [4]:

$$\begin{cases} c_{k+1}^2 - 4c_k c_{k+2} = 0, \\ c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2, \end{cases}$$

and this ellipse is tangent with the line

$$\begin{cases} c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k+1, k+2, \end{cases}$$

and the line

$$\begin{cases} c_l = 0, \\ l \in \{1, 2, \cdots, n\}, l \neq k, k+1, \end{cases}$$

respectively, where $c_l := \sum\limits_{j=0}^{n} (-1)^{l+j} a_j x_{2l-j-1}, l = 1, 2, \cdots, n, a_0 = 1, x_0 = 1, a_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n - 1$.

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[4] When $n = 3$, the ellipse equation is (see [27, 28] for details):
$$(a_2 x_1 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)a_3 x_2 = 0.$$

When $n = 4$, the two ellipse equations are (see [25] [30]-[32] for details):
$$\begin{cases} (a_2 x_1 + x_3 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_3 x_2 \\ \quad - a_2 x_3 - a_4 x_1) = 0, \\ a_4 x_3 = 0, \\ (a_3 x_2 - a_2 x_3 - a_4 x_1)^2 - 4(a_2 x_1 + x_3 \\ \quad - a_1 x_2 - a_3)a_4 x_3 = 0, \\ a_1 - x_1 = 0. \end{cases}$$

When $n = 5$, the three ellipse equations are (see [33] for details):
$$\begin{cases} (a_2 x_1 + x_3 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_5 + a_3 x_2 \\ \quad + a_1 x_4 - a_2 x_3 - a_4 x_1) = 0, \\ a_4 x_3 - a_3 x_4 - a_5 x_2 = 0, \ a_5 x_4 = 0, \\ (a_5 + a_3 x_2 + a_1 x_4 - a_2 x_3 - a_4 x_1)^2 - 4(a_2 x_1 + x_3 \\ \quad - a_1 x_2 - a_3)(a_4 x_3 - a_3 x_4 - a_5 x_2) = 0, \\ a_1 - x_1 = 0, \ a_5 x_4 = 0, \\ (a_4 x_3 - a_3 x_4 - a_5 x_2)^2 - 4(a_5 + a_3 x_2 + a_1 x_4 \\ \quad - a_2 x_3 - a_4 x_1)a_5 x_4 = 0, \\ a_1 - x_1 = 0, \ a_2 x_1 + x_3 - a_1 x_2 - a_3 = 0. \end{cases}$$

When $n = 6$, the four ellipse equations are (see [34] for details):
$$\begin{cases} (a_2 x_1 + x_3 - a_1 x_2 - a_3)^2 - 4(a_1 - x_1)(a_5 + a_3 x_2 + a_1 x_4 \\ \quad - x_5 - a_2 x_3 - a_4 x_1) = 0, \\ a_6 x_1 + a_4 x_3 + a_2 x_5 - a_3 x_4 - a_5 x_2 = 0, \\ a_5 x_4 - a_4 x_5 - a_6 x_3 = 0, \ a_6 x_5 = 0, \\ (a_5 + a_3 x_2 + a_1 x_4 - x_5 - a_2 x_3 - a_4 x_1)^2 - 4(a_2 x_1 + x_3 \\ \quad - a_1 x_2 - a_3)(a_6 x_1 + a_4 x_3 + a_2 x_5 - a_3 x_4 - a_5 x_2) = 0, \\ a_1 - x_1 = 0, \ a_5 x_4 - a_4 x_5 - a_6 x_3 = 0, \ a_6 x_5 = 0, \\ (a_6 x_1 + a_4 x_3 + a_2 x_5 - a_3 x_4 - a_5 x_2)^2 - 4(a_5 + a_3 x_2 + a_1 x_4 \\ \quad - x_5 - a_2 x_3 - a_4 x_1)(a_5 x_4 - a_4 x_5 - a_6 x_3) = 0 \\ a_1 - x_1 = 0, \ a_2 x_1 + x_3 - a_1 x_2 - a_3 = 0, \ a_6 x_5 = 0, \\ (a_5 x_4 - a_4 x_5 - a_6 x_3)^2 - 4(a_6 x_1 + a_4 x_3 + a_2 x_5 \\ \quad - a_3 x_4 - a_5 x_2)a_6 x_5 = 0, \\ a_1 - x_1 = 0, \ a_2 x_1 + x_3 - a_1 x_2 - a_3 = 0, \\ a_5 + a_3 x_2 + a_1 x_4 - x_5 - a_2 x_3 - a_4 x_1 = 0. \end{cases}$$

Proof Since $a(s)$ is Hurwitz stable, by using mathematical induction, and by Lemmas 1 and 2, Lemma 3 is proved by a direct calculation.

For notational simplicity, for $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n, \forall k \in \{1, 2, \cdots, n-2\}$, denote

$$\Omega^a_{ek} := \{ \quad (x_1, x_2, \cdots, x_{n-1}) | c_{k+1}^2 - 4c_k c_{k+2} < 0,$$
$$c_l = 0, l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2\},$$

and

$$\Omega^b_{ek} := \{ \quad (x_1, x_2, \cdots, x_{n-1}) | d_{k+1}^2 - 4d_k d_{k+2} < 0,$$
$$d_l = 0, l \in \{1, 2, \cdots, n\}, l \neq k, k+1, k+2\},$$

where $c_l := \sum_{j=0}^{n} (-1)^{l+j} a_j x_{2l-j-1}, d_l := \sum_{j=0}^{n} (-1)^{l+j} b_j x_{2l-j-1}, l = 1, 2, \cdots, n, a_0 = 1, b_0 = 1, x_0 = 1, a_i = 0$ and $b_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n - 1$.

In what follows, (A, B) stands for the set of points in the line segment connecting the point A and the point B in the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$, not including the endpoints A and B. Denote

$$\Omega^a := \{ \quad (x_1, x_2, \cdots, x_{n-1}) |$$
$$(x_1, x_2, \cdots, x_{n-1}) \in \bigcup_{i=1, i<j \leq n-2}^{n-3} (A_i, A_j),$$
$$\forall A_i \in \Omega^a_{ei}, i \in \{1, 2, \cdots, n-2\}\}$$

and

$$\Omega^b := \{ \quad (x_1, x_2, \cdots, x_{n-1}) |$$
$$(x_1, x_2, \cdots, x_{n-1}) \in \bigcup_{i=1, i<j \leq n-2}^{n-3} (B_i, B_j),$$
$$\forall B_i \in \Omega^b_{ei}, i \in \{1, 2, \cdots, n-2\}\}.$$

Lemma 4 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, if $\Omega^a \cap \Omega^b \neq \phi$, take $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^a \cap \Omega^b$, and let $c(s) := s^{n-1} + (x_1 - \varepsilon)s^{n-2} + x_2 s^{n-3} + \cdots + x_{n-2} s + (x_{n-1} + \varepsilon)$ (ε is a sufficiently small positive number), then for $\frac{c(s)}{a(s)}$ and $\frac{c(s)}{b(s)}$, we have $\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0$.

Proof Suppose $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^a \cap \Omega^b$, let $c(s) := s^{n-1} + (x_1 - \varepsilon)s^{n-2} + x_2 s^{n-3} + \cdots + x_{n-2} s + (x_{n-1} + \varepsilon), \varepsilon > 0, \varepsilon$ sufficiently small.

$\forall \omega \in R$, consider

$$\mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] = \frac{1}{|a(j\omega)|^2}[c_1 \omega^{2(n-1)} + c_2 \omega^{2(n-2)}$$
$$+ \cdots + c_{n-1} \omega^2 + c_n]$$
$$+ \mathrm{Re}[\frac{-\varepsilon(j\omega)^{n-2} + \varepsilon}{a(j\omega)}]$$
$$= \frac{1}{|a(j\omega)|^2}[c_1 \omega^{2(n-1)} + c_2 \omega^{2(n-2)}$$
$$+ \cdots + c_{n-1} \omega^2 + c_n]$$
$$+ \frac{(-\varepsilon)}{|a(j\omega)|^2}(-\omega^{2(n-1)} + \tilde{c}(\omega^2)),$$

where $c_l := \sum_{j=0}^{n} (-1)^{l+j} a_j x_{2l-j-1}, l = 1, 2, \cdots, n, a_0 = 1, x_0 = 1, a_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n - 1$, and $\tilde{c}(\omega^2)$ is a real polynomial with order not greater than $2(n-2)$.

In order to prove that $\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0$, let $t = \omega^2$, we only need to prove that, for any $\varepsilon > 0, \varepsilon$ sufficiently small, the following polynomial $f_1(t)$ satisfies

$$f_1(t) := \quad c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_{n-1} t + c_n$$
$$+ \varepsilon(t^{n-1} - \tilde{c}(t)) > 0, \quad \forall t \in [0, +\infty).$$

Since $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^a$, by the definition of Ω^a, it is easy to know that

$$g_1(t) := c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_{n-1} t + c_n > 0, \quad \forall t \in (0, +\infty).$$

Moreover, we obviously have $f_1(0) > 0$, and for any $\varepsilon > 0$, when t is a sufficiently large or sufficiently small positive number, we have $f_1(t) > 0$, namely, there exist $0 < t_1 < t_2$ such that, for all $\varepsilon > 0, t \in [0, t_1] \cup [t_2, +\infty)$, we have $f_1(t) > 0$.

Denote

$$M_1 = \inf_{t \in [t_1, t_2]} g_1(t),$$
$$N_1 = \sup_{t \in [t_1, t_2]} |t^{n-1} - \tilde{c}(t)|.$$

Then $M_1 > 0$ and $N_1 > 0$. Choosing $0 < \varepsilon < \frac{M_1}{N_1}$, by a direct calculation, we have

$$f_1(t) := \quad c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_{n-1} t + c_n$$
$$+ \varepsilon(t^{n-1} - \tilde{c}(t)) > 0, \quad \forall t \in [0, +\infty).$$

Namely,

$$\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0.$$

Similarly, since $(x_1, x_2, \cdots, x_{n-1}) \in \Omega^b$, there exist $0 < t_3 < t_4$ such that, for all $\varepsilon > 0, t \in [0, t_3] \cup [t_4, +\infty)$, we have $f_2(t) > 0$, where

$$f_2(t) := \quad d_1 t^{n-1} + d_2 t^{n-2} + \cdots + d_{n-1} t + d_n$$
$$+ \varepsilon(t^{n-1} - \tilde{d}(t)),$$

$$d_l := \sum_{j=0}^{n} (-1)^{l+j} b_j x_{2l-j-1}, \quad l = 1, 2, \cdots, n,$$

where $b_0 = 1, x_0 = 1, b_i = 0$ if $i < 0$ or $i > n$, and $x_i = 0$ if $i < 0$ or $i > n - 1$, and $\tilde{d}(\omega^2)$ is a real polynomial with order not greater than $2(n-2)$ which is determined by the following equation:

$$\mathrm{Re}[\frac{-\varepsilon(j\omega)^{n-2} + \varepsilon}{b(j\omega)}] = \frac{(-\varepsilon)}{|b(j\omega)|^2}(-\omega^{2(n-1)} + \tilde{d}(\omega^2)).$$

Denote

$$g_2(t) := d_1 t^{n-1} + d_2 t^{n-2} + \cdots + d_{n-1} t + d_n,$$

$$M_2 = \inf_{t\in[t_3,t_4]} g_2(t),$$

$$N_2 = \sup_{t\in[t_3,t_4]} |t^{n-1} - \tilde{d}(t))|.$$

Then $M_2 > 0$ and $N_2 > 0$. Choosing $0 < \varepsilon < \dfrac{M_2}{N_2}$, we have

$$\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0.$$

Thus, by choosing $0 < \varepsilon < \min\{\dfrac{M_1}{N_1}, \dfrac{M_2}{N_2}\}$, Lemma 4 is proved.

Lemma 5 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n$, if $\lambda b(s) + (1-\lambda)a(s) \in H^n, \lambda \in [0,1]$, then $\Omega^a \cap \Omega^b \neq \phi$

Proof If $\forall \lambda \in [0,1], a_\lambda(s) := \lambda b(s) + (1-\lambda)a(s) \in H^n$, by Lemma 3, for any $\lambda \in [0,1]$, $\Omega_{ek}^{a_\lambda}, k = 1, 2, \cdots, n-2$, are $n-2$ ellipses in the first quadrant of the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$.

$\forall \lambda \in [0,1]$, denote

$$\Omega^{a_\lambda} := \{ \quad (x_1, x_2, \cdots, x_{n-1})| \\ (x_1, x_2, \cdots, x_{n-1}) \in \bigcup_{i=1, i<j\leq n-2}^{n-3}(A_{\lambda i}, A_{\lambda j}), \\ \forall A_{\lambda i} \in \Omega_{ei}^{a_\lambda}, i \in \{1, 2, \cdots, n-2\}\}$$

Apparently, when λ changes continuously from 0 to 1, Ω^{a_λ} will change continuously from Ω^a to Ω^b, and $\Omega_{ek}^{a_\lambda}$ will change continuously from Ω_{ek}^a to $\Omega_{ek}^b, k = 1, 2, \cdots, n-2$.

Now assume $\Omega^a \cap \Omega^b = \phi$, by the definitions of Ω^a and Ω^b and Lemma 3, $\exists u_1 > 0, u_2 > 0, u_1 \neq a_1, u_1 \neq b_1$, and $\exists \tilde{k} \in \{1, 2, \cdots, n-2\}$, such that the following hyperplane L in the R^{n-1} space $(x_1, x_2, \cdots, x_{n-1})$

$$L: \quad \frac{x_1}{u_1} + \frac{x_2}{u_2} + \cdots + \frac{x_{n-1}}{u_{n-1}} = 1$$

separates Ω^a and Ω^b, meanwhile, L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously (or tangent with $\Omega_{e1}^b, \Omega_{e2}^b, \cdots, \Omega_{e(n-2)}^b$ and $\Omega_{e\tilde{k}}^a$ simultaneously).

Without loss of generality, suppose that L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously.

As before, the notation $[x]$ stands for the largest integer that is smaller than or equal to the real number x and $\langle y \rangle_z$ stands for the remainder of the nonnegative integer y divided by the positive integer z.

Since L is tangent with $\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ and $\Omega_{e\tilde{k}}^b$ simultaneously, note that $a(s)$ is Hurwitz stable and $u_1 > 0, u_1 \neq a_1, u_2 > 0$, using mathematical induction, by a lengthy calculation [22, 23, 25] [27, 28] [30]-[34], we know that the necessary and sufficient condition for L being tangent with

$\Omega_{e1}^a, \Omega_{e2}^a, \cdots, \Omega_{e(n-2)}^a$ simultaneously is [5]

$$\sum_{i=0}^{n} (-1)^{[\frac{i+1}{2}]} a_i u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}] - [\frac{i}{2}]} = 0 \qquad (1)$$

and

$$u_j = (-1)^{[\frac{j-1}{2}]} u_1^{\langle j \rangle_2} u_2^{[\frac{j}{2}]}, \quad j = 3, 4, \cdots, n-1, \qquad (2)$$

where $a_0 = 1$.

Since $u_j = (-1)^{[\frac{j-1}{2}]} u_1^{\langle j \rangle_2} u_2^{[\frac{j}{2}]}, \quad j = 3, 4, \cdots, n-1$, L is tangent with $\Omega_{e\tilde{k}}^b$, by a direct calculation, we have

$$\sum_{i=0}^{n} (-1)^{[\frac{i+1}{2}]} b_i u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}] - [\frac{i}{2}]} = 0 \qquad (3)$$

where $b_0 = 1$.

From (1) and (3), we obviously have $\forall \lambda \in [0,1]$,

$$\sum_{i=0}^{n} (-1)^{[\frac{i+1}{2}]} a_{\lambda i} u_1^{\langle i+1\rangle_2} u_2^{[\frac{n}{2}] - [\frac{i}{2}]} = 0 \qquad (4)$$

where $a_{\lambda i} := a_i + \lambda(b_i - a_i), a_0 = 1, b_0 = 1, i = 0, 1, 2, \cdots, n$. (4) and (2) show that L is also tangent with $\Omega_{e\tilde{k}}^{a_\lambda}$ ($\forall \lambda \in [0,1]$), but L separates $\Omega_{e\tilde{k}}^a$ and $\Omega_{e\tilde{k}}^b$, and when λ changes continuously from 0 to 1, $\Omega_{e\tilde{k}}^{a_\lambda}$ will change continuously from $\Omega_{e\tilde{k}}^a$ to $\Omega_{e\tilde{k}}^b$, which is obviously impossible. This completes the proof.

From Theorem 2.4 in [23], or the proof of Lemma 5 in [28], we have

Lemma 6 Suppose $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in H^n, b(s) = s^n + b_1 s^{n-1} + \cdots + b_n \in H^n, c(s) = s^{n-1} + x_1 s^{n-2} + \cdots + x_{n-1}$, if $\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0$, take

$$\widetilde{c}(s) := c(s) + \delta \cdot h(s), \quad \delta > 0, \; \delta \text{ sufficiently small},$$

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[5]When $n = 3$, we have (see [27, 28] for details):

$$u_1 u_2 - a_1 u_2 - a_2 u_1 + a_3 = 0.$$

When $n = 4$, we have (see [25] [30]-[32] for details):

$$\begin{cases} u_1 u_2^2 - a_1 u_2^2 - a_2 u_1 u_2 + a_3 u_2 + a_4 u_1 = 0, \\ u_3 = -u_1 u_2. \end{cases}$$

When $n = 5$, we have (see [33] for details):

$$\begin{cases} u_1 u_2^2 - a_1 u_2^2 - a_2 u_1 u_2 + a_3 u_2 + a_4 u_1 - a_5 = 0, \\ u_3 = -u_1 u_2, \; u_4 = -u_2^2. \end{cases}$$

When $n = 6$, we have (see [34] for details):

$$\begin{cases} u_1 u_2^3 - a_1 u_2^3 - a_2 u_1 u_2^2 + a_3 u_2^2 + a_4 u_1 u_2 - a_5 u_2 - a_6 u_1 = 0, \\ u_3 = -u_1 u_2, \; u_4 = -u_2^2, \; u_5 = u_1 u_2^2. \end{cases}$$

(where $h(s)$ is an arbitrarily given monic n-th order polynomial), then $\dfrac{\widetilde{c}(s)}{a(s)}$ and $\dfrac{\widetilde{c}(s)}{b(s)}$ are both strictly positive real.

The sufficiency of Theorem 1 is now proved by simply combining Lemmas 3-6.

Remark 1 From the proof of Theorem 1, we can see that this paper not only proves the existence, but also provides a design method. In fact, based on the main idea of this paper, we have proposed a geometric algorithm with order reduction for robust SPR synthesis which is very efficient for high order polynomial segments [26].

Remark 2 The method provided in this paper is constructive, and is insightful and helpful in solving more general robust SPR synthesis problems for polynomial polytopes, multilinear families, etc..

Remark 3 Our main result in this paper can also be extended to discrete-time case. In fact, by applying the bilinear transformation of the unit circle into the left half plane, Theorem 1 can be generalized to discrete-time case as well. Moreover, in discrete-time case, the order of the polynomial obtained according to our method is bounded by the order of given polynomial segment [27, 29].

Remark 4 If $\dfrac{c(s)}{a(s)}$ and $\dfrac{c(s)}{b(s)}$ are both SPR, it is easy to know that $\forall \lambda \in [0,1]$, $\dfrac{c(s)}{\lambda a(s) + (1-\lambda)b(s)}$ is also SPR.

Remark 5 The stability of polynomial segment can be checked by many efficient methods, e.g., eigenvalue method, root locus method, value set method, etc. [1, 4, 6].

Remark 6 By using similar method, we can constructively prove the existence of SPR synthesis for low order ($n \leq 4$) interval polynomials. Namely, when $n \leq 4$, the Hurwitz stability of the four Kharitonov vertex polynomials is necessary and sufficient for the existence of a fixed polynomial such that the ratio of this polynomial to any polynomial in the interval polynomial set is SPR invariant [22, 23, 25, 27, 30, 31]. The SPR synthesis for high order interval systems is currently under investigation.

Finally, we give an the example to show that our method is very effective.

Example 1 Suppose $a(s) = s^6 + 12s^5 + 70s^4 + 300s^3 + 500s^2 + 600s + 300$, $b(s) = s^6 + 14s^5 + 60s^4 + 280s^3 + 490s^2 + 650s + 400$. It is easy to see that the convex combination of the two polynomials $a(s)$ and $b(s)$ is robustly Hurwitz stable. Using our method, we can get $(12, 49, , 48, , 99, 0) \in \Omega^a = \{(x_1, x_2, x_3, x_4, x_5)|c_1 = c_2 = c_6 = 0, c_4^2 - c_3c_5 = 8.1656 \times 10^5 - 33670.0x_2 + 1214.1x_4 + 349.41x_2^2 - 28.941x_2x_4 + x_4^2 < 0\}$ such as $\bar{c}(s) := s^5 + 12s^4 + 49s^3 + 48s^2 + 99s$ and $\forall \omega \in R\backslash\{0\}, \mathrm{Re}[\dfrac{\bar{c}(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\dfrac{\bar{c}(j\omega)}{b(j\omega)}] > 0$. Let $c(s) = s^5 + (12-\varepsilon)s^4 + 49s^3 + 48s^2 + 99s + \varepsilon$, ε is a sufficiently small positive number (in this example, $0 < \varepsilon \leq 4$), take $\varepsilon = 2$,

then for $\dfrac{c(s)}{a(s)}$ and $\dfrac{c(s)}{b(s)}$, we have $\forall \omega \in R, \mathrm{Re}[\dfrac{c(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\dfrac{c(j\omega)}{b(j\omega)}] > 0$. Thus, let $\tilde{c}(s) := c(s) + \delta s^6, \delta > 0, \delta$ sufficiently small, e.g., take $\delta \leq 0.06$, then the design requirement has been met.

Note that, in the Example 1, $c(s)$ is not unique.

Acknowledgements

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