# $H^\infty$ PERFORMANCE OF INTERVAL SYSTEMS

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## Abstract

In this paper, we study $H^\infty$ performance of interval systems. We show that, for an interval system, the maximal $H^\infty$ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices. Furthermore, we study robustness of multivariable systems with parametric uncertainties, and establish a multivariable version of Edge Theorem. An illustrative example is presented.

## 1 Introduction

Motivated by the seminal theorem of Kharitonov on robust stability of interval polynomials [1], a number of papers on robustness analysis of uncertain systems have been published in the past few years [2–15]. Kharitonov's theorem states that the Hurwitz stability of the real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight) prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov's theorem is the edge theorem discovered by Bartlett, Hollot and Huang [2]. The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some (affine) dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. Wang and Huang extended Kharitonov's theorem to robust performance of interval systems [8]. They proved that the strict positive realness of an interval transfer function family can be guaranteed by the same property of only eight prescribed critical vertex transfer functions in this family. When the dependency among polynomial coefficients is nonlinear, however, Ackermann shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family [9, 10].

In this paper, we show that, for an interval system, the maximal $H_\infty$ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices. This result is useful in robust performance analysis and $H_\infty$ control design for dynamic systems under parametric perturbations. Furthermore, we study robustness of multivariable systems with parametric uncertainties, and establish a multivariable version of Edge Theorem. An illustrative example is presented.

## 2 System Gain Computation

Denote the $m$-th, $n$-th $(m < n)$ order real interval polynomial families $K_g(s)$, $K_f(s)$ as

$$K_g(s) = \{g(s)|g(s) = \sum_{i=0}^{m} b_i s^i, b_i \in [\underline{b_i}, \overline{b_i}], i = 0, 1, ......, m\}, \tag{1}$$

$$K_f(s) = \{f(s)|f(s) = \sum_{i=0}^{n} a_i s^i, a_i \in [\underline{a_i}, \overline{a_i}], i = 0, 1, ......, n\}. \tag{2}$$

For any $f(s) \in K_f(s)$ , it can be expressed as

$$f(s) = \alpha_f(s^2) + s\beta_f(s^2), \tag{3}$$

where

$$\alpha_f(s^2) = a_0 + a_2 s^2 + a_4 s^4 + a_6 s^6 + ......, \tag{4}$$

$$\beta_f(s^2) = a_1 + a_3 s^2 + a_5 s^4 + a_7 s^6 + ....... \tag{5}$$

For the interval polynomial family $K_f(s)$, define

$$\alpha_f^{(1)}(s^2) = \underline{a_0} + \overline{a_2} s^2 + \underline{a_4} s^4 + \overline{a_6} s^6 + ......, \tag{6}$$

$$\alpha_f^{(2)}(s^2) = \overline{a_0} + \underline{a_2} s^2 + \overline{a_4} s^4 + \underline{a_6} s^6 + ......, \tag{7}$$

$$\beta_f^{(1)}(s^2) = \underline{a_1} + \overline{a_3} s^2 + \underline{a_5} s^4 + \overline{a_7} s^6 + ......, \tag{8}$$

$$\beta_f^{(2)}(s^2) = \overline{a_1} + \underline{a_3} s^2 + \overline{a_5} s^4 + \underline{a_7} s^6 + ......, \tag{9}$$

and denote the four Kharitonov vertex polynomials of $K_f(s)$ as

$$f_{ij}(s) = \alpha_f^{(i)}(s^2) + s\beta_f^{(j)}(s^2), \qquad i, j = 1, 2 \tag{10}$$

For the interval polynomial family $K_g(s)$, the corresponding $\alpha_g^{(i)}(s), \beta_g^{(j)}(s)$ and $g_{ij}(s) \in K_g(s)$ can be defined analogously.

Denote by H the set of all Hurwitz stable polynomials (i.e. all of their roots lie within the open left half of the complex plane).

For the proper stable rational function $\frac{p(s)}{q(s)}$, the $H_\infty$ norm is defined as

$$\left\| \frac{p(s)}{q(s)} \right\|_\infty = \sup \left\{ \left| \frac{p(j\omega)}{q(j\omega)} \right| \mid \omega \in (-\infty\ ,\ +\infty) \right\} \tag{11}$$

Consider the strictly proper open-loop transfer function $P = \frac{g(s)}{f(s)}$, and suppose the closed-loop system is stable under negative unity feedback. Denote its sensitivity function as $S = \frac{1}{1+P} = \frac{f(s)}{f(s)+g(s)}$. Apparently, $||S||_\infty \geq 1$.

For notational simplicity, define

$$J_{i_1 j_1 i_2 j_2}(s) = g_{i_1 j_1}(s) + (1 + \delta e^{j\theta}) f_{i_2 j_2}(s), \\ \delta \in (0,1),\ \ i_1, j_1, i_2, j_2 = 1,2,\ \ \theta \in [-\pi, \pi]. \tag{12}$$

**Lemma A** Suppose $g(s) + f(s) \in H$. Then, for any $\gamma > 1$, we have

$$||S||_\infty < \gamma \Longleftrightarrow g(s) + (1 + \frac{1}{\gamma} e^{j\theta}) f(s) \in H,\ \forall \theta \in [-\pi, \pi].$$

**Lemma B** For any $\delta \in (0,1), \theta \in [-\pi, \pi]$, we have

$$W(s) \subset H \tag{13}$$

$\Longleftrightarrow$

$$J_{1111}, J_{1212}, J_{2222}, J_{2121}, J_{1112}, J_{1222}, \\ J_{2221}, J_{2111}, J_{1211}, J_{2212}, J_{2122}, J_{1121} \in H \tag{14}$$

where $W(s) =: \{g(s) + (1 + \delta e^{j\theta}) f(s) | g(s) \in K_g(s), f(s) \in K_f(s)\}$.

The following theorem shows that, for an interval system, the maximal $H_\infty$ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices.

**Theorem A** Suppose $g_{ij}(s) + f_{ij}(s) \in H,\ i,j = 1,2$. Then

$$\max\{||\frac{f(s)}{f(s)+g(s)}||_\infty | g(s) \in K_g(s), f(s) \in K_f(s)\} = \tag{15}$$

$$\max\{||\frac{f_{i_2 j_2}(s)}{f_{i_2 j_2}(s) + g_{i_1 j_1}(s)}||_\infty | (i_1 j_1 i_2 j_2) = \\ (1111), (1212), (2222), (2121), (1112), (1222), \\ (2221), (2111), (1211), (2212), (2122), (1121)\} \tag{16}$$

## 3 Robust Positivity

**Lemma C** For any fixed $\omega, \beta \in R$, if $f(j\omega) \neq 0$, then

$$\Re \frac{g(j\omega) - \beta f(j\omega)}{f(j\omega)} > 0 \tag{17}$$

if and only if

$$f(j\omega)s + g(j\omega) - \beta f(j\omega) \in H \tag{18}$$

Proof: For any fixed $\omega, \beta \in R$, $g(j\omega) - \beta f(j\omega)$ and $f(j\omega)$ are fixed complex numbers. Thus $f(j\omega)s + [g(j\omega) - \beta f(j\omega)]$ is a first order complex polynomial, and

$$f(j\omega)s + g(j\omega) - \beta f(j\omega) \in H$$

$$\Longleftrightarrow \quad \Re \frac{-[g(j\omega) - \beta f(j\omega)]}{f(j\omega)} < 0 \tag{19}$$

$$\Longleftrightarrow \quad \Re \frac{g(j\omega) - \beta f(j\omega)}{f(j\omega)} > 0$$

This completes the proof.

Consider the interval complex numbers $c_0 + jd_0, c_1 + jd_1$, where $c_i \in [c_i^-, c_i^+], d_i \in [d_i^-, d_i^+], i = 1,2$. Define the sign functions

$$sgn[c_i] = \begin{cases} 1 & c_i = c_i^- \\ \\ 2 & c_i = c_i^+ \end{cases} \tag{20}$$

$$sgn[d_i] = \begin{cases} 1 & d_i = d_i^- \\ \\ 2 & d_i = d_i^+ \end{cases} \tag{21}$$

and define the index sets $I_1, I_2$ as

$$I_1 = \{(1222), (1221), (2221), (2211), (2111), (2112), \\ (1112), (1122), (1211), (2212), (2122), (1121)\}$$

$$I_2 = \{(1112), (1222), (2111), (2221), (1121), (1211), \\ (2122), (2212)\}$$

For any two polynomials $h^{(1)}(s), h^{(2)}(s)$, denote their convex combination as

$$L[h^{(1)}, h^{(2)}] = \{\lambda h^{(1)}(s) + (1-\lambda) h^{(2)}(s) | \lambda \in [0,1]\} \tag{22}$$

**Lemma D** [6] For any fixed $\beta > 0$, the first order complex polynomial set

$$W_1(s) := \{(c_1 + jd_1)(s - \beta) + (c_0 + jd_0) | \\ c_i \in [c_i^-, c_i^+], d_i \in [d_i^-, d_i^+], i = 1,2\} \subset H \tag{23}$$

if and only if

$$\{(c_1 + jd_1)(s - \beta) + (c_0 + jd_0) | \\ (sgn[c_0] \quad sgn[d_0] \quad sgn[c_1] \quad sgn[d_1]) \in I_1\} \subset H \tag{24}$$

**Lemma E** [1, 8] The first order interval complex polynomial set

$$W_2(s) := \{(c_1 + jd_1)s + (c_0 + jd_0) | \\ c_i \in [c_i^-, c_i^+], d_i \in [d_i^-, d_i^+], i = 1,2\} \subset H \tag{25}$$

if and only if

$$\{(c_1 + jd_1)s + (c_0 + jd_0) | \\ (sgn[c_0] \quad sgn[d_0] \quad sgn[c_1] \quad sgn[d_1]) \in I_2\} \subset H \tag{26}$$

**Theorem B** For any fixed $\omega \in R$, if $0 \notin K_f(j\omega)$ and

$$\min\{\Re \frac{g_{i_1 j_1}(j\omega)}{f_{i_2 j_2}(j\omega)} | (i_1 \quad j_1 \quad i_2 \quad j_2) \in I_1\} := \beta_0 > 0 \tag{27}$$

Then

$$\min\{\Re \frac{g(j\omega)}{f(j\omega)} | g(s) \in K_g(s), f(s) \in K_f(s)\} = \beta_0 \tag{28}$$

Proof: Since $g_{ij}(s) \in K_g(s), f_{ij}(s) \in K_f(s), i,j = 1,2$, we have

$$\min\{\Re \frac{g(j\omega)}{f(j\omega)} | g(s) \in K_g(s), f(s) \in K_f(s)\} \leq \beta_0 \tag{29}$$

Suppose

$$\min\{\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}:=\beta_1<\beta_0$$

Since $\beta_0>0$, there exists $\beta_2>0$ such that $\beta_1<\beta_2<\beta_0$. Hence, for any $(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_1$, we have

$$\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}\geq\beta_0>\beta_2>0 \tag{30}$$

Namely

$$\Re\frac{g_{i_1j_1}(j\omega)-\beta_2 f_{i_2j_2}(j\omega)}{f_{i_2j_2}(j\omega)}>0 \tag{31}$$

By Lemma C, for $(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_1$, we have

$$f_{i_2j_2}(j\omega)s+g_{i_1j_1}(j\omega)-\beta_2 f_{i_2j_2}(j\omega)\in H. \tag{32}$$

Consider the first order complex polynomial set

$$\begin{aligned}W_3(s):=\{&f(j\omega)s+g(j\omega)-\beta_2 f(j\omega)|\\&g(s)\in K_g(s), f(s)\in K_f(s)\}\end{aligned} \tag{33}$$

Apparently, when $\omega\geq 0$, we have

$$\alpha_f^{(1)}(-\omega^2)\leq\Re f(j\omega)\leq\alpha_f^{(2)}(-\omega^2) \tag{34}$$

$$\omega\beta_f^{(1)}(-\omega^2)\leq\Im f(j\omega)\leq\omega\beta_f^{(2)}(-\omega^2) \tag{35}$$

$$\alpha_g^{(1)}(-\omega^2)\leq\Re g(j\omega)\leq\alpha_g^{(2)}(-\omega^2) \tag{36}$$

$$\omega\beta_g^{(1)}(-\omega^2)\leq\Im g(j\omega)\leq\omega\beta_g^{(2)}(-\omega^2) \tag{37}$$

By Lemma D, $W_3(s)\subset H$. When $\omega<0$, the two inequalities on the imaginary parts of $f(j\omega), g(j\omega)$ above will be reversed. By Lemma D, $W_3(s)\subset H$. Hence, for any fixed $\omega\in R, f(s)\in K_f(s), g(s)\in K_g(s)$, we have

$$f(j\omega)s+g(j\omega)-\beta_2 f(j\omega)\in H \tag{38}$$

By Lemma C, we have

$$\Re\frac{g(j\omega)-\beta_2 f(j\omega)}{f(j\omega)}>0,\quad\forall f(s)\in K_f(s), g(s)\in K_g(s)$$

Namely

$$\Re\frac{g(j\omega)}{f(j\omega)}>\beta_2,\quad\forall f(s)\in K_f(s), g(s)\in K_g(s) \tag{39}$$

Namely

$$\min\{\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}=\beta_1>\beta_2$$

which contradicts $\beta_1<\beta_2<\beta_0$. This completes the proof.

**Corollary A** If $f_{ij}(s)\in H, i,j=1,2$ and

$$\min\{\inf_{\omega\in R}\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}|(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_1\}:=\gamma_0>0$$

Then

$$\min\{\inf_{\omega\in R}\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}=\gamma_0 \tag{40}$$

Proof: Since $f_{ij}(s)\in H, i,j=1,2$, by Kharitonov Theorem [1], $K_f(s)\subset H$. Hence

$$0\notin K_f(j\omega),\quad\forall\omega\in R \tag{41}$$

Moreover, since $g_{ij}(s)\in K_g(s), f_{ij}(s)\in K_f(s), i,j=1,2$, we have

$$\min\{\inf_{\omega\in R}\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}:=\gamma_1\leq\gamma_0$$

Suppose $\gamma_1<\gamma_0$, since $\gamma_0>0$, there exists $\gamma_2>0$ such that $\gamma_1<\gamma_2<\gamma_0$. Since $\gamma_0>\gamma_2>0$, for any fixed $\omega\in R$, we have

$$\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}>\gamma_2>0,\quad\forall(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_1 \tag{42}$$

By Theorem B, we have

$$\Re\frac{g(j\omega)}{f(j\omega)}>\gamma_2>0,\quad\forall f(s)\in K_f(s), g(s)\in K_g(s) \tag{43}$$

Hence, we have $\inf_{\omega\in R}\Re\frac{g(j\omega)}{f(j\omega)}\geq\gamma_2$. Namely

$$\min\{\inf_{\omega\in R}\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}=\gamma_1\geq\gamma_2$$

which contradicts $\gamma_1<\gamma_2<\gamma_0$. This completes the proof.

By similar analysis, we have

**Corollary B** If $\forall\omega\in[\omega_1,\omega_2], 0\notin K_f(j\omega)$ and

$$\min\{\inf_{\omega\in[\omega_1,\omega_2]}\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}|(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_1\}:=\gamma_0>0$$

Then

$$\min\{\inf_{\omega\in[\omega_1,\omega_2]}\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}=\gamma_0$$

**Theorem C** For any fixed $\omega\in R$, if $0\notin K_f(j\omega)$, then

$$\min\{\Re\frac{g(j\omega)}{f(j\omega)}|g(s)\in K_g(s), f(s)\in K_f(s)\}>0 \tag{44}$$

if and only if

$$\min\{\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}|(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_2\}>0 \tag{45}$$

Proof: Necessity: Obvious.

Sufficiency: Since

$$\min\{\Re\frac{g_{i_1j_1}(j\omega)}{f_{i_2j_2}(j\omega)}|(i_1 \quad j_1 \quad i_2 \quad j_2)\in I_2\}>0 \tag{46}$$

By Lemma C, for any fixed $\omega \in R$

$$f_{i_2 j_2}(j\omega)s + g_{i_1 j_1}(j\omega) \in H, \quad \forall (i_1 \quad j_1 \quad i_2 \quad j_2) \in I_2 \tag{47}$$

Consider the first order interval complex polynomial set

$$W_4(s) := \{f(j\omega)s + g(j\omega) | g(s) \in K_g(s), f(s) \in K_f(s)\} \tag{48}$$

By the proof of Theorem B and by Lemma E, $W_4(s) \subset H$. Hence, for any fixed $\omega \in R, f(s) \in K_f(s), g(s) \in K_g(s)$, we have

$$f(j\omega)s + g(j\omega) \in H \tag{49}$$

By Lemma C, we have

$$\Re \frac{g(j\omega)}{f(j\omega)} > 0, \quad \forall f(s) \in K_f(s), g(s) \in K_g(s) \tag{50}$$

This completes the proof.

## 4 More Technical Tools

For Hurwitz stability of interval matrices, Bialas 'proved' that in order to guarantee robust stability, it suffices to check all vertex matrices [12]. Later, it was shown by Barmish that Bialas' result was incorrect [13]. Kokame and Mori eastblished a Kharitonov-like result on robust Hurwitz stability of interval polynomial matrices [14], and Kamal and Dahleh established some robust stability criteria for MIMO systems with fixed controllers and uncertain plants [15].

In what follows, we will study robustness of a class of MIMO systems with their transfer function matrices described by

$$\mathcal{F}(s) = \left\{ \begin{pmatrix} a_{11}(s) & \ldots & a_{1n}(s) \\ \ldots & \ldots & \ldots \\ a_{n1}(s) & \ldots & a_{nn}(s) \end{pmatrix} : \; a_{ij}(s) \in \mathcal{A}_{ij}(s) \right\}$$
$$\mathcal{A}_{ij}(s) = \text{conv}\,\{b1_{ij}(s), \ldots, bm_{ij}(s)\} \tag{51}$$

where $m$ is a given positive integer.

**Definition 1** A polynomial matrix is a matrix with all its entries being polynomials.

**Definition 2** Suppose $D$ is a simply-connected region in the complex plane. If all the roots of the determinant of a polynomial matrix lie within $D$, then this polynomial matrix is called $D$-stable. A set of polynomial matrices is called robustly $D$-stable, if every polynomial matrix in this set is $D$-stable.

**Definition 3** Suppose $f_1(s), \ldots, f_m(s)$ are $m$ given polynomials, the set

$$\left\{ \sum_{i=1}^{m} \lambda_i f_i(s) : \; \lambda_i \geq 0, \; \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

is called the polynomial polytope generated by $f_1(s), \ldots, f_m(s)$, denoted as conv $\{f_1(s), \ldots, f_m(s)\}$.

**Definition 4** The polynomial $f(s) = a_0 + a_1 s + \ldots + a_n s^n$, with $a_i \in [a_i^L, a_i^U]$ is called an interval polynomial.

**Definition 5** $S_n$ is the set of all bijections from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

**Definition 6** The vertex set and edge set of $\mathcal{A}_{ij}(s)$ are

$$K_{ij}(s) = \{b1_{ij}(s), \ldots, bm_{ij}(s)\}$$

$$E_{ij}(s) = \{ \quad \lambda br_{ij}(s) + (1-\lambda) bt_{ij}(s), \quad \lambda \in [0,1], \; r, t \in \{1, \ldots, m\}\}$$

respectively.

**Definition 7**

$$\mathcal{F}_E(s) = \bigcup_{\sigma \in S_n} \left\{ (a_{ij}(s))_{n \times n} : a_{ij}(s) \left\{ \begin{array}{l} \in E_{ij}(s) \text{ if } i = \sigma(j) \\ \in K_{ij}(s) \text{ if } i \neq \sigma(j) \end{array} \right. \right\} \tag{52}$$

**Lemma 1** (Edge Theorem [2]) Suppose $\Gamma \subset \mathcal{C}$ is a simply-connected region, $\Omega$ is a polynomial polytope without degree dropping. Then, $\Omega$ is $\Gamma$-stable if and only if all the edges of $\Omega$ are $\Gamma$-stable.

**Lemma 2** Suppose $A(s)$ is a given $n \times (n-1)$ polynomial matrix. Then

$$\left\{ \begin{pmatrix} a_{11}(s) & \\ \vdots & A(s) \\ a_{n1}(s) & \end{pmatrix} : \quad \begin{array}{c} a_{i1}(s) \in \mathcal{A}_{i1}(s), \\ i = 1, \ldots, n \end{array} \right\}$$

is robustly $D$-stable $\Leftrightarrow$ for all $i = 1, \ldots, n$,

$$\left\{ \begin{pmatrix} a_{11}(s) & \\ \vdots & A(s) \\ a_{n1}(s) & \end{pmatrix} : \quad \begin{array}{ll} a_{l1}(s) \in K_{lj}(s) & l \neq i \\ a_{l1}(s) \in E_{lj}(s) & l = i \end{array} \right\}$$

is robustly $D$-stable.

Proof: Necessity is obvious, since the later is a subset of the former.

Sufficiency: For any $a_{i1}(s) \in \mathcal{A}_{i1}(s)$, the corresponding matrix is

$$T(s) = \begin{pmatrix} a_{11}(s) & \\ \vdots & A(s) \\ a_{n1}(s) & \end{pmatrix}$$

By Laplace Formula, we can expand the determinant of $T(s)$ along its first column. Then, by convexity and by Lemma 1, we know that $T(s)$ is robustly $D$-stable.

**Lemma 3** Suppose $B(s)$ is a given $(n-1) \times n$ polynomial matrix. $*$ stands for fixed entries in a matrix. Then

$$\left\{ \begin{pmatrix} * \quad a_{1i}(s) \quad * \quad a_{1j}(s) \quad * \\ B(s) \end{pmatrix} : \begin{array}{c} a_{1i}(s) \in \mathcal{A}_{1i}(s) \\ a_{1j}(s) \in \mathcal{A}_{1j}(s) \end{array} \right\}$$

is robustly $D$-stable $\Leftrightarrow$

$$\left\{ \begin{pmatrix} * \quad a_{1i}(s) \quad * \quad a_{1j}(s) \quad * \\ B(s) \end{pmatrix} : \begin{array}{c} a_{1i}(s) \times a_{1j}(s) \in \\ (K_{1i}(s) \times E_{1j}(s)) \cup (E_{1i}(s) \times K_{1j}(s)) \end{array} \right\}$$

is robustly $D$-stable.

Proof: the proof is analogous to the proof of Lemma 2, except that the Laplace expansion is carried out along the row instead of the column.

## 5 Multivariable Edge Theorem

**Theorem 1** $\mathcal{F}(s)$ is robustly $D$-stable if and only if $\mathcal{F}_E(s)$ is robustly $D$-stable.

Proof: Necessity is obvious. To prove sufficiency, we first note that interchanging any two rows (or columns) does not affect the stability of a polynomial matrix (it only changes the sign of the determinant). By Lemma 2

$$\begin{array}{l} \mathcal{F}(s) \text{ is robustly } D \text{ stable} \\ \Leftrightarrow \text{ for all } i=1,\ldots,n, \\ \left\{ \begin{pmatrix} a_{11}(s) & \\ \vdots & A(s) \\ a_{n1}(s) & \end{pmatrix} : \begin{array}{ll} a_{l1}(s)\in K_{lj}(s) & l\neq i \\ a_{l1}(s)\in E_{lj}(s) & l=i \end{array} \right\} \\ \text{ is robustly } D \text{ stable.} \\ \Leftrightarrow \text{ for all } \left\{ \begin{array}{l} i_1=1,\ldots,n \\ i_2=1,\ldots,n \end{array} \right. \\ \left\{ \begin{pmatrix} a_{11}(s) & a_{12}(s) & \\ \vdots & \vdots & A_1(s) \\ a_{n1}(s) & a_{n2}(s) & \end{pmatrix} : \right. \\ \qquad \left. \begin{array}{ll} a_{l1}(s)\in K_{lj}(s) & l\neq i_1 \\ a_{l1}(s)\in E_{lj}(s) & l=i_1 \\ a_{l2}(s)\in K_{lj}(s) & l\neq i_2 \\ a_{l2}(s)\in E_{lj}(s) & l=i_2 \end{array} \right\} \\ \text{ is robustly } D \text{ stable.} \end{array}$$

where $A_1(s)$ is the corresponding $n\times(n-2)$ polynomial matrix. This last equivalence is based on Lemma 2 and the fact that interchanging two columns does not change the stability of a polynomial matrix. Repeating the process above, let $Y_n$ denote the set of all mappings from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$, then

$$\begin{array}{l} \mathcal{F}(s) \text{ is robustly } D \text{ stable} \\ \Leftrightarrow \text{ for all } \eta\in Y_n, \left\{ (a_{ij}(s)) : \begin{array}{ll} a_{ij}(s)\in K_{ij}(s) & i\neq\eta(j) \\ a_{ij}(s)\in E_{ij}(s) & i=\eta(j) \end{array} \right\} \\ \text{ is robustly } D \text{ stable.} \end{array}$$

If there exists an $\eta\in Y_n$ such that $\eta(i_1)=\eta(i_2)=k$, then the corresponding matrix $F(s)=(a_{ij}(s))$ satisfies

$$\begin{array}{l} a_{i_1k}(s)\in E_{i_1k}(s) \\ a_{i_2k}(s)\in E_{i_2k}(s) \end{array}$$

Applying Lemma 2 to column $k$ of $F(s)$, we have

$$\begin{array}{l} \mathcal{F}(s) \text{ is robustly } D \text{ stable} \\ \Leftrightarrow \text{ for all } \sigma\in S_n, \left\{ (a_{ij}(s)) : \begin{array}{ll} a_{ij}(s)\in K_{ij}(s) & i\neq\sigma(j) \\ a_{ij}(s)\in E_{ij}(s) & i=\sigma(j) \end{array} \right\} \\ \text{ is robustly } D \text{ stable.} \\ \Leftrightarrow \mathcal{F}_E(s) \text{ is robustly } D \text{ stable.} \end{array}$$

## 6 Multivariable Interval Model

Interval model, as a simple and effective approximation of uncertain systems, has been the subject of study in robustness analysis for a long time. In a similar vein, we consider the Hurwitz stability of the following uncertain system.

$$\begin{array}{l} \mathcal{G}(s)=\{(c_{ij}(s)) : c_{ij}(s)\in\mathcal{C}_{ij}(s)\} \\ \mathcal{C}_{ij}(s) \text{ are interval polynomials} \end{array} \tag{53}$$

**Definition 8** For the interval polynomial $\mathcal{C}_{ij}(s) = \{\sum_{l=0}^{n} q_l(ij)s^l, \quad q_l(ij) \in [\underline{q}_l(ij),\overline{q}_l(ij)]\}$, its Kharitonov vertex set and Kharitonov edge set are defined respectively as

$$\begin{array}{l} K_{ij}^I(s)=\{c_k^1(s),c_k^2(s),c_k^3(s),c_k^4(s)\} \\ E_{ij}^I(s)=\{\lambda c_k^r(s)+(1-\lambda)c_k^t(s),\ \lambda\in[0,1], \\ \qquad (r,t)\in\{(1,2),(2,4),(4,3),(3,1)\}\} \end{array}$$

where

$$\begin{array}{l} c_k^1(s)=\underline{q}_0(ij)+\underline{q}_1(ij)s+\overline{q}_2(ij)s^2+\overline{q}_3(ij)s^3+\ldots \\ c_k^2(s)=\underline{q}_0(ij)+\overline{q}_1(ij)s+\overline{q}_2(ij)s^2+\underline{q}_3(ij)s^3+\ldots \\ c_k^3(s)=\overline{q}_0(ij)+\underline{q}_1(ij)s+\underline{q}_2(ij)s^2+\overline{q}_3(ij)s^3+\ldots \\ c_k^4(s)=\overline{q}_0(ij)+\overline{q}_1(ij)s+\underline{q}_2(ij)vs^2+\underline{q}_3(ij)s^3+\ldots \end{array}$$

**Definition 9**

$$\mathcal{G}_E(s)=\bigcup_{\sigma\in S_n}\left\{(c_{ij}(s))_{n\times n} : c_{ij}(s)\left\{ \begin{array}{l} \in E_{ij}^I(s) \text{ if } i=\sigma(j) \\ \in K_{ij}^I(s) \text{ if } i\neq\sigma(j) \end{array} \right. \right\} \tag{54}$$

**Lemma 4** (Box Theorem[4—6, 9—11, 14—15]) Suppose $\Delta(s)=\{\delta(s,p)=F_1(s)P_1(s)+\ldots+F_m(s)P_m(s)\}$, $P_i(s)$ is an interval polynomial, $F_i(s)$ is a given fixed polynomial, $i=1,\ldots,m$. And suppose $\Delta(s)$ is degree-invariant. Then, $\Delta(s)$ is Hurwitz stable if and only if $\Delta_E(s)$ is Hurwitz stable, where $\Delta_E(s) = \cup_{l=1}^{m}\{\sum_{i=1}^{l-1}F_i(s)K_{P_i}^0(s)+F_l(s)E_{P_l}^0(s)+\sum_{i=l+1}^{m}F_i(s)K_{P_i}^0(s)\}$ (let $\sum_{i=r}^{t}f_i=0$, if $r>t$).

By resort to the Box Theorem, and following a similar line of arguments as in the proof of Theorem 1, we can get some analogous stability verification results for the interval model.

**Lemma 5** Suppose $A(s)$ is a given $n\times(n-1)$ polynomial matrix. Then

$$\begin{array}{l} \left\{ \begin{pmatrix} c_{11}(s) & \\ \vdots & A(s) \\ c_{n1}(s) & \end{pmatrix} : \begin{array}{c} c_{i1}(s)\in\mathcal{G}_{i1}(s), \\ i=1,\ldots,n \end{array} \right\} \\ \text{is robustly Hurwitz stable.} \\ \Leftrightarrow \text{ for all } i=1,\ldots,n, \\ \left\{ \begin{pmatrix} c_{11}(s) & \\ \vdots & A(s) \\ c_{n1}(s) & \end{pmatrix} : \begin{array}{ll} c_{l1}(s)\in K_{lj}^I(s) & l\neq i \\ c_{l1}(s)\in E_{lj}^I(s) & l=i \end{array} \right\} \\ \text{is robustly Hurwitz stable.} \end{array}$$

**Lemma 6** Suppose $B(s)$ is a given $(n-1)\times n$ polynomial matrix. $*$ stands for fixed entries in a matrix. Then

$$\left\{\begin{pmatrix} * & c_{1i}(s) & * & c_{1j}(s) & * \\ & & B(s) & & \end{pmatrix} : \begin{array}{l} c_{1i}(s) \in \mathcal{C}_{1i}(s) \\ c_{1j}(s) \in \mathcal{C}_{1j}(s) \end{array}\right\}$$

is robustly Hurwitz stable

$$\Leftrightarrow \left\{\begin{pmatrix} * & c_{1i}(s) & * & c_{1j}(s) & * \\ & & B(s) & & \end{pmatrix} : \begin{array}{c} c_{1i}(s) \times c_{1j}(s) \in \\ \left(K^I_{1i}(s) \times E^I_{1j}(s)\right) \cup \left(E^I_{1i}(s) \times K^I_{1j}(s)\right) \end{array}\right\}$$

is robustly Hurwitz stable.

**Theorem 2** $\mathcal{G}(s)$ is robustly Hurwitz stable if and only if $\mathcal{G}_E(s)$ is robustly Hurwitz stable.

**Remark:** Theorem 2 is consistent with the result in [14]. In [14], the authors obtained their result using some theorem in signal processing. Our proof is based on the properties of matrix determinant, hence is more straightforward and self-contained.

## 7 Numerical example

Consider the uncertain polynomial matrix

$$\begin{array}{l}
\mathcal{A}(s) = \{(a_{ij}(s))_{3\times 3}\} \\
a_{11}(s) = \lambda_{11} b1_{11}(s) + (1-\lambda_{11}) b2_{11}(s) \\
a_{12}(s) = \lambda_{12} b1_{12}(s) + (1-\lambda_{12}) b2_{12}(s) \\
a_{13}(s) = \lambda_{13} b1_{13}(s) + (1-\lambda_{13}) b2_{13}(s) \\
a_{21}(s) = \lambda_{21} b1_{21}(s) + (1-\lambda_{21}) b2_{21}(s) \\
a_{22}(s) = \lambda_{22} b1_{22}(s) + (1-\lambda_{22}) b2_{22}(s) \\
a_{23}(s) = \lambda_{23} b1_{23}(s) + (1-\lambda_{23}) b2_{23}(s) \\
a_{31}(s) = \lambda_{31} b1_{31}(s) + (1-\lambda_{31}) b2_{31}(s) \\
a_{32}(s) = \lambda_{32} b1_{32}(s) + (1-\lambda_{32}) b2_{32}(s) \\
a_{33}(s) = \lambda_{33} b1_{33}(s) + (1-\lambda_{33}) b2_{33}(s)
\end{array}$$

Then,

$$\begin{array}{l}
E_{ij}(s) = \{a_{ij}(s)\} \\
K_{ij}(s) = \{b1_{ij}(s), b2_{ij}(s)\} \\
S_3 = \{\sigma_1, \ldots, \sigma_6\} \\
\sigma_1 : 1 \to 1; 2 \to 2; 3 \to 3 \\
\sigma_2 : 1 \to 2; 2 \to 3; 3 \to 1 \\
\sigma_3 : 1 \to 3; 2 \to 1; 3 \to 2 \\
\sigma_4 : 1 \to 1; 2 \to 3; 3 \to 2 \\
\sigma_5 : 1 \to 2; 2 \to 1; 3 \to 3 \\
\sigma_6 : 1 \to 3; 2 \to 2; 3 \to 1
\end{array}$$

Let

$$\mathcal{A}_E(s) = \bigcup_{\sigma \in S_3} \left\{ (a_{ij}(s))_{3\times 3} : a_{ij}(s) \left\{ \begin{array}{l} \in E_{ij}(s) \text{ if } i = \sigma(j) \\ \in K_{ij}(s) \text{ if } i \neq \sigma(j) \end{array} \right. \right\}$$

By Theorem 1, $\mathcal{A}(s)$ is robustly $D$ stable if and only if $\mathcal{A}_E(s)$ is robustly $D$ stable.

## 8 Conclusions

We have shown that, for an interval system, the maximal $H^\infty$ norm of its sensitivity function is achieved at twelve (out of sixteen) Kharitonov vertices. This result is useful in robust performance analysis and $H_\infty$ control design for dynamic systems under parametric perturbations. Furthermore, we have discussed the robust $D$-stability problems for MIMO uncertain systems. The Edge Theorem and Kharitonov Theorem have been generalized to multivariable case.


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