DISCRETE SLIDING MODE CONTROL OF PERMANENT MAGNET STEPPER MOTOR USING FLATNESS PROPERTY.

V. Thakar*, **B. Bandyopadhyay**†
Interdisciplinary Programme In Systems and Control Engineering,
IIT Bombay, Mumbai - 400 076, India
Email: vk@ee.iitb.ac.in*; bijnan@ee.iitb.ac.in†
Phone: 91-22-25767889, Fax: 91-22-25723480

Keywords: Stepper motor, Discrete sliding mode, Reaching law approach, Differentially flat system, Linearizing outputs

Abstract

The paper presents discrete time sliding mode control (DSMC) for permanent magnet (PM) stepper motor which is known to be differentially flat system. Discretization of continuous time model of the stepper motor has been carried out and linearizing outputs are selected. Discrete state variables and control inputs are obtained in terms of linearizing outputs and their higher order differences. Linearized equations for the system are thus obtained and auxilary control is designed using reaching law approach. Then the actual control is obtained in terms of flat output and state variables. It is shown that system states reach to zero from given initial conditions.

1 Introduction

Sliding Mode Control (SMC) is a robust control scheme based on the concept of changing the controller's structure, with reference to the motion of the states of the system along the predefined manifold in order to obtain desired response. In order to obtain sliding motion, high speed, discontinuous switching controller is used to change the structure of the system. In sliding mode, system response is governed by the sliding surface[10]. In Discrete Sliding Mode, control action can only be activated at sampling instants and hence only Quasi Sliding Mode is possible. Gao[4] proposed reaching law approach which ensures that system trajectory will hit the switching manifold and undergoes zigzag motion about the manifold and remains within a Quasi Sliding Mode Band(QSMB). The sliding mode control can be applied for the control of various systems such as power converters, motors drives and robots [5, 11]. Stepper motors are electromagnetic incremental motion systems which converts digital input into analog angular motion of the rotor in steps. They have certain significant advantages. They can be easily interfaced with solid state electronic devices. They are normally operated in open loop. They are open loop stable but response is poor at high stepping rate which may even lead to oscillatory response and loss of synchronism. Hence close loop control of stepper motor becomes essential for certain applications[6, 12]. M. Zribi and J. Chassion[12] proposed method for position control of PM stepper motor using exact linearization. Marc Bodson et al[1]suggested a model based control law using exact linearization method for control of PM stepper motor in which nonlinear observer was used for speed estimation. M. Zribi et al[13] developed a sliding mode control scheme for PM stepper motor in which the motor is shown to posses a differential flatness property. In [8] the sliding mode control for flat systems is first presented. Flat systems, as introduced by fliss and his coworkers[3], are dynamic systems which are linearizable to controllable linear system by means of endogenous feedback [13]. The flat systems have finite set of differentially independent outputs known as linearizing outputs. The system is differentially flat if all of its variables can be expressed as differential function of linearizing outputs. It is shown in [13] that all the state variables and control inputs of the stepper motor model can be expressed in terms of the output and their higher order derivatives where winding current and angular position are considered as control and linearizing outputs.

In this paper we present discrete sliding mode control for PM stepper motor which uses flatness property of the motor. This paper is organized as follows. Section 2 presents the nonlinear mathematical model of the stepper motor and simplified nonlinear model obtained using direct-quadrature transformation(D-Q) as given in [1, 13]. Discrete model of the stepper motor with linearizing output is introduced in Section 3. Representation of discrete state variables and control inputs in terms of the linearizing outputs and linear model of the system are presented in section 4. Design of discrete sliding mode control using reaching law approach has been included in Section 5. Section 6 contains the simulation results followed by the concluding remarks.

2 Mathematical model of the PM Stepper Motor

The motor model as given in[1],[13] is as follows

$$\frac{di_a}{dt} = \frac{[v_a - Ri_a + k_m \omega sin(N_r\theta)]}{L},$$

$$
\begin{aligned}
\frac{di_b}{dt} &= \frac{[v_b - Ri_b - k_m cos(N_r\theta)]}{L}, \\
\frac{d\omega}{dt} &= \frac{[-k_m i_a sin(N_r\theta) + k_m i_b cos(N_r\theta) - B\omega)]}{J}, \\
\frac{d\theta}{dt} &= \omega. \qquad (1)
\end{aligned}
$$

Where v_a and v_b are the phase voltages, i_a and i_b are the phase currents, ω is the angular velocity of the rotor, θ is the angular position of the motor shaft, R is the resistance of the phase winding, L is the inductance of the phase winding, B is the viscous friction coefficient, J is the inertia of rotor and load, N_r is the number of rotor teeth and K_m is the motor torque constant. The model neglects the small magnetic coupling between the phase windings, small change in inductance as function of rotor position, the detent torque and variation in inductance due to magnetic saturation[1].

The system represented by Equation (1) may be transformed into appropriate nonlinear form known as direct- quadrature form. The transformation is such that the nonlinearity may be cancelled by state feedback[1]. The transformation and transformed system are given as,

$$
\begin{bmatrix} i_d \\ i_q \\ \omega \\ \theta \end{bmatrix} = \begin{bmatrix} cos(Nr\theta) & sin(N_r\theta) & 0 & 0 \\ -sin(N_r\theta) & cos(N_r\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ \omega \\ \theta \end{bmatrix}, \qquad (2)
$$

and

$$
\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \begin{bmatrix} cos(N_{r\theta}) & sin(N_r\theta) \\ -sin(N_r\theta) & sin(N_r\theta) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}. \qquad (3)
$$

where i_d is the direct current, i_q is the quadrature current, v_d is the direct voltage and v_q is the quadrature voltage. It is shown in [1] that the application of D-Q transformation to Equation (1) results in the following system of equations.

$$
\begin{aligned}
\frac{di_d}{dt} &= \frac{(v_d - Ri_d + N_r L\omega i_q)}{L}, \\
\frac{di_q}{dt} &= \frac{(v_q - Ri_q + N_r L\omega i_d - k_m\omega)}{L}, \\
\frac{d\omega}{dt} &= \frac{(k_m i_q - B\omega)}{J}, \\
\frac{d\theta}{dt} &= \omega. \qquad (4)
\end{aligned}
$$

3 Discrete model of the PM stepper motor

Discrete model of the stepper motor is obtained by discretizing the system represented by Equation (4) with sampling time τ and discrete state space representation of the system is given as follows.

$$
\begin{aligned}
x_1(k+1) &= (1-k_1)x_1(k) + k_5 x_2(k)x_3(k), \\
&\quad +u_1(k) \\
x_2(k+1) &= (1-k_1)x_2(k) - k_5 x_1(k)x_3(k) \\
&\quad -k_2 x_3(k) + u_2(k), \\
x_3(k+1) &= k_3 x_2(k) + (1-k_4)x_3(k), \\
x_4(k+1) &= k_6 x_3(k) + x_4(k), \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(k) &= i_d(k), x_2(k) = i_q(k), x_3(k) = \omega(k), x_4(k) = \theta(k), \\
x(k) &= [x_1(k)x_2(k)x_3(k)x_4(k)]', \\
k_1 &= \frac{R\tau}{L}, k_2 = \frac{k_m\tau}{L}, k_3 = \frac{k_m\tau}{J}, k_4 = \frac{B\tau}{J}, k_5 = N_r\tau, \\
k_6 &= \tau, u_1 = \frac{v_d(k)\tau}{L}, u_2 = \frac{v_q(k)\tau}{L}.
\end{aligned}
$$

Now it is possible to define linearizing outputs and controlled output for the system as

$$y_1(k) = x_1(k),$$

and

$$y_2(k) = x_4(k).$$

Now all the system state variables and control inputs can be represented in terms of y_1, y_2 and their higher order differences.

4 System representation in terms of linearizing outputs

The state variables are represented in terms of linearizing outputs and their higher order differences as follows.

$$
\begin{aligned}
x_1(k) &= y_1(k), \\
x_2(k) &= \frac{y_2(k+2) - y_2(k+1)}{k_3 k_6} - \\
&\quad \frac{(1-k_4)(y_2(k+1) - y_2(k))}{k_3 k_6}, \\
x_3(k) &= \frac{y_2(k+1) - y_2(k)}{k_6}, \\
x_4(k) &= y_2(k). \qquad (6)
\end{aligned}
$$

The control inputs can also be expressed in terms of linearizing outputs from the Equation (5) as follows.

$$
\begin{aligned}
u_1(k) &= x_1(k+1) - (1-k_1)x_1(k) \\
&\quad -k_5 x_2(k)x_3(k), \\
u_2(k) &= x_2(k+1) - (1-k_1)x_2(k) \\
&\quad +k_5 x_1(k)x_3(k) + k_2 x_3(k). \qquad (7)
\end{aligned}
$$

Substituting the expressions from Equation (6) into Equation (7), control inputs can be represented in terms of linearizing outputs and their higher order differences as follows.

$$u_1(k) = y_1(k+1) - (1-k_1)y_1(k) - k_5 Y,$$

where

$$\begin{aligned}
Y &= \left[\frac{y_2(k+2)-y_2(k+1)}{k_3k_6}\right] - \\
&\quad \left[\frac{(1-k_4)(y_2(k+1)-y_2(k)}{k_3k_6}\right] \times \\
&\quad \left[\frac{y_2(k+1)-y_2(k)}{k_6}\right], \\
u_2(k) &= Y_1 - Y_2 \\
&\quad + k_5 y_1(k)\left[\frac{y_2(k+1)-y_2(k)}{k_6}\right] \\
&\quad + k_2\left[\frac{y_2(k+1)-y_2(k)}{k_6}\right],
\end{aligned} \tag{8}$$

where

$$Y_1 = \frac{\begin{array}{c}[y_2(k+3)-y_2(k+2)- \\ (1-k_4)(y_2(k+2)-y_2(k+1))]\end{array}}{k_3k_6}, \tag{9}$$

$$\begin{aligned}
Y_2 = (1-k_1)&\left[\frac{y_2(k+2)-y_2(k+1)}{k_3k_6}\right] \\
-&\left[\frac{(1-k4)(y_2(k+1)-y_2(k))}{k_3k_6}\right].
\end{aligned}$$

From the expressions given by the Equation (9) for u_1 and u_2, the linearized equations of for the system are given by

$$\begin{aligned}
\frac{y_1(k+1)-y_1(k)}{k_6} &= v_1 \\
\frac{y_2(k+3)-3y_2(k+2)+3y_2(k+1)-y_2(k)}{k_6^3} &= v_2
\end{aligned} \tag{10}$$

where v_1 and v_2 are auxiliary inputs.

Simplifying the equations obtained above in terms of the system states, we get the following linear state equations.

$$\begin{aligned}
x_1(k+1) &= x_1(k) + k_6 v_1(k), \\
x_2(k+1) &= (1+k_4)x_2(k) - \frac{k_4^2}{k_3}x_3(k) + \frac{k_6^2}{k_3}v_2(k), \\
x_3(k+1) &= k_3 x_2(k) + (1-k_4)x_3(k), \\
x_4(k+1) &= k_6 x_3(k) + x_4(k).
\end{aligned} \tag{11}$$

The above system can be expressed as

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau v(k), \tag{12}$$

$$y(k) = Cx(k). \tag{13}$$

with

$$\begin{aligned}
\Phi_\tau &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+k_4 & \frac{-k_4^2}{k_3} & 0 \\ 0 & k_3 & 1-k_4 & 0 \\ 0 & 0 & k_6 & 1 \end{bmatrix}, \\
\Gamma_\tau &= \begin{bmatrix} k_6 & 0 \\ 0 & \frac{k_6^2}{k_3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
C &= [1000; 0001].
\end{aligned}$$

5 Discrete Sliding Mode Control Design

Design of sliding mode control involves two step procedure. firstly a suitable switching surface is designed and then a sliding mode control is designed to obtain sliding motion along the designed switching manifold. Different methods to design switching hyperplane are discussed in [2, 9]. In this design, method proposed in [2] is used to design switching planes.

5.1 Reaching Law Approach

The reaching law for the discrete sliding mode control as proposed in[4] is

$$s(k+1) - s(k) = -q\tau s(k) - \varepsilon\tau sgn(s(k)) \tag{14}$$

where the $\tau > 0$, sampling period, $\epsilon > 0, q > 0$ and $1 - q\tau > 0$.

The inequality for τ must hold to guarantee the motion of system trajectory towards the switching plane and that it will cross the switching plane in finite time. The presence of signum function ensures the sliding motion about the switching plane in zigzag pattern, restricting the trajectory in a specified band known as quasi sliding mode band QSMB.

5.2 Switching Hyper Plane Design

Switching plane equation is given as

$$s(k) = c^T x(k) = 0. \tag{15}$$

For the design of the switching hyperplanes, the system given in Equation(11) is transformed into suitable normal form by reversing the order of the system equations. The system thus takes the following form as

$$z(k+1) = \bar{\Phi}_\tau\, z(k) + \bar{\Gamma}_\tau\, v(k), \tag{16}$$

with

$$\bar{\Phi}_\tau = \begin{bmatrix} 1 & k_6 & 0 & 0 \\ 0 & 1-k_4 & k_3 & 0 \\ 0 & -\frac{k_4^2}{k_3} & 1+k_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\bar{\Gamma}_\tau = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{k_6^2}{k3} \\ k_6 & 0 \end{bmatrix},$$

Let us assume,

$$\bar{\Phi}= \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} z = \begin{bmatrix} z1 & z2 \end{bmatrix}' \, and \, \bar{\Gamma}= \begin{bmatrix} 0 & \Gamma_2 \end{bmatrix}'$$

Thus, the system given in Equation(16) can be written as

$$\begin{aligned} z_1(k+1) &= \Phi_{11}z_1(k) + \Phi_{12}z_2(k), \\ z_2(k+1) &= \Phi_{21}z_1(k)+\Phi_{22}z_2(k) + \Gamma_2 v(k). \end{aligned} \quad (17)$$

The switching planes for the system (16) becomes

$$s(k) = \bar{c}^T z(k), \quad (18)$$

$$= [\bar{c}_1^T \bar{c}_2^T] z(k). \quad (19)$$

with

$$\bar{c}_1^T = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad (20)$$

and

$$\bar{c}_2^T = \begin{bmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{bmatrix}.$$

So,

$$\bar{c}_1^T z_1 + \bar{c}_2^T z_2 = 0,$$

or

$$z_2 = -\bar{c}_2^T \bar{c}_1^T z_1.$$

from the Equation (16)and Equation (18) the following equation is obtained

$$z_1(k+1) = (\Phi_{11} - \Phi_{12} \bar{c}_2^{T^{-1}} \bar{c}_1^T) z_1(k)$$

.

Then, $\bar{c}^T$ is obtained by assigning the eigenvalues of

$$(\Phi_{11} - \Phi_{12} \bar{c}_2^{T^{-1}} \bar{c}_1^T) \quad (21)$$

in a desired location. Thus, c^T can be easily obtained.

5.3 Control Law Design

The auxiliary control is obtained for the system represented by Equation (11) using reaching law approach given in Equation (14). Simplified representation of the control is given in [7] which has the following form

$$v(k) = Fx(k) + \gamma sgn(s(k)), \quad (22)$$

where

$$\begin{aligned} F &= -(c^T \bar{\Gamma})^{-1}[(c^T \bar{\Phi} - c^T I + q\tau c^T)], \\ \gamma &= -(c^T \bar{\Gamma})^{-1}\varepsilon\tau. \end{aligned}$$

So the actual control is

$$\begin{aligned} u_1(k) &= v_1(k)k_6 + k_1x_1(k) \\ &\quad -k_5x_2(k)x_3(k), \\ u_2(k) &= v_2(k)\frac{k_6^2}{k_3} + (k_1+k_4)x_2(k) + \\ &\quad k_2x_3(k) - \frac{k_4^2}{k_3}x_3(k) + k_5x_1(k)x_3(k). \end{aligned} \quad (23)$$

6 Simulation

The simulation study is carried out for the PM stepper motor, with the same data same as used in [13]with $R = 19.1388$ $L = 40mH$, $km = 0.1349Nm/A$, $J = 4.1295 \times 10^{-4} k_g m^2$, $B = 0.0013Nm/rad/sec$ and $Nr = 50$. The parameters for the control law are as $\tau = 0.1, \varepsilon = 0.02, q = 2$. The objective is to bring the system states and outputs from any initial conditions to zero in finite time. Figure 1 shows responses of the phase currents i_a and i_b, angular velocity ω, angular position θ. Plot of voltages v_a, v_b and switching surfaces s_1 and s_2 are shown in Figure 2.

7 Conclusion

The discrete multivariable sliding mode control for PM stepper is designed using the property of differential flatness. Selecting the winding current and angular position of the shaft as an output, all the state variables and inputs of the discrete system can be expressed in terms of the these outputs, i.e. linearizing outputs and their higher order differences. The linear representation of the system is obtained while representing the control inputs in terms of linearizing outputs. The auxiliary control is obtained using reaching law approach for the linear model obtained using differential flatness property of the stepper motor model. Finally the actual control is obtained for the stepper motor. The simulation results shows the effectiveness of the discrete sliding mode control for PM stepper motor which brings the system states to zero from initial values.

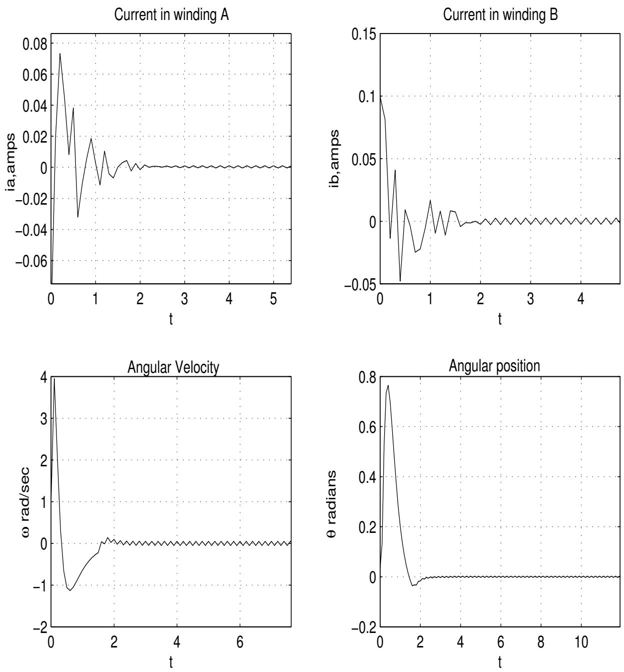

Figure 1: Response of the system states

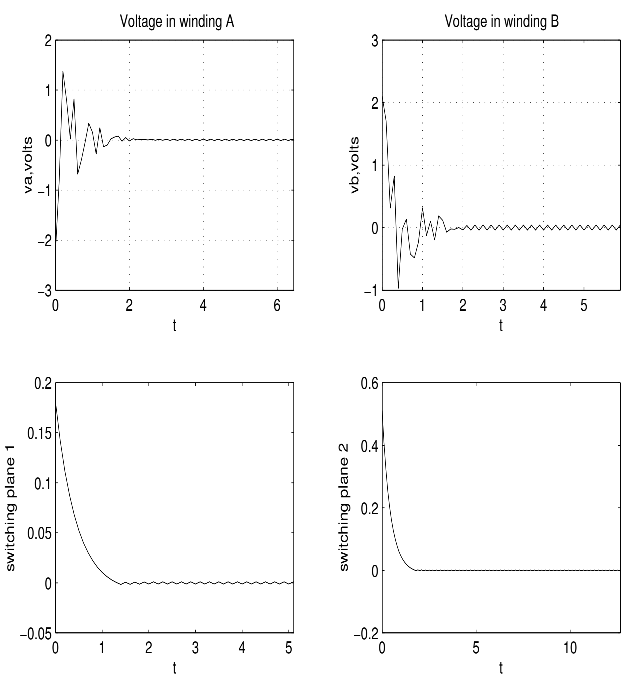

Figure 2: Plot of winding voltages and switching surfaces

References

[1] M Bodson, J. N. Chiasson, R. T. Novotnak and R. B. Rekowski, 'High performance nonlinear feedback control of a permanent magnet stepper motor', *IEEE Transactions on Control System Technology*, vol-1, no-1, pp. 5-13, (March 1993)

[2] C. M. Dorling and A. S. Zinober, 'Two approches to hyperplane design in multivariable structure control system', *International Journal of Control*, vol. 44, no.1, pp. 65-82,(1986)

[3] M. Fliess, J. Levine, P. Martin and P. Rouchon, 'Flatness and defect of nonlinear systems:Introductory theory and example', *International Journal of Control*, vol-61, no 6, pp. 1327-1362, (1995)

[4] W. Gao, Y. Wang and A. Homaifa,'Discrete time variable structure control systems ' *IEEE Transaction on Industrial Electronics*, vol. 42, no.2, pp. 117-122, (April 1995).

[5] J. H. Hung, W. Gao and J. C. Hung, 'Variable structure control: A survey', *IEEE Transaction on Industrial Electronics*, vol. 40, pp.2-21, (February 1993).

[6] T. Kenjo and A. Sugawara, 'The stepping motors and their microprocessor controls',second edition, Oxford science publication,(1994)

[7] C. M. Saaj, B. Bandyopadhyay and H. Unbehauen,' A new algorithm for discrete-time sliding mode control using fast output sampling feedback', *IEEE Transaction on Industrial Electronics*, vol. 49, no.3, pp. 518-523, (June 2002)

[8] H. Sira-Ramirez,' On the sliding mode control of multivariable nonlinear system', *International Journal of Control*, vol. 64, no.4, pp 745-765,(1996)

[9] S. K. Spurgeon 'Hyperplane design techniques for discrete-time variable structure control system' *International Journal of Control*, vol. 55, no-2, pp. 445-456,(1992)

[10] V. I. Utkin, 'Variable structure systems with sliding modes' *IEEE Transaction on Automatic Control,* vol. AC-22, pp. 212-222, (April 1977)

[11] V. I. Utkin, 'Sliding mode control of an electromechanical systems', Francis and Taylor Publications, (1999).

[12] M. Zribi and J. Chiasson, 'Position control of PM stepper motor by exact linearization' *IEEE Transactions on Automatic Control*, vol. 36, pp. 620- 625, no.5, (May 1991)

[13] M. Zribi, H. Sira-Ramirez and A. Ngai,'Sliding mode control of PM stepper motor from the perspective of differentially flat system', Invited paper, F1009-3,*The European Control Conference*, Germany,(1999)