ON MODIFICATION OF A CLASS OF LYAPUNOV-BASED ROBUST CONTROLLERS SUBJECT TO BOUNDED INPUT

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Abstract

In this paper, we consider a modification design of a class of Lyapunov-based robust controllers subject to bounded input. Our modification shows benefits in enhancing the input utilization and in retaining the stability and the robustness of the original control. An estimation of the stabilization region is proposed to explore region where the control is modified. It results in an estimate showing singularity. This estimate is utilized to determine the design parameter for the local, semiglobal and global stabilization.

1 Introduction

Controlling systems subject to bounded input is very common in practice. Disregards for the limitation quite often causes control saturation and damage in system components. Moreover, the performance deterioration and, for some cases, the closed-loop instability may be incurred. The problem of bounded input is related to that of input saturation and bounded control. In literature, many strategies have been proposed to deal with this kind of problems [1]-[13]. For input saturation, the approach of semiglobal stabilization is a notion which copes with a local stabilization within an arbitrary compact set in the state space [10]. However, this notion requires the system being null-controllable [12], or specifically being stabilizable and having all its eigenvalues in the closed left half-plane [10]. Therefore, if the system is not null-controllable, the stabilization subject to input constraint will be local. Under this circumstance, many researchers seek to estimate the stabilization region and adjust the design parameter for enlarging the region (see [5], [11] and references therein). It is known that the more the stability region is enlarged, the lower the design gain is needed. However, a lower design gain could incur a lower control magnitude and lead to less robustness and regulation performance, particularly in the neighborhood of the state origin. To avoid this drawback, the low-high gain design [10] and the ARE-based gain scheduling [13],[1] are proposed to enhance the input utilization for a better regulation and robustness. Basically, there are two approaches to dealing with the stabilization subject to bounded input. One is to consider the constraint at the beginning design stage, which yields several results of the semiglobal stabilization (see [5], [10], and references therein). The other is to neglect the constraint at the initial design stage and do some modification later on to satisfy the constraint. In this article, we adopt the latter approach and consider the modification of a class of Lyapunov-based robust controllers when the input needs to be bounded. We will demonstrate that the semiglobal stabilization can also be achieved using our approach. It is known that the Lyapunov-based robust control is related to the sliding mode control [14]. Therefore, it is adequate to consider the modification of this kind of control subject to bounded input. Given an original Lyapunov-based controller that neglects the input constraint, our modification comprises two stages: the first stage is to reshape the original control into a form of norm-bound that satisfies the constraint and preserves the direction of the original control. The preservation is mainly to keep the same sign of the Lyapunov function derivative and thus tending to retain the stability already established. The second stage is to utilize some technique extended from [4] to enhance the input utilization. The stability and robustness developed in the first modification stage are enhanced as well. The design parameter is obtained from the sets of solution data that solve both an algebraic Riccati equation (ARE) and an inequality derived from the estimation of stabilization region. This estimation explores the area where the control is modified and yields an estimate showing singularity. The design parameter is determined by this estimate for the local, semiglobal and global stabilization.

2 Problem formulations and system assumptions

2.1 Problem formulations

Consider the uncertain system

$$\dot{x}(t) = (A + \Delta A(t))x(t) + B(u(t) + h(t, x(t))) \ , \quad x(0) = x_0 \in \mathrm{O} \tag{1}$$

subject to the input constraint

$$|u_i(t)| \le \bar{u}_i \ , \forall i = 1,2,3,...,m \ , \quad \forall t \in \Re_+ \tag{2}$$

and the specified initial set $\mathrm{O} \subset \Re^n$. The constant matrices $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$ are known. The uncertainty matrix $\Delta A : \Re_+ \to \Re^{n \times n}$ and the nonlinear function $Bh : \Re_+ \times \Re^n \to \Re^n$ respectively describe the parametric deviation and the matching disturbance. Concerning the specified input constraint and the initial set, the goal of our control design is two-fold: one is the state regulation, that is to drive the system state from the specified initial set O to the vicinity of the state origin $x = 0$, and the other is to utilize the input capacity as much as possible.

2.2 Notations and assumptions

For the input constraint (2), define a diagonal matrix

$$\bar{U} := diag\left\{ \bar{u}_i \Big|_{i=1}^{m} \right\} \tag{3}$$

and two nonlinear vector functions, namely, the unit-ball saturation function

$$S_{ball}(z) := \begin{cases} z & , \ if \ \|z\| \le 1 \\ z/\|z\| & , \ if \ \|z\| > 1 \end{cases} \ , \quad z \in \Re^m ,$$

and the unit-box saturation function

$$S_{box}(z) := [sat(z_1), sat(z_2), ..., sat(z_m)]' ,$$

$$sat(z_i) := \begin{cases} -1, & if \ z_i < -1 \\ z_i, & if \ |z_i| \le 1 \\ 1, & if \ z_i > 1 \end{cases} \ , \quad z_i \in \Re .$$

Some assumptions for the system (1) are given as follows.

(A1) *Structured uncertainty decomposition*: There exist constant matrices D, E and uncertainty matrices $\tilde{A}_m(t)$, $F(t)$, $F'(t)F(t) \le I$, $\forall t \in \Re_+$ such that D is linearly independent of B and

$$\Delta A(t)x = [B \ D]\begin{bmatrix} \tilde{A}_m(t) \\ F(t)E \end{bmatrix} x = B\tilde{A}_m(t)x + DF(t)Ex \ , \forall t \in \Re_+ \ , \tag{4}$$

where $B\tilde{A}_m(t)x$ and $DF(t)Ex$ respectively are referred to as the matching and mismatching uncertainties.

(A2) *Quadratic stabilization* [9]: Given the D, E in the assumption (A1) and A, B of the system (1), there exists some $¢ > 0$ such that the ARE

$$A'X + XA + XDD'X - \frac{1}{¢}XBB'X + E'E + ¢I = 0$$

has a solution $X = X' > 0$.

(A3) In the neighborhood of the state origin $x = 0$, magnitude of the uncertainty $\tilde{A}_m(t)x + h(t,x)$ is less than that of the input capacity/constraint $\bar{U}$. Specifically, there exist constants $0 \le k_0 < 1$ and $0 \le k_1$ such that the ratio $\bar{U}^{-1}\left(\tilde{A}_m(t)x + h(t,x)\right)$ satisfies

$$\left\| \bar{U}^{-1}\left(\tilde{A}_m(t)x + h(t,x)\right) \right\| \le k_0 + k_1\|x\| \ , \forall t \in \Re_+, \forall x \in \Re^n . \tag{5}$$

The assumption (A3) simply means that, under the matching disturbance $\tilde{A}_m(t) + h(t,x)$, the state regulation to an arbitrarily small residual set containing $x = 0$ can be guaranteed.

3 The controller design

3.1 A robust control design that neglects input constraint

In general, if the input constraint is neglected, then there exist many strategies that can robustly stabilize the system (1). In this article, we consider the method of the Lyapunov-based control [6] by which the control can be designed as

$$u(t) = \bar{U} \, p_u(x(t)), \forall t \in \Re_+$$

$$p_u(x) = -\frac{1}{«}\bar{U}B'Px - k_0 S_{ball}(£^{-1}\bar{U}B'Px) \ , \tag{6}$$

where $£ > 0$, $« > 0$ and $P = P' > 0$ are the design parameters. Among them, the $£$ can be assigned freely, and the $«$ and P are taken from a set of solution data

($\sim \geq 0, « > 0$, $k_\Delta > 0$, $P(\sim, «, k_\Delta) = P'(\sim, «, k_\Delta) > 0$) that solves the following ARE

$$(A + \sim I)'P + P(A + \sim I) \; - \frac{1}{«} PB\overline{U}^2 B'P + PDD'P + E'E + « \, (k_1^{\,2} + k_\Delta)I = 0 \quad (7)$$

The solvability of ARE (7) can be verified with the assumption (A2). For the system (1) under the control (6), a global regulation is achievable. Since manipulating the derivative of a Lyapunov function candidate $V(x) = x'Px$ yields the following global inequality

$$\hat{V}(x) \leq -2\tilde{\sim} V(x) + \frac{£\, k_0}{2}, \quad \forall x \in \Re^n, \tilde{\sim} := \sim + \frac{«k_\Delta}{2\, a_{\max}(P)},$$

which ensures the controlled system behaves *globally exponential convergent with rate* $\tilde{\sim}$ *to within*

$$\Omega(£) := \left\{ x \in \Re^n \middle| x'Px \leq £k_0 / 4\tilde{\sim} \right\}, \qquad (8)$$

where the residual set $\Omega(£)$, sometimes referred to as the region of ultimate boundedness, can be set arbitrarily small by adjusting $£$.

3.2 First stage modification

When regarding the input constraint and the specified initial set, the control like (6) is theoretically inapplicable; because the control $u(t) = \overline{U}p_u(x(t))$ that starts with some initial states will possibly violate the constraint (2) at the initial time and/or during the transient. The following norm-type modification

$$u(t) = \overline{U}\, S_{ball}\,(p_u(x(t))), \forall t \in \Re_+ \qquad (9)$$

provides a way to reshape the original control (6) so that the violation of the input constraint is avoidable and in addition, direction of the original control can be preserved that in turn keeps the Lyapunov derivative as the same sign. When the modification (9) is applied, the rest of design work is to adjust the design parameter $(\sim, « , k_\Delta, P)$ of ARE (7) to ensure that the related stabilization region can contain the specified initial set O . A manipulation similar to [7] shows that the system (1) under the control (9) has an estimated stabilization region

$$\Theta(\sim, «, P) := \left\{ x \in \Re^n \middle| x'Px \leq \quad (\sim, «, P) \right\},$$

$$(\sim, «, P) := a_{\max}\left(B\overline{U}^2 B'P\right) \left(\frac{«(1 - k_0)}{a_{\max}\left(B\overline{U}^2 B'P\right) - \tilde{\sim}«} \right)^2 \qquad (10)$$

and has a region of ultimate boundedness same as (8). Therefore if additionally, the design parameter $(\sim, « , k_\Delta, P)$ can be selected to satisfy

$$\underset{x \in O}{Max}\left(x'Px\right) < \quad (\sim, «, P), \qquad (11)$$

or in other words the specified initial set O can be verified being contained in the stabilization region $\Theta(\sim, «, P)$, then the control (9) constructed with this admissible $(\sim, « , k_\Delta, P)$ can achieve the regulation and as well satisfy the constraint. It can be shown that if a lower k_Δ and/or $\sim$ is chosen then a smaller eigenvalue $a_{\max}(P(\sim, «, k_\Delta))$ as well as a larger stabilization region (10) can be obtained for solving (11). Two special cases below can be observed from the singularity of estimation (10):

(1) *Semiglobal stabilization*: Suppose there is a sequence of sets of solution data $(\sim, «, k_\Delta, P(\sim, «, k_\Delta))$ taking

$$a_{\max}\left(B\overline{U}^2 B'P\right) - \tilde{\sim}« \to 0^+ . \qquad (12)$$

Then the right-hand side of (11) goes unbounded, thus for any assigned compact initial set O , there always exist sets of solution data $(\sim, «, k_\Delta, P(\sim, «, k_\Delta))$ that solves (11). Thus, feature of the semiglobally practical stabilization can be concluded. It demonstrates in our numerical example that this property (12) is inherent in the null controllable system.

(2) *Global stabilization*: Suppose there exists a set of solution data $(\sim, «, k_\Delta, P(\sim, «, k_\Delta))$ such that

$$a_{\max}\left(B\overline{U}^2 B'P\right) - \tilde{\sim}« \leq 0 . \qquad (13)$$

Then similar to [7], it can be shown that the control (9) constructed with such a set of data can globally and practically stabilize the system (1). However this inequality (13) requires the matrix A to be Hurwitz. Finally, it is worth of noting that for the systems that are not null controllable, the trial for (11) will fail in general, provided the initial set O is assigned arbitrarily [12].

3.3 Second stage modification

As previously mentioned, enlarging the stabilization region incurs a lower control magnitude and leads to a poorer regulation, particularly near the state origin. For dealing with such drawback, a second-stage modification that strengthens the first-stage modification (9) and, in the meanwhile, satisfies the constraint (2) is proposed as

$$u(t) = p(x(t)), \forall t \in \Re_+ ,$$

$$p(x) = \overline{U}S_{box}\left(S_{ball}\left(p_u(x)\right) + H\, p_u(x)\right), \forall x \in \Re^n , \qquad (14)$$

where $H = diag\left\{ h_i > 0 \Big|_{i=1}^{m} \right\}$ is a design gain matrix that can be set arbitrary. There are two advantages in this advanced modification: namely, enhancing the input utilization and retaining the robustness and stability established in the first-stage modification. These two advantages will be explained in the later in this article. Theorem below, extended from [4], gives a foundation to view the properties inherent in the new control (14).

Theorem 1. *Suppose the functions* $f:\Re^n \to \Re^m$, $g:\Re^n \to \Re^m$ *are continuous with* $f(0)=0$, $g(0)=0$. *Then for a given diagonal matrix* $L = diag\left\{ a_i > 0|_{i=1}^{m} \right\}$ *, there exists a continuous diagonal matrix function* $N(x) = diag\left\{ n_i(x) \ge 0 \ |_{i=1}^{m}, \ \forall x \in \Re^n \right\}$ *such that*

$$S_{box}\left(S_{ball}\left(f(x)\right) + Lg(x)\right) = S_{ball}\left(f(x)\right) + N(x)g(x), \ \forall x \in \Re^n .$$

Proof. The proof is omitted for brevity.

3.4 Properties inherent in the second stage modification

Based on Theorem 1, the modification (14) immediately can be rearranged as

$$p(x) = \overline{U}S_{ball}\left(p_u(x)\right) + \overline{U}Z(x)p_u(x) = \overline{U}S_{ball}\left(p_u(x)\right) + Z(x)\overline{U}p_u(x), \quad (15)$$

where $Z(x) = diag\left\{ z_i(x) \ge 0 \ |_{i=1}^{m}, \forall x \in \Re^n \right\}$ exists and is continuous. From (15), the second-stage modification (14) can be viewed as the first-stage modification (9) plus the original control (6) weighted with $Z(x)$. From other aspect, since all the functions of (14) are continuous with $p_u(0)=0$, $S_{ball}(0) = S_{box}(0) = 0$, thus near the state origin $x=0$, the modification (14) can be approximated as

$p(x) = \overline{U}\ p_u(x) + H\overline{U}\ p_u(x) \ \cong H\overline{U}\ p_u(x)$, if $H >> 0$.

That is to say, the modification (14) can also be viewed as the original control (6) weighted with a high gain H if near $x=0$. The properties of the modification (14) are presented as follows.

(P1) *Enhancing the input utilization:* To see this, observe that in (14), the gap between $S_{ball}(.)$ and $S_{box}(.)$ is compensated with an additional term $H\,p_u(x)$. Through the operation $S_{box}(.)$, if the term $H\,p_u(x)$ is large enough then the modification control (14) will be forced to perform with a full utilization $\overline{U}$. Figure 1 gives a two-dimensional illustration for the successive operations of (14). In Figure1, Γ, and ¥ denote respectively the bound of input, the unit box, and the unit ball. Other vectors are denoted respectively as $a := p_u(x)$, $b := S_{ball}(p_u(x))$, $c := H\,p_u(x)$, $d := S_{ball}(p_u(x)) + Hp_u(x)$, $e := S_{box}(S_{ball}(p_u(x)) + Hp_u(x))$, and $f := p(x)$. By the successive operations $a \to b \to d \to e \to f$ we observe that the initial $p_u(x)$ will finally reach to $p(x)$ which is the second-stage modification performing with a full utilization $\overline{U}$. This feature can be achieved by simply choosing the matrix H high enough.

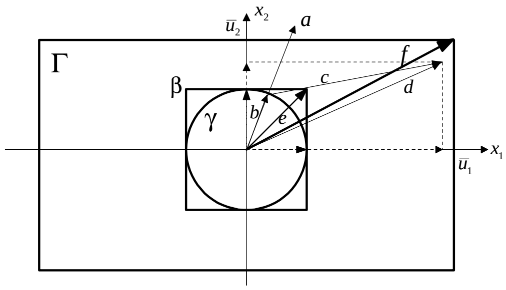

Figure 1. An illustration for the successive operations of (14)

(P2) *Retaining the robustness and stability established by the first-stage modification:* To see this, take a set of solution data $(\sim, «, k_\Delta, P)$ of the ARE (7) and choose a Lyapunov function candidate as $V(x) = x'Px$. For the system (1) under the control (14), the derivative of $V(x)$ is given by

$$\dot{V}(x) = x'(A'P + PA)x + 2x'PDF(t)Ex \ +2x'PB\ p(x)$$
$$+2x'PB\overline{U}\overline{U}^{-1}\left(\tilde{A}_m(t)x + h(t,x)\right). \quad (16)$$

Direct manipulation yields the following results:

$$2x'PDF(t)Ex \le 2\|D'Px\|\|Ex\|$$
$$\le x'PDD'Px + x'E'Ex, \forall t \in \Re_+, \forall x \in \Re^n \quad (17)$$

$$2x'PB\overline{U}\overline{U}^{-1}\left(\tilde{A}_m(t)x + h(t,x)\right) \le 2\|\overline{U}B'Px\|\left(k_0 + k_1\|x\|\right)$$
$$\le 2k_0\|\overline{U}B'Px\| + 2\left\|\sqrt{1/«}\,\overline{U}B'Px\right\|\left\|\sqrt{«}\,k_1 x\right\|$$
$$\le 2k_0\|\overline{U}B'Px\| + \frac{1}{«}x'PB\overline{U}^2B'Px + «k_1^2x'x . \quad (18)$$

Thus, substituting from (17), (18), (15) and (7) into (16) gives

$$\dot{V}(x) \le -2\tilde{\sim}x'Px + \frac{2}{«}x'PB\overline{U}^2B'Px + 2k_0\|\overline{U}B'Px\|$$
$$+2x'PB\overline{U}S_{ball}(p_u(x)) + 2x'PB\overline{U}Z(x)p_u(x), \quad (19)$$

where $\tilde{\sim} := \sim + \dfrac{«k_\Delta}{2a_{\max}(P)}$. Since by (6) we have

$$x'PB\overline{U}Z(x)p_u(x) =$$
$$-\frac{1}{«}x'PB\overline{U}Z(x)\overline{U}B'Px \ - \ k_0x'PB\overline{U}Z(x)S_{ball}(£^{-1}\overline{U}B'Px) \le 0 . \quad (20)$$

Thus, by substituting (20) into (19), we finally arrive at

$$\dot{V}(x) \le \tilde{W}(x) \le W(x), \quad \forall x \in \Re^n, \quad (21)$$

where

$$\tilde{W}(x) := -2\tilde{\sim}\, x'Px + \frac{2}{«} x'PB\bar{U}^2 B'Px + 2k_0 \left\|\bar{U}B'Px\right\|$$

$$+2x'PB\bar{U}S_{ball}(p_u(x)) + 2x'PB\bar{U}Z(x)p_u(x) \,,$$

$$W(x) := -2\tilde{\sim} x'Px + \frac{2}{«} x'PB\bar{U}^2 B'Px + 2k_0 \left\|\bar{U}B'Px\right\| \; +2x'PB\bar{U}S_{ball}(p_u(x)) \,.$$

Let us investigate the estimated Lyapunov derivative in (21). Since $\tilde{W}(x)$ and $W(x)$ can be viewed as the estimates of derivative $\hat{V}(x)$ respectively under the modification controls (14) and (9), and also from (21) we have $W(x) < 0 \Rightarrow \tilde{W}(x) < 0$, therefore we can conclude that if Θ is a Lyapunov stability region established by first-stage modification (9) then certainly, it is a Lyapunov stability region established by the second-stage modification (14). We also conclude that property (P2) is true.

Theorem below summaries our main results in this paper.

Theorem 2. *Consider the system (1) under the bounded control (14) with a solution data (* $\sim \ge 0, « > 0$ *,* $k_\Delta > 0$ *,* $P = P' > 0$ *) of the ARE (7) to be chosen. And also consider the residual set* $\Omega(£)$ *in (8) that can be set arbitrarily small by adjusting* $£$ *. Then, for the specified initial set* O *, the controlled system is:*

(i) Globally and practically stable to within $\Omega(£)$ *: if there exists a set of solution data that satisfies (13) and is chosen to construct the control (14).*

(ii) Semi-globally and practically stable to within $\Omega(£)$ *: if there exists a sequence of sets of solution data that satisfies (12) and a set satisfying (11) is chosen to construct the control (14).*

(iii) (Locally and) practically stable to within $\Omega(£)$ *: if there exists a set of solution data that satisfies (11) and is chosen to construct the control (14).*

3.5 Example: (*The semi-global stabilization)*

Consider a linearied satellite in orbit [2] subject to the specified bounded input $|u_1(t)| \le 15$, $|u_2(t)| \le 20$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\mu^2 & 0 & 0 & 2\mu \\ 0 & 0 & 0 & 1 \\ 0 & -2\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix},$$

with $\mu = 1$, where states $x_1 = °$, $x_2 = ȓ$, $x_3 = ¯$, and $x_4 = ¯ȓ$ are the polar coordinates. The initial state and the disturbances are assumed to be $x_0' = [15 \quad 15 \quad 0 \quad 0]$, $h_1(t) = \sin(4®t)$ and $h_2(t) = 1 + 0.5\cos(4®t)$ respectively. Verification shows that the system is null controllable with eigenvalues $ª(A) = \{0, 0, j, -j\}$. Direct manipulation gives $\bar{U} = diag(15, 20)$, $D = 0$, $E = 0$, $k_0 = 0.1003$, $k_1 = 0$. Let us choose $\sim = 0, « = 50$ and let k_Δ be a variable to find the associated solution P that solves both the ARE (7) and (11). Figure 2 depicts the curves of the two sides of (11) and as well the singularity of (12) as $k_\Delta \to 0$. In accordance, we pick $£ = 0.4498$, $H = diag(80,5)$, and, in ARE (7), $\sim = 0$, $« = 50$, $k_\Delta = 0.1 \times 10^{-5}$,

$$P = \begin{bmatrix} 0.2142 & 0.0168 & -0.0051 & 0.1117 \\ 0.0168 & 0.0056 & -0.0007 & 0.0083 \\ -0.0051 & -0.0007 & 0.0003 & -0.0024 \\ 0.1117 & 0.0083 & -0.0024 & 0.0591 \end{bmatrix}$$

as the admissible parameters to construct the control (14). Figure 3 and Figure 4 demonstrate the history of the bounded control and the state. For a satellite escaping away from its normal orbit, the controls u_1 and u_2 respectively try to pull the satellite back and slow down the angular speed. Both use the full capacity during the transient time. Also notice that the control u_1 still employs its full capacity even when it switches to push the satellite back to the orbit.

4 Conclusions

In this article, we present a modification of a class of Lyapunov-based controller subject to bounded input. This modification comprises two stages. The first stage is to reshape the original control for preserving the original control direction and satisfying the constraint. The second stage is to use a structure to enhance the input utilization and retaining the stability and robustness of the control developed in the first stage. An estimation of stability region is also proposed that suggests the conditions for local, semiglobal and global stabilization.

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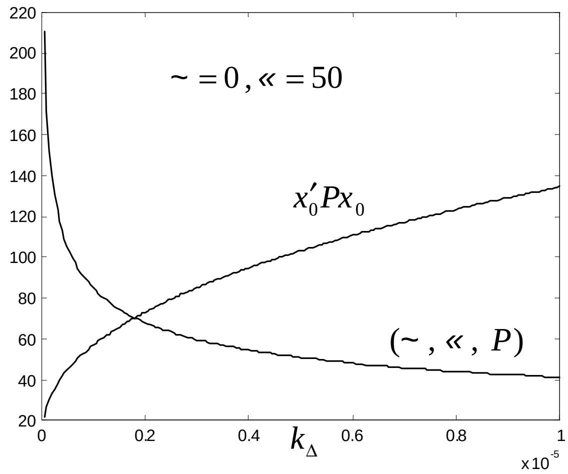

Figure 2. The curves of (11) with respect to variable k_Δ .

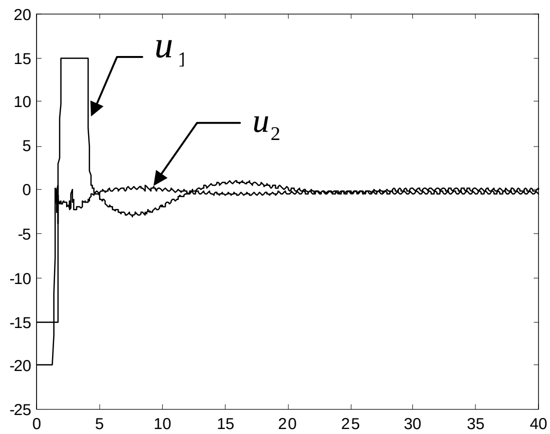

Figure 3. The control history

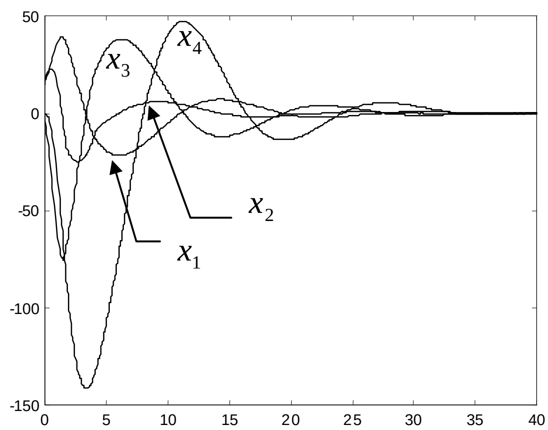

Figure 4. The state history.