GUARANTEED STABILIZED PLANTS

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Abstract

A notion of a guaranteed stabilized plant as a plant for that there exists a stabilizing controller delivered some phase and gain margins of system stability is introduced. A problem of finding a condition under that the plant is guaranteed stabilized is formulated.
It is shown that two classes of plants (minimum-phase and asymptotically stable) are guaranteed stabilized.

1. Introduction

Several branches of a linear system robustness analysis may be extracted.

The first branch [10], [7] proceeds from direct indices which are system coefficients boundaries such that the system keeps stability.

The second branch is based on indirect indices such as stability margins (phase and gain margins) that is a description "distance" a Nyquist curve from critical point $(-1, j0)$ [6]. A merit of these indeces is that they may be examined by an experiment and that is why they is widely used in practice [11].

In accordance with the second direction a system is robust if it has sufficient margins of stability. It is arisen a question: whether for any plant it may be found a controller that provides system robustness?

Below, the plants having this property are refered to as guaranteed stabilized one's. To find the guaranteed stabilized plants a radius of stability margin [2] is used. The radius is a generalization of the notions of the phase and gain margins and it is explicitly determined by the system transfer function. This allows to obtain an analytical description of the guaranteed stabilized plants.

2. Statement of the problem

Consider a stable system with constant coefficients described by the following differential equations

$$\begin{array}{c} y^{(n)} + d_{n-1}y^{(n-1)} + \cdots + d_0 y = \\ k_m u^{(m)} + \cdots + k_0 u, \quad m < n-1 \end{array} \tag{1}$$

$$\begin{array}{c} g_{n_c}u^{(n_c)} + \cdots + g_0 u = \\ = r_{m_c}y^{(m_c)} + \cdots + r_0 y, \quad m_c \leq n_c \end{array} \tag{2}$$

where $y(t)$ is a measured output of the completely controlled plant (1), $u(t)$ is an output of controller (2), $y^{(i)}$, $u^{(j)}$ $(i=\overline{1,n},\ j=\overline{1,m})$ are the derivatives of these functions.

An open-loop transfer function of the system is:

$$w(s) = -\frac{k(s)}{r(s)} \cdot \frac{r(s)}{g(s)} \tag{3}$$

where

$$\begin{array}{cc} d(s) = \sum_{i=0}^{n} d_i s^i, & k(s) = \sum_{i=0}^{m} k_i s^i, \\ g(s) = \sum_{i=0}^{n_c} g_i s^i, & r(s) = \sum_{i=0}^{m_c} r_i s^i, \end{array} \tag{4}$$

Definition 2.1 A number

$$r = min_{0 \leq \omega < \infty} |1 + w(j\omega)| \tag{5}$$

is named a stability margin radius of system (1), (2).

The stability margin radius is a radius of a circle whose centre is critical point $(-1, j0)$. It is a generalization of the phase and gain margins notion. Thus, if this radius $r = 0.75$ then the phase margin equals $42°$ and the gain margin is 1.75. The value $r = 1$ fits the values: $60°$ and 2.

On the other hand the stability margin radius is a conversion of the H_∞ norm of a sensitivity function of the system (1), (2). This property allows to find a maximum [5] of the radius.

Definition 2.2 The system (1), (2) is named the robust (in the sense of the stability margin radius) if

$$r^2 \geq \eta^2 \quad (\eta \doteq 0.75 \div 1) \tag{6}$$

Definition 2.3 Plant (1) is named a guaranteed stabilized one if there exists a controller (2) such that system (1), (2) is robust.

Problem 2.1 Find conditions under which plant (1) is guaranteed stabilized.

3. Robust controllers set

Definition 3.1 Controller (2) is named robust if it provides robustness of system (1), (2).

Assertion 3.1 All set of the robust controllers is described by the following polynomials

$$\begin{aligned} g(s) &= g^0(s)b(s) + k(s)a(s), \\ r(s) &= r^0(s)b(s) + d(s)a(s) \end{aligned} \tag{7}$$

in which:

i polynomials $g^0(s)$ and $r^0(s)$ are a solution of the following Bezout-Identity

$$d(s)g^0(s) - k(s)r^0(s) = \psi(s). \tag{8}$$

ii $a(s)$ is a polynomial, $b(s)$ and $\psi(s)$ are Hurwitz polynomials (e.g. all roots of these polynomials have negative real parts).

iii polynomials $a(s)$, $b(s)$ and $\psi(s)$ have to satisfy the following inequality

$$\begin{aligned} \psi(-s)\psi(s)b(-s)b(s) \geq \eta^2 d(-s)d(s)\cdot \\ \cdot[g^0(-s)b(-s) + k(-s)a(-s)]\cdot \\ \cdot[g^0(s)b(s) + k(s)a(s)], \\ s = j\omega, \quad 0 \leq \omega < \infty \end{aligned} \tag{9}$$

iiii degrees of polynomials $a(s)$, $b(s)$ and $\psi(s)$ have to meet condition $m_c \leq n_c$ of the controller realizability

$$\begin{aligned} deg[g^0(s)b(s) + k(s)a(s)] \geq \\ \geq deg[r^0(s)b(s) + d(s)a(s)] \end{aligned} \tag{10}$$

Proof. It is well known [4] that the identity (8) and condition (ii) is a description of all set of stabilizing controllers.

The relations (iii) and (iiii) are extracted a subset of robust controllers.

Inequality (9) is a simple consequence of the expression

$$[1 + w(-j\omega)][1 + w(j\omega)] \geq \eta^2 \tag{11}$$

that follows from definition 2.1 and 2.2.

In fact, the open-loop transfer function of system (1), (2) with controller (2) whose coefficients are delivered by expressions (7) is

$$w(s) = -\frac{k(s)[r^0(s)b(s) + d(s)a(s)]}{d(s)[g^0(s)b(s) + k(s)a(s)]}. \tag{12}$$

After substituting this expression into inequality (11) it is obtained that

$$\begin{aligned} [d(-s)g^0(-s)b(-s) - k(-s)r^0(-s)b(-s)]\cdot \\ \cdot[d(s)g^0(s)b(s) - k(s)r^0(s)b(s)] \geq \\ \geq \eta^2 d(-s)d(s)[g^0(-s)b(-s) + k(-s)a(-s)]\cdot \\ \cdot[g^0(s)b(s) + k(s)a(s)] \end{aligned} \tag{13}$$

Inequality (9) follows from this expression if one takes into account identity (8).

4. Classes of guaranteed stabilized plants

Consider two classes of the plants: minimum-phase plants (e.g. the plants whose polynomial $k(s)$ is Hurwitz polynomial and $d(s)$ is an arbitrary polynomial) and asymtotically stable plants ($d(s)$ is Hurwitz polynomial and $k(s)$ is an arbitrary polynomial).

Going to a solution of problem 2.1 it need to remark that for its solution it is sufficiently to find only one controller from the set described by assertion 3.1.

Assertion 4.1 All minimum-phase plants are guaranteed stabilized.

Proof Consider two cases of a structure of polynomial $k(s)$: $m = n - 1$ and $m \leq n - 1$.

<u>Case $m = n - 1$</u>

Let

$$a(s) = 0, \quad b(s) = 1 \tag{14}$$

and a polynomial $\psi(s)$ is determined as

$$\psi(s) = k(s)\delta(s), \tag{15}$$

where $\delta(s)$ is Hurwitz polynomial that is a solution of the following identity

$$\delta(-s)\delta(s) = [d(-s)d(s) + q^0], \qquad (16)$$

where q^0 is a positive number.

Polynomials (7) of controller (2) are

$$g(s) = g^0(s) = k(s), \quad r(s) = r^0(s) = \delta(s) - d(s). \qquad (17)$$

Taking into account the expression (15) the inequality (9) may be rewritten as

$$\frac{\psi(-s)\psi(s)}{d(-s)d(s)g(-s)g(s)} = \\ = \frac{k(-s)k(s)[d(-s)d(s) + q^0]}{g(-s)g(s)d(-s)d(s)} \geq \eta^2 \qquad (18)$$
$$s = j\omega \quad 0 \leq \omega < \infty.$$

A substitution of the first from the equalities (17) gives

$$\left| 1 + \frac{q^0}{d(-j\omega)d(j\omega)} \right| \geq 1 \geq \eta^2, \quad 0 \leq \omega < \infty. \qquad (19)$$

The assertion proof for this case has been obtained early [1] on the base of the LQ-optimization.

Case $m \leq n - 1$.

Let

$$\psi(s) = k(s)\varepsilon(s)\delta(s), \qquad (20)$$

where a polynomial $\varepsilon(s)$ of a degree $p = n - m - 1$ has the following structure

$$\varepsilon(s) = \prod_{i=1}^{p}(T_i s + 1) = \varepsilon_p s^p + \cdots + \varepsilon_1 s + 1 \qquad (21)$$

in which positive numbers T_i $(i = \overline{1,p})$ satisfy the inequalities

$$1 > T_1 > T_2 > \cdots > T_p. \qquad (22)$$

A solution of the Bezout-Identity

$$d(s)g(s) - k(s)r(s) = k(s)\varepsilon(s)\delta(s) \qquad (23)$$

is sought in the following form

$$g(s) = k(s)m(s), \quad (deg[m(s)] = p), \qquad (24)$$

where a polynomial $m(s)$ is a solution of the Bezout-Identity

$$d(s)m(s) - r(s) = \varepsilon(s)\delta(s). \qquad (25)$$

Taking into account the structure of polynomials $g(s)$ and $\psi(s)$ the inequality (9) is rewritten as

$$\frac{\varepsilon(-s)\varepsilon(s)[d(-s)d(s) + q^0]}{m(-s)m(s)d(-s)d(s)} \geq \eta^2 \qquad (26)$$
$$s = j\omega, \quad 0 \leq \omega < \infty$$

Now, in order to prove assertion 4.1 it is sufficiently to show that

$$\frac{\varepsilon(-s)\varepsilon(s)}{m(-s)m(s)} \geq \eta^2 \quad s = j\omega, \quad 0 \leq \omega < \infty. \qquad (27)$$

In connection with it a dependence of coefficients of the polynomial $m(s) = m_p s^p + \cdots + m_1 s + m_0$ on coefficients of polynomial $\varepsilon(s)$ must be studied.

Lemma 4.1 There exists sufficiently small values T_i $(i = \overline{1,p})$ such that the coefficients of the polynomials $m(s)$ and $\varepsilon(s)$ link by the following relations

$$m_i = \varepsilon_i(1 + e_i) \quad (i = \overline{1,p}) \qquad (28)$$

in which e_i $(i = \overline{1,p})$ are some numbers satisfied the inequalities

$$|e_i| \leq e_i^* \quad (i = \overline{1,p}) \qquad (29)$$

where e_i^* $(i = \overline{1,p})$ are any specified positive numbers.

A proof of the assertion is given in Appendix.

It is obviously that the numbers e_i^* $(i = \overline{1,p})$ may be took such that the inequality (27) holds.

Assertion 4.2 All asymptotically stable plants are guaranteed stabilized.

Proof Let,for simplicity, all roots of polynomial $k(s)$ have the positive real parts ($k(-s)$ is Hurwitz polynomial).

Let the equalities (14) hold and a polynomial $\psi(s)$ is

$$\psi(s) = d(s)k(-s)v(s)q, \qquad (30)$$

where q is a sufficiently large positive number, $v(s)$ is a Hurwitz polynomial of a degree $p_1 \geq n - m$.

One of solutions of Bezout-Identity (8) is

$$g(s) = g^0(s) = k(-s)v(s)q + k(s), \\ r(s) = r^0(s) = d(s) \qquad (31)$$

Taking into account the expressions (30) and (31) the robustness condition (9)is rewritten as

$$\frac{d(-s)d(s)k(s)k(-s)v(-s)v(s)q^2}{d(-s)d(s)[k(s)k(-s)v(-s)v(s)q^2 + \rho(s)]} \geq \eta^2$$
$$\rho(s) = k(s)k(s)v(-s)q + \\ + k(-s)k(-s)v(s)q + k(s)k(-s). \qquad (32)$$

If the number q is sufficiently large then this inequality holds.

5. Unstable and nonminimum-phase plants

Consider the plant (1) whose polynomials have the roots with positive real parts.

In order to prove that such plants may be nonguaranteed stabilized it is sufficiently to find only one example of plant (1) for that there does not exist a controller from the set described by assertion 3.1.

Such plant [7], [9] has the following transfer function

$$w_0 = \frac{s-1}{s^2-s-2}. \tag{33}$$

This plant is unstable and nonminimum-phase.

A controller that delivers a maximum of the margin radius for this plant has been found in paper [5]. This value $r = 0.32 < 0.75$ and therefore the plant (33) is not the guaranteed stabilized plant.

A technique proposed in this paper allows to find a controller that provides a maximal radius and to find out whether a plant is garanteed stabilized or not.

6. Examples

Example 1 Consider a minimum-phase, unstable plant described by the following transfer function

$$w_0(s) = \frac{k(s)}{d(s)} = \frac{5(0.4s+1)}{(s+0.2)(s^2-6s+25)}. \tag{34}$$

Taking the polynomial $\varepsilon(s) = 0.003s + 1$ and the number $q_0 = 10^4$ the polynomial $\psi(s)$ is written in accordance with (20) as

$$\psi(s) = 5(0.4s+1)(0.003s+1)\delta(s), \tag{35}$$

where

$$\begin{aligned}\delta(-s)\delta(s) &= (-s+0.2)(s+0.2)\cdot\\ &\cdot(s^2-6s+25)(s^2+6s+25)+10^4.\end{aligned} \tag{36}$$

After a calculation of the polynomial $\delta(s)$ the following polynomial $\psi(s)$ was found:

$$\begin{aligned}\psi(s) = 0.006s^5 + 2.07s^4 + 23.9s^3 + 147s^2\\ +450.6s + 500.62.\end{aligned} \tag{37}$$

Solving Bezout-Identity (8) delivers

$$\begin{aligned}g(s) &= 0.006s^2 + 2.105s + 5.225\\ r(s) &= -15.39s^2 - 25.18s - 94.9.\end{aligned} \tag{38}$$

The stability margin radius of the system consisted of the plant (1) with transfer function (34) and the controller (2) with the polynomials (38) is $r = 0.99$.

Example 2 Consider an asymtotically stable, nonminimum-phase plant described by the following transfer function [8]:

$$w_0 = \frac{k(s)}{d(s)} = \frac{5(-0.4s+1)}{(s+0.2)(s^2+6s+25)}. \tag{39}$$

Taking the polynomial $v(s) = (0.3s+1)(0.5s+1)$ and the number $q = 10$ the polynomial $\psi(s)$ is written as

$$\begin{aligned}\psi(s) = 50(s+0.2)(s^2+6s+25)(0.4s+1)\cdot\\ \cdot(0.3s+1)(0.5s+1).\end{aligned} \tag{40}$$

Formulae (31) give

$$\begin{aligned}r(s) &= (s+0.2)(s^2+6s+25)\\ g(s) &= 50(0.4s+1)(0.3s+1)(0.5s+1)\\ &\quad -2s+5.\end{aligned} \tag{41}$$

The stability margin radius of the system (39), (41) is $r = 0.99$.

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7. Appendix

7.1 Proof of lemma 4.1.

For simplicity, the lemma will be proved for a case when $n = 4$ and $p = 2$. In this case Bezout-Identity (25) is

$$\begin{aligned}(s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0)(m_2 s^2 + m_1 s + m_0) - \\ -(r_3 s^3 + r_2 s^2 + r_1 s + r_0) = (\varepsilon_2 s^2 + \varepsilon_1 s + 1)\cdot \\ \cdot (s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0),\end{aligned} \tag{42}$$

and polinomial $\varepsilon(s) = (T_1 s + 1)(T_2 s + 1)$.

The identity (42) delivers the following equations for coefficients of the polynomial $m(s)$

$$\begin{aligned} m_2 &= \varepsilon_2 \\ m_1 &= \varepsilon_1 - d_3 m_2 + \delta_3 \varepsilon_2 = \varepsilon_1[1 + e_1] \\ m_0 &= 1 + e_0 \end{aligned} \tag{43}$$

where

$$e_1 = \frac{(\delta_3 - d_3) T_1 T_2}{T_1 + T_2}, \tag{44}$$

$$e_0 = (\delta_2 - d_2) T_1 T_2 + \delta_3 (T_1 + T_2) - d_3 (T_1 + T_2)(1 + e_1). \tag{45}$$

It follows from the expression (44) that for any small specified number e_1^* may be always found a number $T_2 < T_1$ such that $|e_1| < e_1^*$. Analogously, it follows from (45) that may be always pointed out $T_1 < 1$ such that $|e_0| < e_0^*$. The lemma proof when $n > 4$ and $p > 2$ is analogous.