NEW RESULTS ON THE CONVEX DIRECTION WITH RESPECT TO A GIVEN HURWITZ POLYNOMIAL

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Keywords: Convex direction, Hurwitz stability, Interlacing property, Positive pair, Even-odd part.

Abstract

Convex direction with respect to a given Hurwitz polynomial $f_1(s)$ is a local version of convex direction given by Rantzer. In this paper, a new sufficient condition is given to judge and construct a convex direction with respect to $f_1(s)$.

1 Introduction

In robustness analysis and synthesis, finite test is desired, such as the Kharitonov theorem for interval polynomials[7]. Because the stability domain in the coefficient space of polynomials is not convex, generally speaking, finite test does not hold. A. C. Bartlett, C. V. Hollot and L. Huang proved the edge theorem for polynomial polytope with respect to any simply connected domain[2]. Convex direction, as an instrument to jump from an edge result to an extreme point result, play an important role[3, 6]. Given a polynomial $f_2(s)$, $f_2(s)$ is called a convex direction if any Hurwitz polynomial $f_1(s)$ (i.e., all roots of $f_1(s)$ lie in the open left half plane) such that $f_1(s) + \mu f_2(s)$ is Hurwitz stable for every $\mu \in [0, 1]$ is equivalent to that $f_1(s) + f_2(s)$ is so. A polynomial is Hurwitz stable if it is a Hurwitz polynomial. If we restrict the arbitrariness of $f_1(s)$, then the definition above gives a local version, which is called the convex direction with respect to $f_1(s)$ in the following. The concept is less restrictive and thus less conservative for a variety of robust stability applications, such as judging the stability of a given segment. Thus, it is of interest to pay attention to such problem.

There are several versions on the definition of convex direction with respect to $f_1(s)$, where $f_1(s)$ is a Hurwitz polynomial. Here we employ the definition given in Ref. [1]. Suppose that $f_1(s)$ is Hurwitz stable. A polynomial $f_2(s)$ is called a convex direction with respect to $f_1(s)$, if for any $\mu \in [0, 1], \deg(f_1(s) + \mu f_2(s)) = \deg(f_1(s))$ and $f_1(s) + \mu f_2(s)$ is Hurwitz stable is equivalent to $f_1(s) + f_2(s)$ is so, where the denotation $\deg(\cdot)$ stands for the degree of polynomial.

The research on convex direction with respect to $f_1(s)$ follows two lines. Starting with the definition, one line is similar to the study of convex direction, and a frequency-related condition is given, which is necessary and sufficient[1]. As pointed in the case of convex direction, this condition is checked by sweeping ω from 0 to ∞. Thus, the computation efforts should be demanding. The other one is based on the analysis of root loci, where two different definitions on convex direction with respect to $f_1(s)$ are given[4, 5]. The interest of results in Ref. [4] is to give some useful descriptions on the characterization of convex direction with respect to $f_1(s)$. Furthermore, in Ref. [5] an algorithm is given to construct a set of convex direction with respect to $f_1(s)$.

In this paper, the problem of convex direction with respect to $f_1(s)$ is studied in terms of root loci, too. Two classes of convex directions with respect to $f_1(s)$ are described, which enlarges the set defined in Ref. [5]. An interesting example will be given in the end. It is useful to show that there exists other class of convex directions with respect to $f_1(s)$ except those pointed in the Theorem.

2 Preliminaries

Assumption: *$f_1(s)$ is a Hurwitz polynomial and $f_2(s)$ is an arbitrary polynomial such that $f_1(s)+\mu f_2(s)$ has the same degree as $f_1(s)$ for any $\mu \in [0, 1]$, and $f_1(s)+f_2(s)$ is Hurwitz stable.*

Definition 1. *The polynomial $f_2(s)$ is called a convex direction with respect to $f_1(s)$, if that $f_1(s)$ and $f_1(s) + f_2(s)$ are Hurwitz stable is equivalent to that $f_1(s) + \mu f_2(s)$ is Hurwitz stable for all $\mu \in [0, 1]$.*

This definition is a local version of convex direction given by Rantzer[3].

Denote the even-odd parts associated with $f_1(s)$, $f_2(s)$ and $f_\mu(s) = f_1(s)+\mu f_2(s)$ by $(h_1(\lambda), g_1(\lambda))$, $(h_2(\lambda), g_2(\lambda))$ and $(h_\mu(\lambda), g_\mu(\lambda))$, i.e.,

$$\begin{aligned} f_1(s) &= h_1(s^2) + sg_1(s^2), \\ f_2(s) &= h_2(s^2) + sg_2(s^2), \\ f_\mu(s) &= h_\mu(s^2) + sg_\mu(s^2). \end{aligned}$$

This gives that

$$h_\mu(\lambda) = h_1(\lambda) + \mu h_2(\lambda), \quad g_\mu(\lambda) = g_1(\lambda) + \mu g_2(\lambda)$$

We recall that a polynomial is Hurwitz stable if and only if $(h(\lambda), g(\lambda))$ forms a positive pair, namely,

Lemma 1. [6] *Let $(h(\lambda), g(\lambda))$ be the even-odd parts associated with the given real polynomial $f(s) = h(s^2) + sg(s^2)$. Then $f(s)$ is Hurwitz stable if and only if the roots of $h(\lambda)$ and $g(\lambda)$, says $\{x_i\}$ and $\{y_i\}$, satisfy $x_{i+2} < y_{i+1} < x_{i+1} < y_i < x_i < 0$.*

By Assumption and Lemma 1, it is easy to see that all roots of $h_1(\lambda), h_1(\lambda)+h_2(\lambda), g_1(\lambda)$ and $g_1(\lambda)+g_2(\lambda)$ are distinct, real and negative.

At the beginning, we develop some characterizations of the even-odd parts of $f_1(s) + \mu f_2(s)$ for $\mu \in [0, 1]$.

Proposition 1a. *Suppose that $x_0^{(1)}$ is a root of $h_1(\lambda)$ and $x_0^{(2)}$ a root of $h_1(\lambda)+h_2(\lambda)$ such that $h_1(\lambda)$ and $h_1(\lambda)+h_2(\lambda)$ have no roots between $x_0^{(1)}$ and $x_0^{(2)}$. Let $\Delta = \{\beta : \beta \in [0,1], h_2(\beta x_0^{(1)} + (1-\beta)x_0^{(2)}) = 0\}$ and denote m the number of elements in Δ, then m is odd or $m = 0$.*

Proof: For all $\beta \in (0,1), x(\beta) = \beta x_0^{(1)} + (1-\beta)x_0^{(2)}, h_1(x(\beta)) \neq 0, h_1(x(\beta)) + h_2(x(\beta)) \neq 0$, if $m \neq 0$ there exists at least a root of $h_2(\lambda)$. Suppose m is even, without loss of generality, we assume that $x_0^{(1)} < x_0^{(2)}$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_m, \beta_i \in \Delta$. There must exist an $\epsilon > 0$ such that

$$h_2(\lambda^*)h_2(\lambda^{**}) > 0, \forall \lambda^* \in (x_0^{(1)}, x_0^{(1)}+\epsilon), \ \lambda^{**} \in (x_0^{(2)}-\epsilon, x_0^{(2)})$$

Considering the equation $h_1(\lambda) + \mu h_2(\lambda) = 0$, since $h_1(\lambda)$ and $h_1(\lambda)+h_2(\lambda)$ have no other roots in $(x_0^{(1)}, x_0^{(2)})$ we have

$$\lim_{\lambda \to x(\beta_1)^-} \left(-\frac{h_1(\lambda)}{h_2(\lambda)}\right) \cdot \lim_{\lambda \to x(\beta_m)^+} \left(-\frac{h_1(\lambda)}{h_2(\lambda)}\right) > 0.$$

It follows that there exists a root of $h_1(\lambda)$ on $\left(x(\beta_m), x_0^{(2)}\right)$ or of $h_1(\lambda)+h_2(\lambda)$ on $\left(x_0^{(1)}, x(\beta_1)\right)$. We have reached a contradiction. This completes the proof.

Proposition 2a. *If $x_i^{(1)}$ is a root of equation $h_1(\lambda) = 0$, then there exists a number $x_i^{(2)} < 0$ such that $h_1(x_i^{(2)}) + h_2(x_i^{(2)}) = 0$. Furthermore, for every $\lambda \in \left(\min\{x_i^{(1)}, x_i^{(2)}\}, \max\{x_i^{(1)}, x_i^{(2)}\}\right)$, all of $h_1(\lambda), h_2(\lambda)$ and $h_1(\lambda)+h_2(\lambda)$ are nonzero, and $x_i(\mu)$ is continuous on $\mu \in [0,1]$ such that*

$$\lim_{\mu \to 0} x_i(\mu) = x_i^{(1)}, \quad \lim_{\mu \to 1} x_i(\mu) = x_i^{(2)},$$

where $x_i(\mu)$ satisfies the equation $h_1(\lambda) + \mu h_2(\lambda) = 0$.

Proof: Under our assumption, without loss of generality, let $x_n^{(1)} < \ldots < x_2^{(1)} < x_1^{(1)} < 0, x_n^{(2)} < \ldots < x_2^{(2)} < x_1^{(2)} < 0$ be roots of $h_1(\lambda)$ and $h_1(\lambda)+h_2(\lambda)$, respectively. Denote $x_i(\mu)$ the ith root of $h_1(\lambda) + \mu h_2(\lambda)$. By continuity, $x_i(\mu)$ runs from $x_i^{(1)}$ to $x_j^{(2)}$ when μ changes from 0 to 1. If $i \neq j$, then there exists $i_1, i_2 \in \{1, \ldots, n\}, \mu_1, \mu_2 \in [0,1]$ such that

$$x_{i_1}(\mu_1) = x_{i_2}(\mu_2) \triangleq x*.$$

Since $h_1(x*) + \mu_1 h_2(x*) = 0$ and $h_1(x*) + \mu_2 h_2(x*) = 0$, we have $\mu_1 = \mu_2$. This shows that $x*$ is a root of $h_1(\lambda) + h_2(\lambda)$ with algebraic multiplicity 2. In this case, by continuity, $x_i^{(2)}$ lies between $x_{i_1}^{(1)}$ and $x*$. This will reach a contradiction, because of existence of $\mu_3 \neq \mu_4 \in (0,1)$ such that $x_{i_1}(\mu_3) = x_{i_1}(\mu_4)$. Thus, $i = j$, that is to say, $x_i(\mu)$ runs from $x_i^{(1)}$ to $x_i^{(2)}$ when μ changes from 0 to 1. The left statement is obvious.

The dual of Propositions 1a and 2a are given as follow:

Proposition 1b. *Suppose that $y_0^{(1)}$ is a root of $g_1(\lambda)$ and $y_0^{(2)}$ a root of $g_1(\lambda)+g_2(\lambda)$ such that $h_1(\lambda)$ and $g_1(\lambda)+g_2(\lambda)$ have no roots between $y_0^{(1)}$ and $y_0^{(2)}$. Let $\Delta = \{\beta : \beta \in [0,1], g_2(\beta y_0^{(1)} + (1-\beta)y_0^{(2)}) = 0\}$ and denote m the number of elements in Δ, then m is odd or $m = 0$.*

Proposition 2b. *If $y_i^{(1)}$ is a root of equation $g_1(\lambda) = 0$, then there exists a number $y_i^{(2)} < 0$ such that $g_1(y_i^{(2)}) + g_2(y_i^{(2)}) = 0$. Furthermore, for every $\lambda \in \left(\min\{y_i^{(1)}, y_i^{(2)}\}, \max\{y_i^{(1)}, y_i^{(2)}\}\right)$, all of $g_1(\lambda), g_2(\lambda)$ and $g_1(\lambda)+g_2(\lambda)$ are nonzero, and $y_i(\mu)$ is continuous on $\mu \in [0,1]$, and*

$$\lim_{\mu \to 0} y_i(\mu) = y_i^{(1)}, \quad \lim_{\mu \to 1} y_i(\mu) = y_i^{(2)},$$

where $y_i(\mu)$ satisfies the equation $g_1(\lambda) + \mu g_2(\lambda) = 0$.

Remark 1. *By proposition 2a, together with its proof, and proposition 2b, it is shown that for any $\mu \in [0,1]$, $\{x_i(\mu)\}$ and $\{y_i(\mu)\}$ are distinct, real and negative.*

Remark 2. *For any $\mu \in [0,1]$, denote $x(\mu)$ the solution of equation $h_1(\lambda) + \mu h_2(\lambda) = 0$. Then $\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)|_{\lambda = x(\mu)} \neq 0$. The similar statement is true for $g_1(\lambda) + \mu g_2(\lambda) = 0$.*

By contrary, if $\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)|_{\lambda = x(\mu)} = 0$, then it follows that $x(\mu)$ is a root of $h_1(\lambda) + \mu h_2(\lambda)$ with algebraic multiplicity 2. This contradicts to Remark 1.

Lemma 2. *Denote $x(\mu)$ the solution of equation $h_1(\lambda) + \mu h_2(\lambda) = 0$ and $y(\mu)$ the solution of equation $g_1(\lambda) + \mu g_2(\lambda) = 0$. Then*

$$\frac{d}{d\mu}(x(\mu)) = -\frac{1}{\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)}|_{\lambda = x(\mu)}$$

$$\frac{d}{d\mu}(y(\mu)) = -\frac{1}{\frac{d}{d\lambda}\left(\frac{g_1(\lambda)}{g_2(\lambda)}\right)}|_{\lambda = y(\mu)}$$

Proof: Starting with the equation $h_1(x(\mu)) + \mu h_2(x(\mu)) = 0$. The chain rule now reads

$$\frac{dh_1(x(\mu))}{d(x(\mu))} \cdot \frac{d(x(\mu))}{d\mu} + h_2(x(\mu)) + \mu \frac{dh_2(x(\mu))}{d(x(\mu))} \cdot \frac{d(x(\mu))}{d\mu} = 0$$

which gives that

$$\frac{d(x(\mu))}{d\mu} = -\frac{h_2(\lambda)}{\frac{d}{d\lambda}h_1(\lambda) + \mu \frac{d}{d\lambda}h_2(\lambda)}|_{\lambda = x(\mu)}$$

Since $\mu = -\frac{h_1(x(\mu))}{h_2(x(\mu))}$, the result follows. The dual statement can be confirmed similarly.

3 Main Result

The main result on our point can be stated as follows.

Theorem 1. *$f_2(s) = h_2(s^2) + sg_2(s^2)$ is a convex direction with respect to $f_1(s) = h_1(s^2) + sg_1(s^2)$ if one of the following two conditions holds:*
a) $h_1(s^2) + h_2(s^2) + sg_1(s^2)$ and $h_1(s^2) + s(g_1(s^2) + g_2(s^2))$ are Hurwitz stable;
b) For any $i, j \in \{1, \ldots, n\}, 0 \le j - i \le 1$ such that

$$\begin{array}{l} x_i^{(1)} > y_j^{(1)} > x_i^{(2)} > y_j^{(2)} \\ x_i^{(2)} < y_j^{(2)} < x_i^{(1)} < y_j^{(1)}, \end{array} \tag{1}$$

we have

$$\begin{array}{ll} \overline{H_i} < \underline{G_j} + (x_i^{(1)} - y_j^{(1)}), & \text{if } i = j; \\ \underline{H_i} > \overline{G_j} + (x_i^{(1)} - y_j^{(1)}), & \text{if } i = j + 1, \end{array}$$

where $x_i^{(1)}, x_i^{(2)}, y_i^{(1)}$ and $y_i^{(2)}$ are the i-th roots of $h_1(\lambda), h_1(\lambda) + h_2(\lambda), g_1(\lambda)$ and $g_1(\lambda) + g_2(\lambda)$, respectively, and

$$\begin{array}{l} \overline{x_i} = \max\{x_i^{(1)}, x_i^{(2)}\}, \ \ \underline{x_i} = \min\{x_i^{(1)}, x_i^{(2)}\} \\ \overline{y_i} = \max\{y_i^{(1)}, y_i^{(2)}\}, \ \ \underline{y_i} = \min\{y_i^{(1)}, y_i^{(2)}\} \\ \overline{H_i} = \max_{\lambda \in [\underline{x_i}, \overline{x_i}]} \left\{ \frac{1}{\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)} \right\} \\ \underline{H_i} = \min_{\lambda \in [\underline{x_i}, \overline{x_i}]} \left\{ \frac{1}{\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)} \right\} \\ \overline{G_i} = \max_{\lambda \in [\underline{y_i}, \overline{y_i}]} \left\{ \frac{1}{\frac{d}{d\lambda}\left(\frac{g_1(\lambda)}{g_2(\lambda)}\right)} \right\} \\ \underline{G_i} = \min_{\lambda \in [\underline{y_i}, \overline{y_i}]} \left\{ \frac{1}{\frac{d}{d\lambda}\left(\frac{g_1(\lambda)}{g_2(\lambda)}\right)} \right\} \end{array}$$

Proof: a) By hypothesis, all of $h_1(s^2) + h_2(s^2) + sg_1(s^2), h_1(s^2) + s(g_1(s^2) + g_2(s^2)), f_1(s)$ and $f_1(s) + f_2(s)$ are Hurwitz stable. It follows from interlacing property that

$$\begin{array}{l} x_{i+2}^{(2)} < y_{i+1}^{(1)} < x_{i+1}^{(2)} < y_i^{(1)} < x_i^{(2)} < 0, \\ x_{i+2}^{(1)} < y_{i+1}^{(2)} < x_{i+1}^{(1)} < y_i^{(2)} < x_i^{(1)} < 0, \\ x_{i+2}^{(1)} < y_{i+1}^{(1)} < x_{i+1}^{(1)} < y_i^{(1)} < x_i^{(1)} < 0, \\ x_{i+2}^{(2)} < y_{i+1}^{(2)} < x_{i+1}^{(2)} < y_i^{(2)} < x_i^{(2)} < 0. \end{array}$$

Set $\overline{x_l} = \min\{x_l^{(1)}, x_l^{(2)}\}, \underline{x_l} = \max\{x_l^{(1)}, x_l^{(2)}\}$, and similarly define $\overline{y_l}, \underline{y_l}$ for $l = i, i+1, i+2$. This shows that

$$\overline{x_{i+2}} < \underline{y_{i+1}} < \overline{y_{i+1}} < \underline{x_{i+1}} < \overline{x_{i+1}} < \underline{y_i} < \overline{y_i} < \underline{x_i} < 0 \tag{2}$$

To show that $f_2(s)$ is a convex direction with respect to $f_1(s)$, it suffices to show that $f_1(s) + \mu f_2(s)$ is Hurwitz stable for any $\mu \in [0, 1]$. For any $\mu \in [0, 1]$, the associated even-odd parts of $f_1(s) + \mu f_2(s)$ are $h_1(s^2) + \mu h_2(s^2)$ and $g_1(s^2) + +\mu g_2(s^2)$, denote their roots $x_i(\mu), y_i(\mu)$, respectively. Then $x_i(\mu) \in [\underline{x_i}, \overline{x_i}], y_i(\mu) \in [\underline{y_i}, \overline{y_i}]$. By (2), it follows that

$$x_{i+2}(\mu) < y_{i+1}(\mu) < x_{i+1}(\mu) < y_i(\mu) < x_i(\mu) < 0.$$

This shows that $f_1(s) + \mu f_2(s)$ is Hurwitz stable.

b) Without loss of generality, assume that $j = i \in \{1, \ldots, n\}$ satisfying $x_i^{(1)} > y_j^{(1)} > x_i^{(2)} > y_j^{(2)}$. By the mean value theorem, there exist two numbers $\mu_1, \mu_2 \in [0, 1]$ such that

$$\begin{array}{l} x_i(\mu) = x_i^{(1)} + \mu \left(\frac{d}{d\mu}(x_i(\mu)|_{\mu=\mu_1}) \right) \\ y_i(\mu) = y_i^{(1)} + \mu \left(\frac{d}{d\mu}(y_i(\mu)|_{\mu=\mu_2}) \right) \end{array}$$

Applying Lemma 2, our hypothesis gives that

$$\begin{array}{l} x_i(\mu) - y_i(\mu) = (x_i^{(1)} - y_i^{(1)}) + \\ \mu \left(\frac{-1}{\frac{d}{d\lambda}\left(\frac{h_1(\lambda)}{h_2(\lambda)}\right)}|_{\lambda = x_i(\mu_1)} + \frac{1}{\frac{d}{d\lambda}\left(\frac{g_1(\lambda)}{g_2(\lambda)}\right)}|_{\lambda = y_i(\mu_2)} \right) \\ > (x_i^{(1)} - y_i^{(1)}) + \mu \left(-\overline{H_i} + \underline{G_i}\right) \\ > (1 - \mu)(x_i^{(1)} - y_i^{(1)}) \\ > 0 \end{array}$$

Similar proofs can be given for the other cases. Thus for any $\mu \in [0, 1]$

$$x_{i+2}(\mu) < y_{i+1}(\mu) < x_{i+1}(\mu) < y_i(\mu) < x_i(\mu) < 0.$$

This shows that $f_1(s) + \mu f_2(s)$ is Hurwitz stable. The proof is completed.

Remark 3. *The first condition in Theorem 1 is exactly equivalent to the result given in Ref.* [5]. *This point can be seen by our proof.*

Example 1. *Let*

$$\begin{array}{l} f_1(s) = s^3 + 6s^2 + 11s + 6, \\ f_2(s) = s^3 - 3s^2 - s, \\ f_3(s) = s^3 - 3s^2 + 81s + 30, \\ f_4(s) = s^3 - 3s^2 + 17s + 30. \end{array}$$

It is easy to verify that $f_1(s), f_1(s) + f_i(s) (i = 2, 3, 4)$ are Hurwitz stable. Direct verification shows that $f_2(s)$ satisfies the first condition of the Theorem 1, $f_3(s)$ violates the first condition but satisfies the second one of the Theorem 1, while $f_4(s)$ satisfies neither one of Theorem 1. Applying the Theorem 1, it follows that $f_2(s)$ and $f_3(s)$ are convex direction with respect to $f_1(s)$, and whether $f_4(s)$ is a convex direction with respect to $f_1(s)$ can not be confirmed in this case by the Theorem 1.

In fact, from Fig. 1, 2, 3, it can be seen that all of $f_i(s), i = 2, 3, 4$ are convex direction with respect to $f_1(s)$.

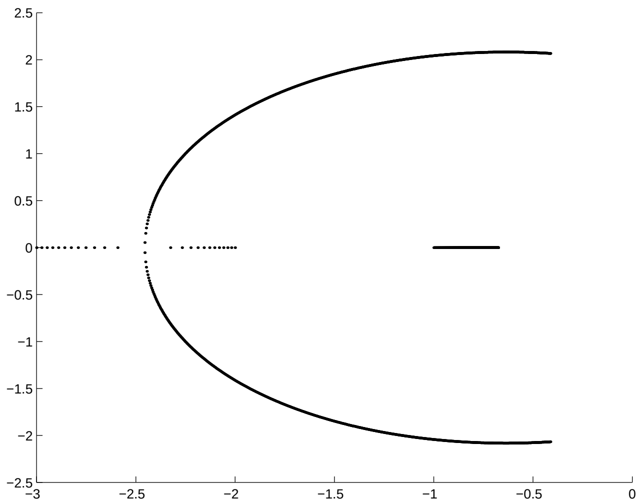

Fig. 1

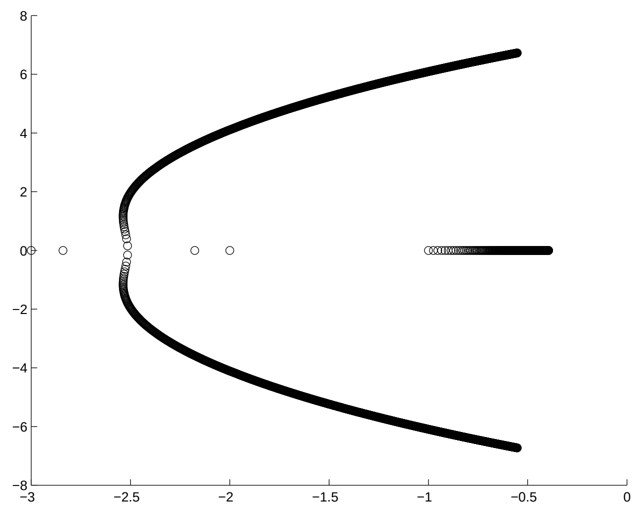

Fig. 2

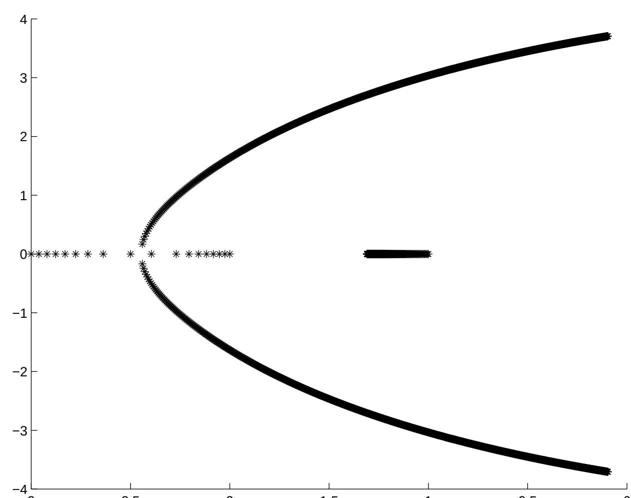

Fig. 3

Acknowledgements

The research work was supported by National Natural Science Foundation of China (No. 60204006 and No. 69925307), the Knowledge Innovation Pilot Program of the Chinese Academy of Sciences, and National Key Basic Research Special Fund and National Hi-Tech R&D 863 Project (No. 2002AA755002).

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