REGULARIZING FOR POLYNOMIAL MATRICES AND ITS APPLICATIONS

Wataru Kase* and Yasuhiko Mutoh†

* Department of Electrical Electric Systems Eng., Osaka Institute of Technology
5-16-1 Omiya, Asahi-ku, Osaka, 535-8585, JAPAN email: kase@ee.oit.ac.jp.
† Department of Mechanical Engineering, Sophia University
7-1 Kioi-cho, Chiyoda-ku, Tokyo, 102-8554, JAPAN, email: y_mutou@sophia.ac.jp

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Abstract

Recently, we have proposed a solution of Diophantine equation, which plays a central role of the stabilizing problem in the design of linear control systems. But the method is limited to the case where the transfer function matrix is strictly proper and the denominator polynomial matrix is column proper. In this note, we proposed a solution of general Diophantine equation which does not exist such limitations. In this extension, we proposed an interactor like polynomial matrix, called a regularizing polynomial matrix.

1 Introduction

Recently, we have proposed a solution of Diophantine equation, which plays a central role of the stabilizing problem in the design of linear control systems. But the method is limited to the case where the transfer function matrix is strictly proper and the denominator polynomial matrix is column proper. In this note, we proposed a solution of general Diophantine equation which does not exist such limitations. In this extension, we proposed a regularizing polynomial matrix for non-row/column proper polynomial matrices.

The regularizing matrix is almost equivalent to an interactor matrix [1]. A derivation of the interactor is much complex. Although the algebraic equation, which the interactor should be satisfied, was shown in [4], the solution in [4] was not adequate for computer calculations. The authors proposed a solution of the equation by using Moore-Penrose pseudoinverse[6]. Since a function to calculate the pseudoinverse is available in some standard softwares for control engineering, the method is adequate for computer calculations.

One of the most related problem to the regularizing matrix is row/column properizing by elementary operations. Since the regularizing matrix is not unimodular, non-unimodular part of the regularizing matrix must be eliminate to find the elementary operation (unimodular) matrix. To do this, a calculation of the minimal bases in polynomial vector space will be used. A calculation of the invariant zeros will be also discussed.

2 Regularizing Polynomial Matrix

Consider the following $m \times m$ polynomial matrix $D(z)$:

$$\begin{aligned} D(z) &= D_0 + zD_1 + \cdots + z^n D_n \\ &= S^n_{I_m}(z)\boldsymbol{D} \end{aligned} \qquad (1)$$

where

$$\boldsymbol{D} = \begin{bmatrix} D_0 \\ D_1 \\ \dots \\ D_n \end{bmatrix}, \qquad (2)$$
$$S^n_{I_m}(z) = \begin{bmatrix} I_m & zI_m & \cdots & z^n I_m \end{bmatrix}.$$

The problem consider in this section is to find a polynomial matrix $L(z)$

$$\begin{aligned} L(z) &= L_0 + zL_1 + \cdots + z^w L_w \\ &= S^w_{I_m}(z)\boldsymbol{L}, \quad w \le n \end{aligned} \qquad (3)$$

which makes n-th degree's coefficient matrix of $D(z)L(z)$ be nonsingular. We call the above $L(z)$ as a regularizing matrix. The existence of such matrix is clear by considering the interactor for $D(z)/z^n$.

$D(z)L(z)$ can be written by

$$\begin{aligned} D(z)L(z) &= (D_0 + zD_1 + \cdots + z^n D_n)S^w_{I_m}(z)\boldsymbol{L} \qquad (4) \\ &= S^{n+w}_{I_m}(z) \begin{bmatrix} D_0 & 0 & \dots & 0 \\ D_1 & D_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_w & D_{w-1} & \dots & D_0 \\ \vdots & \vdots & \ddots & \vdots \\ D_n & D_{n-1} & \dots & D_{n-w} \\ 0 & D_n & \dots & D_{n-w+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_n \end{bmatrix} \boldsymbol{L}. \end{aligned}$$

Assume that the n-th degree's coefficient of $D(z)L(z)$ is the identity matrix. If $L(z)$ is the regularizing matrix, then the following equality must hold from the above relation:

$$\boldsymbol{T}\boldsymbol{L} = \boldsymbol{J} \tag{5}$$

where

$$\boldsymbol{T} = \begin{bmatrix} D_n & D_{n-1} & \dots & D_{n-w} \\ 0 & D_n & \dots & D_{n-w+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_n \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

In order to confirm the solvability of the above equation, the degree w is defined as the minimum integer which satisfies the following condition:

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{T} & \boldsymbol{J} \end{bmatrix} = \operatorname{rank} \boldsymbol{T}. \tag{7}$$

Using Moore-Penrose pseudoinverse to solve eqn.(7), we have a regularizing matrix which has the following property:

Lemma 1 Let

$$L^{\sim}(z) = L^T(z^{-1}) = z^{-1}L_1^T + z^{-2}L_2^T + \cdots + z^{-w}L_w^T, \tag{8}$$

If $\boldsymbol{L}$ is given by

$$\boldsymbol{L} = \boldsymbol{T}^{\dagger}\boldsymbol{J}, \tag{9}$$

then the following relation hold:

$$L^{\sim}(z)L(z) = \boldsymbol{L}^T\boldsymbol{L}. \tag{10}$$

The above Lemma can be proved on the same line as is shown in [6]. It means that the regularizing matrix by eqn.(9) has the all-pass property in discrete-time, and all zeros are lie at origin.

3 Applications

3.1 Row/Column Properizing by Elementary Operations

Row/column properizing of polynomial matrices by elementary operations needs much complex procedure [2], [9]. Although a simple method proposed in [5], the method restricted to the case where a given polynomial matrix is "regular". Therefore, by using the regularizing matrix presented in the previous section, we can obtain a general method.

For a given polynomial matrix $D(z)$, let $L(z)$ be the regularizing matrix. Since $D(z)L(z)$ is both row and column proper, the controllability canonical realization of $L(z)\{D(z)L(z)\}^{-1}$, say $(A,\ B,\ C)$, can be easily obtain. Then, the following relation holds [5]:

$$\begin{aligned} S_{I_m}^{\nu}(z)L(z) = {} & \boldsymbol{O}_{\nu}(C,A)S^{\mu_i-1}(z) \\ & +\boldsymbol{T}_{\nu-1}(A,B,C)S_{I_m}^{\nu-1}(z)\{D(z)L(z)\}, \end{aligned} \tag{11}$$

where μ_i's are the controllability indices of $(A,\ B)$ and ν is the maximum observability index of (C, A), and

$$\boldsymbol{O}_{\nu}(C,A) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\nu} \end{bmatrix}, \quad S^{\mu_i}(z) = \text{block diag}\begin{Bmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{\mu_i} \end{Bmatrix},$$

$$\boldsymbol{T}_{\nu}(A,B,C) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\nu}B & CA^{\nu-1}B & \cdots & CB \end{bmatrix}. \tag{12}$$

Choosing the left null bases for $\boldsymbol{O}_{\nu}(C, A)$, say $\boldsymbol{M}$, we can obtain

$$\boldsymbol{M}S_{I_m}^{\nu}(z)L(z) = \boldsymbol{M}\boldsymbol{T}_{\nu-1}(A,B,C)S_{I_m}^{\nu-1}(z)\{D(z)L(z)\}. \tag{13}$$

Let c_i denote the i-th row vector in C, and ν_i denote the i-th observability indices of $(C,\ A)$. Define $\hat{\boldsymbol{O}}$ and $\tilde{\boldsymbol{O}}$

$$\hat{\boldsymbol{O}} = \begin{bmatrix} c_1 \\ \vdots \\ c_1A^{\nu_1-1} \\ c_2 \\ \vdots \\ c_mA^{\nu_m-1} \end{bmatrix}, \quad \tilde{\boldsymbol{O}} = \begin{bmatrix} c_1A^{\nu_1} \\ \vdots \\ c_mA^{\nu_m} \end{bmatrix}. \tag{14}$$

Then there exist a matrix Λ such that

$$\tilde{\boldsymbol{O}} = \Lambda\hat{\boldsymbol{O}}. \tag{15}$$

Using Λ, define

$$\boldsymbol{M} = \begin{bmatrix} -\Lambda & I_m \end{bmatrix} U, \tag{16}$$

where U is a row selection matrix such that

$$U\boldsymbol{O}_{\nu} = \begin{bmatrix} \hat{\boldsymbol{O}} \\ \tilde{\boldsymbol{O}} \end{bmatrix}. \tag{17}$$

Lemma 2 Choose linearly independent row vectors from the top of $\boldsymbol{O}_{\nu}$, say $c_iA^{\nu_i}$ is row span of c_1, …, c_m, c_1A, …, c_mA, $c_1A^{\nu_i}$, …, $c_{i-1}A^{\nu_i}$ ($\sum_{i=1}^m \nu_i = n$, $\nu_i \geq 0$). Then, $\tilde{D}(z) := \boldsymbol{M}S_{I_m}^{\nu}(z)$ is both row and column proper matrix.

Lemma 3 Choose linearly independent row vectors c_1, …, $c_1A^{\nu_1-1}$, c_2, …, $c_2A^{\nu_2-1}$, c_3, …, $c_mA^{\nu_m-1}$, where $c_1A^{\nu_1}$ is row span of c_1, …, $c_1A^{\nu_1-1}$, $c_2A^{\nu_2}$ is row span of c_1, …, $c_1A^{\nu_1-1}$, c_2, …, $c_2A^{\nu_2-1}$, …, and $c_mA^{\nu_m}$ is row span of c_1, …, $c_mA^{\nu_m-1}$. Then, $\tilde{D}(z) := \boldsymbol{M}S_{I_m}^{\nu}(z)$ is a lower triangular column proper matrix.

In both cases, $U(z) := \boldsymbol{M}\boldsymbol{T}_{\nu-1}(A,B,C)S_{I_m}^{\nu-1}(z)$ is a unimodular matrix from the choice of $\tilde{\boldsymbol{O}}$.

Example 1 Consider the following polynomial matrix

$$
\begin{aligned}
D(z) &= \begin{bmatrix} z^3+5z^2+8z+4 & -z^3-7z^2-14z-8 \\ -z^3-8z^2-19z-12 & z^3+10z^2+33z+36 \end{bmatrix} \\
&= z^3 \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{D_3} + z^2 \underbrace{\begin{bmatrix} 5 & -7 \\ -8 & 10 \end{bmatrix}}_{D_2} \\
&\quad + z \underbrace{\begin{bmatrix} 8 & -14 \\ -19 & 33 \end{bmatrix}}_{D_1} + \underbrace{\begin{bmatrix} 4 & -8 \\ -12 & 36 \end{bmatrix}}_{D_0}.
\end{aligned}
$$

In this case, $w = 3$ and $\boldsymbol{T}$ and $\boldsymbol{J}$ can be defined by

$$
\boldsymbol{T} = \begin{bmatrix} D_3 & D_2 & D_1 \\ 0 & D_3 & D_2 \\ 0 & 0 & D_3 \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix}.
$$

Then, the coefficient matrix $\boldsymbol{L}$ of the regularizing matrix $L(z)$ is given by

$$
\boldsymbol{L} = \boldsymbol{T}^{\dagger}\boldsymbol{J} = \frac{1}{4}\begin{bmatrix} -2 & -3 \\ 2 & 3 \\ 4 & 5 \\ 0 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}
$$

and $D(z)L(z)$ can be calculated by

$$
D(z)L(z) = \begin{bmatrix} z^3-7z-6 & -4.5z^2-13.5z-9 \\ 2z^2+14z+24 & z^3+10z^2+33z+36 \end{bmatrix}.
$$

Calculating a minimal degree left annihilating matrix $\begin{bmatrix} U(z) & -\tilde{D}(z) \end{bmatrix}$ of $\begin{bmatrix} D(z)L(z) \\ L(z) \end{bmatrix}$ like [5], we have

$$
\begin{aligned}
\tilde{D}(z) &= \begin{bmatrix} 3z^2+13z+10 & -2z-8 \\ 2z+2 & 3z^2+17z+20 \end{bmatrix}, \\
U(z) &= \begin{bmatrix} 1.5z+5.5 & 1.5z+1 \\ 1.5z+6.5 & 1.5z+2 \end{bmatrix}.
\end{aligned}
$$

3.2 A Calculation of Invariant Zeros

Let $(A,\ B,\ C,\ D)$ denote a minimal realization of $G(z) = N(z)D^{-1}(z)$, where $D(z)$ and $N(z)$ are right coprime polynomial matrices. Let $P(s)$ be the system matrix for $G(z)$ which is given by

$$
P(s) = \begin{bmatrix} zI_n - A & B \\ -C & D \end{bmatrix}. \tag{18}
$$

If there exist a scalar λ and vector $v \neq 0$ which satisfies

$$
\text{rank } P(\lambda)v = 0, \tag{19}
$$

then λ is called the invariant zero, and v is called corresponding invariant zero vector. If D has full column rank, then an invariant zero of $G(z)$ is given as an unobservable mode of $(D^{\perp}C,\ A - BD^{\dagger}C)$ (see Lemma 4.1 in [8]). In this subsection, it will be considered the case where D does not have full rank.

It is well known that

$$
\begin{bmatrix} zI_n - A & B \\ -C & 0 \end{bmatrix} \simeq \begin{bmatrix} I_n & 0 \\ 0 & N(z) \end{bmatrix} \tag{20}
$$

where $\simeq$ means the eqivalence under some elementary operations [9]. Therefore, the invariant zeros can be calculated by finding a scalar λ' and vector $v' \neq 0$ which satisfies

$$
N(\lambda')v' = 0. \tag{21}
$$

Now, There exists a unimodular matrix $U(z)$ such that

$$
U(z)N(z) = \begin{bmatrix} R(z) \\ 0 \end{bmatrix}, \tag{22}
$$

where $R(z)$ is nonsingular. The above $U(z)$ can be found by applying the regularizing polynomial matrix $L(z)$ for $N(z)$, and then finding the minimal annihilating matrix by using Lemma 3. That is, the problem is to find the zeros of $R(z)$.

Example 2 Consider the following system:

$$
A = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 1&0 \\ 0&0 \\ 0&0 \\ 0&1 \end{bmatrix}, \quad C = \begin{bmatrix} 1&0&0&0 \\ 1&-1&1&0 \\ 1&0&0&0 \end{bmatrix}.
$$

The transfer function matrix of the above system is given by

$$
G(z) = C(zI-A)^{-1}B = \begin{bmatrix} 1 & 0 \\ 1 & z-1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z^3 \end{bmatrix}^{-1}
$$

and the numertor polynomial matrix of the above $G(z)$ is row equivalent to

$$
\begin{bmatrix} 1 & 0 \\ 0 & z-1 \\ 0 & 0 \end{bmatrix}
$$

and thus $G(z)$ has the invariant zero at $z = 1$.

3.3 A Solution of General Diophantine Equation

The problem considered in this note is to find polynomial matrices $X_d(z) \in \boldsymbol{R}^{m\times p}[z]$ and $Y_d(s) \in \boldsymbol{R}^{m\times m}[z]$ satisfying

$$
\begin{bmatrix} X_d(z) & Y_d(z) \end{bmatrix} \begin{bmatrix} D(z) \\ N(z) \end{bmatrix} = H_d(z)F(z) \tag{23}
$$

for given polynomial matrices $D(z) \in \boldsymbol{R}^{m\times m}[z]$, $N(z) \in \boldsymbol{R}^{p\times m}[z]$ and $H_d(z)$, $F(z) \in \boldsymbol{R}^{m\times m}[z]$. The above equation is called Diophantine equation. In order to solve the above equation, it is assumed without loss of generality that $D(z)$ is nonsingular.

Let $\boldsymbol{X}_d$, $\boldsymbol{Y}_d$ and $\boldsymbol{H}_d$ denote the parameters satisfying

$$\begin{aligned}
X_d(z) &= X_0 + zX_1 + \cdots + z^{d-2}X_{d-2} \\
&= \begin{bmatrix} X_0 & X_1 & \dots & X_{d-2} \end{bmatrix} \begin{bmatrix} I_p \\ zI_p \\ \vdots \\ z^{d-2}I_p \end{bmatrix} \\
&:= \boldsymbol{X}_d S_{I_p}^{d-2}(z), \\
Y_d(z) &= Y_0 + zY_1 + \cdots + z^{d-1}Y_{d-1} \\
&= \begin{bmatrix} Y_0 & Y_1 & \dots & Y_{d-1} \end{bmatrix} \begin{bmatrix} I_m \\ zI_m \\ \vdots \\ z^{d-1}I_m \end{bmatrix} \\
&:= \boldsymbol{Y}_d S_{I_m}^{d-1}(z), \\
H_d(z) &= H_0 + zH_1 + \cdots + z^{d-1}H_{d-1} \\
&= \begin{bmatrix} H_0 & H_1 & \dots & H_{d-1} \end{bmatrix} \begin{bmatrix} I_m \\ zI_m \\ \vdots \\ z^{d-1}I_m \end{bmatrix} \\
&:= \boldsymbol{H}_d S_{I_m}^{d-1}(z). \qquad (24)
\end{aligned}$$

If it is also assumed that $\begin{bmatrix} F(z) \\ N(z) \end{bmatrix} D^{-1}(z)$ is strictly proper, then the following result can be obtained easily [7].

Lemma 4 Let $(A,\ B,\ C)$ and $(A,\ B,\ F)$ denote the minimal realization of $N(z)D^{-1}(z)$ and $F(z)D^{-1}(z)$ respectively. Then, there exist $X_d(z)$ and $Y_d(z)$ which satisfy
1) Diophantine equation (23).
2) $H_d^{-1}(z)X_d(z)$ is strictly proper.
3) $H_d^{-1}(z)Y_d(z)$ is proper.
if and only if the following relation holds:

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{X}_d & \boldsymbol{Y}_d \end{bmatrix} \begin{bmatrix} I_{m(d-1)} & 0 \\ \boldsymbol{T}_{d-2}(A,B,C) & \boldsymbol{O}_{d-1}(C,A) \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{H}_d\boldsymbol{T}_{d-2}(A,B,F) & \boldsymbol{H}_d\boldsymbol{O}_{d-1}(F,A) \end{bmatrix}. \qquad (25)
\end{aligned}$$

In order to see the above lemma, $D(z)$ should be column proper. If not, consider the regularizing matrix $L(z)$ for $D(z)$. Then, the controllability canonical realization $(\hat{A}, \hat{B}, \begin{bmatrix} \hat{C} \\ C_F \end{bmatrix})$ for $\begin{bmatrix} N(z) \\ F(z) \end{bmatrix} L(z)\{D(z)L(z)\}^{-1}$ can be obtain easily. Of course, it may not be minimal. Calculating the minimal bases $\begin{bmatrix} D^T(z) & N^T(z) & F^T(z) \end{bmatrix}^T$ like in eqn.(13), a minimal realization can be obtained.

4 Conclusions

In this note, a regularizing polynomial matrix for non-row or column proper polynomial matrix was proposed. A calculation method of the regularizing matrix was proposed. This method is easy and adequate for computer calculations. As applications, row/column properizing by elementary operations, a calculation of invariant zeros, and a solution of general Diophantine equation were considered. These are adequate for computer calculations again, since they carry out by using the state space representations. Numerical examples was presented to confirm the proposed method.

References

[1] W. A. Wolovich and P. L. Falb, Invariants and canonical forms under dynamic compensations, *SIAM J. Contr. and Optim.*, vol.**14**, no.6, pp.996-1008, 1976.

[2] W. A. Wolovich, *Linear multivariable systems*, Springer-Verlag, Berlin, 1974.

[3] Y. Mutoh and P.N. Nikiforuk, Suboptimal perfect model matching for a plant with measurement noise and its application to MRACS, *IEEE Trans. Automat. Contr.*, vol.**39**, pp.422-425, 1994.

[4] Y. Mutoh and R. Ortega, Interactor structure estimation for adaptive control of discrete-time multivariable nondecouplable systems, *Automatica*, vol.**29**, no.3, pp.635-647, 1993.

[5] W. Kase, A solution of polynomial matrix equations using extended division algorithm and another description of all-stabilizing controllers, *Int. J. Systems Science*, vol.**30**, pp.95-104, 1999.

[6] W. Kase, Y. Mutoh and M. Teranishi, A simple derivation of interactor matrix and its applications, *Proc. of the 38th IEEE Conf. on Dicision and Contr.*, pp.493-498, 1999.

[7] W. Kase, Y. Azuma and Y. Mutoh, A description of all stabilizing controllers and its applications, *Proc. of IFAC 15th Triennial World Congress*, 2002.

[8] H. Kimura, Y. Lu and R. Kawatani, On the structure of H^∞ control systems and related extensions, *IEEE Trans. Automat. Contr.*, vol.**36**, no.6, pp.653-667, 1991.

[9] T. Kailath, *Linear systems*, Prentice-Hall, 1980.