SOME REMARKS ON STATIC OUTPUT FEEDBACK STABILIZATION PROBLEM: NECESSARY CONDITIONS FOR MULTIPLE DELAY CONTROLLERS

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Abstract

This paper focuses on the static output feedback stabilization problem for a class of SISO systems in the case of multiple delay controllers. We are interested in giving *necessary conditions* for the existence of such stabilizing controllers. Illustrative examples (a chain of integrators, or a chain of oscillators) are presented and discussed.

1 Introduction

It is commonly known in the control literature that the existence of a delay in some control scheme may induce *instability* or *bad performances*. At the same time, there exist simple dynamical systems (second-order oscillators) for which a delay in the output feedback control law may induce a *stabilizing effect*, see, for example, [1, 8]. To the best of authors' knowledge, this second idea, *to induce delays in the feedback laws for control purposes*, was not sufficiently exploited in the literature. Some remarks in this sense can be found in [6], or more recently in [9] (multiple delay blocks to stabilize a chain of integrators).

Related to the remarks above, several questions arise in a natural manner. First, if one delay block is *sufficient* or *not* to stabilize a given system if the controller without delay does not ensure such a property. Second, if one delay block is *not sufficient*, then the use of multiple delay blocks may guarantee the *stability* of the closed-loop scheme or *not*. Some motivation examples (second-order integrator, and a chain including two oscillators) are presented below.

The aim of the paper is to discuss more in detail the static output feedback stabilization problem for SISO systems using an algebraic approach. More explicitly, *necessary* conditions for the existence of multiple delay controllers will be expressed in terms of Hurwitz stability of some polynomials associated with the stabilization problem. The key idea is a connection between the stability of some quasipolynomial and its corresponding derivatives.

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1.1 Motivation examples

In this subsection we present two examples which clarify some essential aspects of the problem under consideration.

Example 1 *We start with the second-order integrator*

$$\ddot{x}(t) = u(t), \qquad y(t) = x(t).$$

It is well known that the integrator can not be stabilized with a static output control of the form $u(t) = -kx(t)$*. Let us consider a control action of the form*

$$u(t) = -k_1 x(t-\tau_1) - k_2 x(t-\tau_2), \quad 0 \le \tau_1 < \tau_2,$$

then the closed loop characteristic function is

$$h(s) = s^2 + k_1 e^{-\tau_1 s} + k_2 e^{-\tau_2 s}.$$

For $k_1 = k_2 = 0$*, the function has the double root at* $s = 0$*. Let us try to select the control coefficients,* k_1*,* k_2*, such that this multiple root moves to the left half plane. For example, we can select the coefficients in such a way that*

$$\begin{cases} h(-\varepsilon) = \varepsilon^2 + k_1 e^{\tau_1 \varepsilon} + k_2 e^{\tau_2 \varepsilon} = 0 \\ h'(-\varepsilon) = -2\varepsilon - k_1 \tau_1 e^{\tau_1 \varepsilon} - k_2 \tau_2 e^{\tau_2 \varepsilon} = 0. \end{cases}$$

Then

$$k_1^{(0)} = -\varepsilon \frac{2 + \tau_1 \varepsilon}{e^{\tau_1 \varepsilon}(\tau_2 - \tau_1)}, \quad k_2^{(0)} = -\varepsilon \frac{2 - \tau_1 \varepsilon}{e^{\tau_2 \varepsilon}(\tau_2 - \tau_1)}.$$

For this choice of coefficients, $h(s)$ *has a double root at* $s = -\varepsilon$*. By direct calculation it is possible to verify that for sufficiently small* $\varepsilon > 0$*, the function*

$$h(s) = s^2 + k_1^{(0)} e^{-\tau_1 s} + k_2^{(0)} e^{-\tau_2 s}$$

has no roots in the closed right half complex plane, i.e., the control stabilizes the integrator.

Some useful observations: First, we need two delay terms in control in order to stabilize the double integrator. Second, there is no restriction for delays, except that they are distinct. Third, there is no control with one delay term which stabilizes the double integrator. Some comments in this sense, using different approaches, can be found in [7], [2].

Example 2 *Let us consider the oscillator*

$$\left[\frac{d^4}{dt^4} + (\omega_1^2+\omega_2^2)\frac{d^2}{dt^2} + \omega_1^2\omega_2^2\right] : x(t) = u(t), \quad y(t) = x(t),$$

when $0 < \omega_1 < \omega_2$*. Once again, there is no static output control* $u(t) = -ky(t)$ *which stabilizes the oscillator. Let us consider a delay control of the form* $u(t) = -ky(t-\tau)$*. The characteristic function of the closed loop system is*

$$h(s) = s^4 + (\omega_1^2+\omega_2^2)s^2 + \omega_1^2\omega_2^2 + ke^{-\tau s}.$$

For $k = 0$ *the function has two pairs of pure imaginary roots:* $\pm j\omega_1$, $\pm j\omega_2$*. One can ask: What happens with the roots when* k *moves from zero? To answer the question we consider two root functions,* $s_1(k)$*, and* $s_2(k)$*, such that* $s_i(0) = j\omega_i$, $i = 1,2$*. Then*

$$h(s_i(k)) \equiv 0, \quad i = 1,2.$$

Differentiating the identity with respect to k *at the point* $k = 0$*, we arrive at the following equalities:*

$$\left[-4j\omega_i^3 + 2j(\omega_1^2+\omega_2^2)\omega_i\right] s_i'(0) + e^{-j\tau\omega_i} = 0,$$

for $i = 1,2$*. Therefore,*

$$s_1'(0) = j\frac{e^{-j\tau\omega_1}}{2\omega_1(\omega_2^2-\omega_1^2)} = \frac{\sin(\tau\omega_1) + j\cos(\tau\omega_1)}{2\omega_1(\omega_2^2-\omega_1^2)},$$

and

$$s_2'(0) = -j\frac{e^{-j\tau\omega_2}}{2\omega_2(\omega_2^2-\omega_1^2)} = \frac{-\sin(\tau\omega_2) - j\cos(\tau\omega_2)}{2\omega_2(\omega_2^2-\omega_1^2)}.$$

If $\sin(\tau\omega_1) < 0$*, and* $\sin(\tau\omega_2) > 0$*, then*

$$\Re(s_1'(0)) < 0, \text{ and } \Re(s_2'(0)) < 0,$$

it means that these two roots move to the left as k *increases from zero. It is easy to check by direct calculations that for all sufficiently small* $k > 0$*, all roots of* $h(s)$ *lie in the open left half complex plane. It means that control with just one delay term may stabilize the oscillator.*

Useful observations: First, to stabilize the oscillators we need just one delay term in control. Second, there are restrictions on the choice of delay, it should be such that $\sin(\tau\omega_1) < 0$, and $\sin(\tau\omega_2) > 0$. Third, in the limit case, when $\omega_1 = \omega_2$, a control with one delay term can not stabilize the oscillator.

1.2 Paper organization

The paper is organized as follows: Section 2 includes some prerequisites. The main results are presented and discussed in Sections 3 and 4 for the case of a single delay block, and of multiple delay blocks, respectively. Applications of the theory will be presented in Section 5, including the stabilization of chains of integrators, second-order, and oscillatory systems, respectively. Concluding remarks end the paper. The notations are standard.

2 Prerequisites

In the sequel, we shall use the following result (for the proof, see [3], the full version of the paper):

Lemma 2.1 *Consider the quasipolynomial*

$$h(s) := \sum_{i=0}^{n}\sum_{l=1}^{r} h_{il}s^{n-i}e^{\tau_l s},$$

such that $\tau_1 < \tau_2 < \ldots \tau_r$*, with main term* $h_{0r} \neq 0$*, and* $\tau_1 + \tau_r > 0$*. If* $h(s)$ *is stable*[1]*, then* $h'(s)$ *is also a stable quasipolynomial.*

Remark 2.1 *This Lemma is a special case of a more general result on stability of entire functions given in [5].*

3 Necessary conditions: single delay block

Define now the following class of quasipolynomials:

$$h(s) := q(s) + ke^{-s\tau}, \tag{1}$$

where: $q(s) = s^n + q_1 s^{n-1} + \ldots + q_n$ is a given *unstable* polynomial of degree n, and (k,τ), is a given delay block. Note that (1) represents the closed-loop characteristic function of the transfer $H_{yu}(s) = \dfrac{1}{q(s)}$ subject to the controller: $u(t) = -ky(t-\tau)$.

We have the following result:

Proposition 3.1 *If there is no* $\tau \geq 0$*, such that* $\tau q(s) + q'(s)$ *is Hurwitz stable polynomial, then a delay block* (k,τ) *can not stabilize the transfer:*

$$H_{yu}(s) = \frac{1}{q(s)}.$$

Proof: Consider a delay block (k,τ) as a controller for the transfer H_{yu} given in the proposition, and assume that *there exists* such a *stabilizing pair*. The closed-loop characteristic function becomes: $h_1(s) = q(s) + ke^{-s\tau}$, and it should be stable. Note that the quasipolynomials $h_1(s)$ and $h_2(s) = h_1(s)e^{s\tau} =$

[1] The roots of the transcendental equation $h(s) = 0$ are located in $\mathbf{C}^-$.

$q(s)e^{s\tau} + k$ have the same roots, which implies also the *stability* of the quasipolynomial $h_2(s)$.
Apply now Lemma 2.1 to the quasipolynomial $h_2(s)$. It follows that the stability of $h_2(s)$ implies the stability of $h_2'(s)$, that is the stability of: $h_2'(s) := [q'(s) + \tau q(s)]\, e^{s\tau}$. In conclusion, a *necessary* condition for getting closed-loop stability using only one delay block is given by the existence of a *positive* τ such that the polynomial $\tau q(s) + q'(s)$ is Hurwitz stable. This ends the proof.

A natural consequence of the Proposition 3.1 is the following:

Corollary 3.2 *If one of the following statements holds:*
a) at least for one k coefficients $q_k, q_{k+1} \leq 0$, or
b) the polynomial $q(s)$ has at least one unstable root with the multiplicity ≥ 2, then the system:

$$H_{yu}(s) \quad := \quad \frac{1}{q(s)}$$

can never be stabilized by a controller including a single delay block.

The method above can be easily extended to a more general SISO system. In this sense, consider the following strictly proper transfer function:

$$H_{yu}(s) := \frac{p(s)}{q(s)}, \tag{2}$$

p and q coprime polynomials, with $deg(p) = m < n = deg(q)$ subject to the one delay block controller: $u(t) = -ky(t - \tau)$, and let us apply the same method but for the closed-loop characteristic function:

$$h_1(s) = q(s) + kp(s)e^{-s\tau}.$$

We assume that the transfer function $H_{yu}(s)$ is *unstable*.

We have the following result:

Proposition 3.3 *A necessary condition for a delay block (k, τ) to stabilize the transfer $H_{yu}(s)$ defined by (2) is given by the existence of $\tau > 0$ such that n-degree polynomial*

$$Q(s) = \left(\frac{d}{ds} + \tau\right)^{m+1} : q(s)$$

is Hurwitz stable.

Proof. Let quasipolynomial $h_1(s) = q(s) + kp(s)e^{-s\tau}$ be Hurwitz stable, then the following quasipolynomial $f_1(s) := q(s)e^{s\tau} + p(s)k$, obtained by multiplying $h_1(s)$ with $e^{s\tau}$, is also Hurwitz stable. By Lemma 2.1 all $m + 1$ first derivatives of $f_1(s)$ are also Hurwitz stable. Now,

$$\frac{d^{m+1}}{ds^{m+1}} f_1(s) = \frac{d^{m+1}}{ds^{m+1}} \left[q(s)e^{\tau s}\right] = Q(s)e^{\tau s}.$$

Therefore, the polynomial $Q(s)$ should be also Hurwitz stable.

As in the previous case, we have the following:

Corollary 3.4 *If one of the following statements holds:*
a) For at least one k the coefficients $q_k, q_{k+1}, \ldots q_{k+m+1} \leq 0$, or
b) The polynomial $q(s)$ has at least one unstable root with the multiplicity $\geq m + 2$, then the system:

$$H_{yu}(s) \quad := \quad \frac{p(s)}{q(s)},$$

with $deg(q(s)) = n > deg(p(s)) = m$ can never be stabilized by a controller including a single delay block.

4 Necessary conditions: multiple delay blocks

Consider the following SISO strictly proper transfer function:

$$H_{yu}(s) := \frac{p(s)}{q(s)}, \tag{3}$$

p and q coprime polynomials, with $deg(p) = m < n = deg(q)$, and assume also that the transfer function H_{yu} is *unstable*. Furthermore, assume that the conditions in Proposition 3.3 are not satisfied, which is equivalent to say that a *delay block* is *not sufficient* to control the transfer H_{yu} defined above.

A natural question arises: *If one delay block is not sufficient to stabilize, is it reasonable to consider a controller involving multiple delay blocks to stabilize the original system?* As seen in [9], the answer is positive in the case of a chain of integrators, but their conditions are only sufficient (the necessity was not discussed, and it was proposed as a *conjecture*). It seems clear that the procedure presented above can be easily extended to handle the case of multiple delay blocks, by *special elimination algorithm*, presented in the following subsection.

4.1 Elimination procedure

Introduce the following controller:

$$u(t) := -\sum_{i=1}^{l} k_i y(t - \tau_i),$$

including $l \geq 1$ distinct delay blocks (k_i, τ_i) ($i = 1, \ldots, l$), with $0 \leq \tau_1 < \ldots < \tau_l$. Simple computations lead to the following closed-loop characteristic function

$$h_0(s) = q(s) + p(s) \sum_{i=1}^{l} k_i e^{-s\tau_i}. \tag{4}$$

As specified in the Introduction, we are interested in finding necessary conditions on delay blocks (k_i, τ_i), $i = 1, \ldots, l$, such that the closed-loop system (4) is asymptotically stable.

Assume that $h_0(s)$ is Hurwitz stable, then quasipolynomial

$$f_0(s) \quad = \quad e^{s\tau_l} h_0(s) = q(s)e^{s\tau_l} + \sum_{i=1}^{l-1} k_i p(s) e^{s(\tau_l - \tau_i)} + k_l p(s),$$

is also Hurwitz stable. By Lemma 2.1 it implies the stability of the first $(m+1)$ derivatives of the quasipolynomial. So, quasipolynomial

$$h_1(s) = e^{-\tau_l s} \frac{d^{m+1}}{ds^{m+1}} [f_0(s)] = q_1(s) + \sum_{i=1}^{l-1} k_i p_{1i}(s) e^{-\tau_i s},$$

is also Hurwitz stable. Here

$$q_1(s) = \left(\frac{d}{ds} + \tau_l \right)^{m+1} : q(s)$$

is a n-degree polynomial, while

$$p_{1i}(s) = \left(\frac{d}{ds} + \tau_l - \tau_i \right)^{m+1} : p(s), \ i = 1, ..., l-1,$$

are m-degree polynomials. Quasipolynomial $h_1(s)$ includes only $(l-1)$ delay blocks. This procedure allows eliminate one by one all delay blocks.

4.2 Basic lemma

An appropriate application of the elimination procedure above leads to the following lemma (see [3] for the proof):

Lemma 4.1 *The Hurwitz stability of $h_0(s)$ implies the stability of n-degree polynomial*

$$Q(s) = \left(\frac{d}{ds} + \tau_1 \right)^{m+1} \left(\frac{d}{ds} + \tau_2 \right)^{m+1} ... \left(\frac{d}{ds} + \tau_l \right)^{m+1} : q(s).$$

Furthermore, consider the following polynomial

$$\begin{aligned} \varphi(z) &= (z+\tau_1)^{m+1} (z+\tau_2)^{m+1} ... (z+\tau_l)^{m+1} \\ &= \alpha_0 + \alpha_1 z + ... + \alpha_N z^N, \end{aligned}$$

where $N = (m+1)l$. Then:

$$Q(s) = \sum_{\nu=0}^{\min\{n,N\}} \alpha_\nu q^{(\nu)}(s).$$

From the lemma above, we have the following result:

Proposition 4.1 *A controller involving l delay blocks cannot stabilize system (3) if delays $\tau_1, \tau_2, ..., \tau_l$ are such that polynomial*

$$\begin{aligned} Q(s) &= \left(\frac{d}{ds} + \tau_1 \right)^{m+1} \left(\frac{d}{ds} + \tau_2 \right)^{m+1} ... \left(\frac{d}{ds} + \tau_l \right)^{m+1} : q(s) \\ &= \sum_{\nu=0}^{\min\{n,N\}} \alpha_\nu q^{(\nu)}(s). \end{aligned}$$

is not Hurwitz stable.

Corollary 4.2 *Let $N < n$, then if one of the following statements holds:*
a) For at least one k the coefficients $q_k, q_{k+1}, \ldots, q_{k+N} \leq 0$,
or
b) The polynomials $q(s)$ has at least one unstable root with the multiplicity $\geq N+1$, then the system:

$$H_{yu}(s) := \frac{p(s)}{q(s)}$$

can never be stabilized by a controller including l delay blocks.

5 Examples

In the sequel, we shall consider various simple illustrative examples: optimizing the number of delay blocks for stabilizing a class of second-order systems, a chain of integrators [9], or for stabilizing various (linear) oscillatory systems in function of the number of oscillating modes.

5.1 Second-order systems

We consider the stabilizability of the second-order system

$$H_{yu}(s) = \frac{1}{q(s)} := \frac{1}{s^2 + a_1 s + a_2} \tag{5}$$

as a function of its parameters a_1 and a_2.

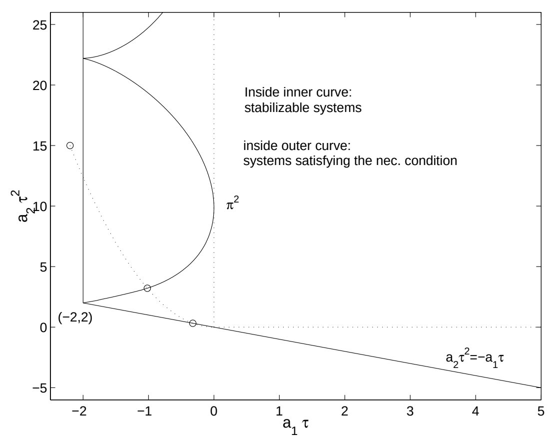

Figure 1: Values of (a_1, a_2) for which a stabilizing controller of the form $u = -ky(t-\tau)$ exists and for which the condition (6) is satisfied. Here τ is fixed and k is the controller parameter. When the delay is also considered as a controller parameter, the stabilizability can be determine as follows: given any (a_1, a_2) the curve $\tau \to (a_1\tau, a_2\tau^2)$ is a half parabola (indicated with dotted line). Iff this curve intersects the stability region, the system is stabilizable (for those delay values corresponding to points on the curve inside the stability region).

With a controller consisting of one delay block, $u = -ky(t-\tau)$, the necessary stability condition following from Lemma 2.1 and Proposition 3.1 reads as

$$\tau q(s) + q'(s) \text{ stable } \leftrightarrow \begin{cases} a_1\tau > -2, \\ a_2\tau^2 > -a_1\tau. \end{cases} \tag{6}$$

For *fixed* τ one can deduce from Figure 1 the values of (a_1, a_2) where the condition (6) is satisfied and also the exact region where stabilization is possible (meaning that a stabilizing value of k exists).

An important observation is that the values of (a_1, a_2) where (6) is satisfied for *some* positive value of τ, namely

$$\{a_1 > 0\} \cup \{a_2 > a_1^2/2\},$$

correspond to the *exact* stabilizability conditions when τ is also a controller parameter and, hence, for the system under consideration, Proposition 3.1 yields *necessary and sufficient* stabilizability conditions. Conservatism only lies in the fact the set of feasible delay values allowed by (6) is generally too large, as follows from Figure 1.

Using two delay blocks in the controller, $u(t) = -k_1\, y(t - \tau_1) - k_2\, y(t - \tau_2)$, the system (5) is stabilizable for any a_1 and a_2.

5.2 Chain of integrators

We shall completely characterize the stabilizability of the n-th order integrator using delayed output feedback. In this sense, we shall use the following result (see [3] for the proof):

Lemma 5.1 *Let $p(s)$ be a Hurwitz stable polynomial of degree n, then for every $\tau > 0$, $\mu > 0$ and $\gamma > 0$ polynomial*

$$q(s) = (\tau s + \mu)p(s) + \gamma s p'(s)$$

is also Hurwitz stable.

In [9], the authors proposed a *conjecture* concerning the minimal number of delay blocks *necessary* to stabilize a chain of integrators. Based on the results above, we can give positive answer to this conjecture. More explicitly, we can prove that:

Proposition 5.1 *A chain of n integrators ($n \geq 2$) can neither be stabilized with a chain of less than n delay blocks, nor with a proportional+delay compensator with less than $(n-1)$ delays.*

Proof: Despite the fact that this statement can be deduced directly from Corollary 4.3, which states that a controller including $(n-1)$ delay blocks is not sufficient to stabilize $H_{yu}(s) = 1/q(s)$, where $q(s) = s^n$, since $q_n = q_{n-1} = \dots q_1 = 0$, we supply the statement by an independent proof.

First, we observe that the closed loop characteristic function of the n-th order integrator with a static output controller with m delay terms ($m < n$) is

$$h_1(s) = s^n + \sum_{i=1}^{m} k_i e^{-\tau_i s}, \text{ here } 0 \leq \tau_1 < \tau_2 < \dots < \tau_m.$$

The zero set of $h(s)$ coincides with that of

$$f_1(s) = e^{\tau_m s} h_1(s) = e^{\tau_m s} s^n + \sum_{i=1}^{m-1} k_i e^{(\tau_m - \tau_i)s} + k_m.$$

If $f_1(s)$ is stable then, by Lemma 2.1,

$$\frac{df_1(s)}{ds} = (\tau_m s + n)s^{n-1} e^{\tau_m s} + \sum_{i=1}^{m-1} k_i(\tau_m - \tau_i) e^{(\tau_m - \tau_i)s}$$

should also be stable. It means that

$$h_2(s) = e^{-\tau_m s}\frac{df_1(s)}{ds} = p_1(s)s^{n-1} + \sum_{i=1}^{m-1} k_i(\tau_m - \tau_i)e^{-\tau_i s}$$

is also stable. In the last expression, polynomial $p_1(s) = \tau_m s + n$ is Hurwitz. Stability of $h_2(s)$ implies that of $f_2(s) = e^{\tau_{m-1} s} h_2(s)$, and by Lemma 2.1, stability of

$$\frac{df_2(s)}{ds} = \left[(\tau_{m-1}s + n - 1)\, p_1(s + s p_1'(s)\right] s^{n-2} e^{\tau_{m-1} s} + \sum_{i=1}^{m-2} k_i(\tau_m - \tau_i)(\tau_{m-1} - \tau_i)e^{(\tau_{m-1} - \tau_i)s}.$$

So, function $h_3(s) = e^{\tau_{m-1} s}\frac{df_2(s)}{ds}$ becomes:

$$h_3(s) = p_2(s)s^{n-2} + \sum_{i=1}^{m-2} k_i(\tau_m - \tau_i)(\tau_{m-1} - \tau_i)e^{-\tau_i s}$$

is also stable. Polynomial $p_2(s) = (\tau_{m-1}s + n - 1)\, p_1(s) + s p_1'(s)$ is Hurwitz by Lemma 5.1. Repeating this procedure m times we arrive at the conclusion that polynomial

$$h_{m+1}(s) = e^{\tau_1 s}\frac{df_m(s)}{ds} = p_m(s)s^{n-m}$$

should be Hurwitz stable. But it is not true because $m < n$ and the $h_{m+1}(s)$ still has a root at $s = 0$.

Remark 5.2 *In [9], where this result was also mentioned as a conjecture, it was shown that n delay blocks, or a proportional+delay controller with $n-1$ delays are* sufficient*. Two different approaches were presented. Both are constructive and rely on frequency-domain techniques: on a derivative feedback approximation idea, and a pole placement idea, respectively.*

5.3 Stabilizing oscillatory systems

We start with the observation that, given frequencies $0 < \omega_1 < \omega_2 < \dots < \omega_n$, the chain of n oscillators

$$\prod_{\nu=1}^{n}\left(\frac{d^2}{dt^2} + \omega_\nu^2\right) : x(t) = u(t), \quad y(t) = x(t), \tag{7}$$

with control $u(t) = -ky(t - \tau)$, has the closed loop characteristic function

$$h(s) = \prod_{\nu=1}^{n}\left(s^2 + \omega_\nu^2\right) + ke^{-\tau s} := q(s) + ke^{-s\tau}. \tag{8}$$

Then the *necessary* stability condition of Proposition 3.1, expressed by the Hurwitz stability of

$$q'(s) + \tau q(s) \tag{9}$$

for some $\tau > 0$, is always satisfied. To see this, examine the root locus of the polynomial

$$\mu q'(s) + q(s) \tag{10}$$

as a function of the parameter $\mu = 1/\tau$.

Define the n root functions $s_\nu(\mu)$ such that $s_\nu(0) = j\omega_\nu,\ \nu = 1, \ldots, n$. A simple calculation yields that $s'_\nu(0) = -1$ and therefore, the polynomial (10) is stable for sufficiently small values of μ. This implies the stability of (9) for large values of τ.

A *sufficient* stabilizability condition is provided in the following proposition:

Proposition 5.3 *Given a set of frequencies* $0 < \omega_1 < \omega_2 < \ldots < \omega_n$. *If there exists* $\tau > 0$ *such that*

$$(-1)^\nu \sin(\tau\omega_\nu) > 0, \text{ for } \nu = 1, 2, ..., n, \tag{11}$$

then for sufficiently small $k > 0$, *all roots of (8) have negative real part.*

Proof: In order to check the statement we introduce n root functions, $s_\nu(k)$, $\nu = 1, 2, ..., n$, of (8) such that $s_\nu(0) = j\omega_\nu$, $\nu = 1, 2, ..., n$. Substitute $s_\nu(k)$ in (8) we obtain the identity

$$\prod_{i=1}^{n} \left(s_\nu^2(k) + \omega_i^2\right) + ke^{-\tau s_\nu(k)} \equiv 0.$$

After differentiation of the identity with respect to k, and substitution $k = 0$, we arrive at the equality

$$\left[2j\omega_\nu \prod_{i\neq\nu} \left(-\omega_\nu^2 + \omega_i^2\right)\right] s'_\nu(0) + e^{-j\tau\omega_\nu} = 0.$$

Some simple computations lead to:

$$s'_\nu(0) \quad = \quad \frac{\sin(\tau\omega_\nu) + j\cos(\tau\omega_\nu)}{2\omega_\nu \prod_{i\neq\nu}\left(-\omega_\nu^2 + \omega_i^2\right)}. \tag{12}$$

Taking in mind the fact that $0 < \omega_1 < \omega_2 < \ldots < \omega_n$, we can conclude that

$$\text{sign}\left\{\Re(s'_\nu(0))\right\} = (-1)^{\nu-1}\text{sign}\left\{\sin(\tau\omega_\nu)\right\}.$$

Condition (11) implies that

$$\Re(s'_\nu(0)) < 0, \text{ for all } \nu = 1, 2, ..., n.$$

This observation proves that all n roots of $h(s)$ move from the imaginary axis to the left as k increases from zero. For sufficiently small positive k all other roots of $h(k)$ remains in the left half plane of the complex plane.

Remark 5.4 *For the case where not all the frequencies* ω_ν *in (7) are different, a straightforward application of Corollary 4.3 yields: When the uncontrolled system (7) has an imaginary eigenvalue with multiplicity larger than* l, *it can not be stabilized with an output feedback controller with only* l *delay terms. For example, if in (7)* $\omega_1 = \omega_2 = \ldots = \omega_n$, *then we need not less than* n *delay blocks to stabilize the system.*

6 Concluding remarks

This paper addressed the static output feedback stabilization problem if multiple delays are used in defining the control law. Several necessary conditions are derived, using some properties of the derivative of a stable quasipolynomial. Various examples (second-order SISO systems, chain of integrators, or oscillators) are considered, and largely discussed.

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