# ROBUST PREDICTIVE CONTROL FOR LINEAR SYSTEMS SUBJECT TO NORM-BOUNDED MODEL UNCERTAINTY

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## Abstract

A novel robust predictive control algorithm is presented for input-saturated uncertain discrete-time linear systems described by structured norm-bounded uncertainties. The solution is based on the minimization, at each time instant, of an LMI convex optimization problem obtained by a recursive use of the S-procedure. Stability and feasibility are proved and comparisons with robust multi-model (polytopic) MPC algorithms are reported.

## 1 Introduction

Model predictive control (MPC) has become an attractive feedback strategy for systems subject to input and state/output inequality constraints [1]. More recently, a notable amount of research has been devoted to extending the basic nominal MPC strategies to uncertain linear systems. Traditionally, research on robust minmax MPC control has mainly focused on polytopic or multi-model uncertain linear systems [2, 3]. The reason is that the uncertain polytopic paradigm is well suited to be used within predictive control strategies because the propagation of the effects of the uncertainty over the control horizon is not usually conservative, especially if formulated in closed-loop fashion [4, 1], and tight convex sets of state predictions can easily be formulated via LMI constraints [2]. However, their large computational burdens still prevent their use in practical problems. Efforts at removing or ameliorating this situation have been recently undertaken e.g. in [5], where the idea was to move as much computational burden as possible off-line.In this paper, instead, we propose a general $N$ free moves robust MPC strategy for uncertain *norm-bounded* (NB) linear systems [6]. On this subject, fewer contributions have appeared in the MPC literature. Kothare et al. [2] gave the first constructive solution for the case $N=0$. More recently, in [7] a robustness analysis tool for optimization-based control strategies has been proposed, postulating the existence of robust MPC schemes for NB uncertainty. The proposed method is based on the minimization, at each time step, of an upper bound of the worst-case infinite horizon quadratic cost under LMI constraints derived off-line by a recursive use of the S-procedure [8]. Unlike the polytopic uncertain description, it is found here that the number of LMIs grows only *linearly* with the control horizon $N$. A similar approach has been developed by the authors in [9] but the novelty here is twofold: first, the conditions over the upper bound to the quadratic cost and the input constraints are derived in a more compact form; then, by means of the Choleski factorization, numerical pitfalls are avoided. For the sake of completeness, in order to cover a lack in the numerical simulations of [9], extensive statistical tests have been performed in the final example in terms of both control performance and computational burdens.

## 2 Problem Formulation

Consider the following discrete-time linear system with uncertainties or perturbations appearing in the feedback loop

$$\left\{ \begin{array}{lcl} x(t+1) & = & \Phi x(t) + G u(t) + B_p p(t) \\ y(t) & = & C x(t) \\ q(t) & = & C_q x(t) + D_q u(t) \\ p(t) & = & (\Delta q)(t) \end{array} \right. \tag{1}$$

with $x \in \mathbb{R}^{n_x}$ denoting the state, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_u}$ the output, $p, q \in \mathbb{R}^{n_p}$ additional variables accounting for the uncertainty. For a more extensive discussion about this type of uncertainty see Boyd et al. [6]. It is further assumed that the plant input is subject to the following ellipsoidal constraint

$$u(t) \in \Omega_u, \quad \Omega_u \triangleq \{u \in \mathbb{R}^{n_u} \;:\; u^T Q_u u \leq \bar{u}^2\}, \tag{2}$$

with $Q_u = Q_u^T > 0$ and $\bar{u} > 0$. The aim is to find a state-feedback regulation, $u(t) = g(x(t))$, which possibly asymptotically stabilizes (1) subject to (2). We recall now some properties on quadratic stabilizability which are relevant for our subsequent developments. The family of systems (1) is said to be robust quadratically stabilizable if there exists a constant state-feedback control law $u = Kx$ such that all the closed loop trajectories asymptotically converge to zero. In [6] it has been shown that a linear state-feedback law is able to quadratically stabilize an uncertain system of the form (1) if there exists a matrix $P = P^T > 0$ and a scalar $\lambda > 0$ such that the following linear matrix inequality is satisfied

$$\left[ \begin{array}{cc} \Phi_K^T P \Phi_K - P + K^T R_u K + R_x + \lambda C_K^T C_K & \Phi_K^T P B_p \\ B_p^T P \Phi_K & B_p^T P B_p - \lambda I_{n_x} \end{array} \right] \leq 0 \tag{3}$$

where $\Phi_K \triangleq \Phi + GK$, $C_K \triangleq C_q + D_q K$ and $\lambda \in \mathbb{R}$, and $R_x \geq 0$, $R_u > 0$ are given symmetric matrices used in (4). Accordingly, the following sets $S_t \triangleq \left\{ p \mid \|p\|_2^2 \leq \|C_K x(t)\|_2^2 \right\}$, representing the plant uncertainty domains at each time instant $t$ can be determined and a bound on the following quadratic performance index

$$J(x(0), u(\cdot)) \triangleq \max_{p(k) \in S_k} \sum_{k=0}^{\infty} \left\{ \|x(k)\|_{R_x}^2 + \|Kx(k)\|_{R_u}^2 \right\}, \tag{4}$$

results to be given by $J(x(0),Kx(\cdot)) \leq x(0)^T P x(0)$. Moreover, the following ellipsoidal set

$$C(P,\rho) \triangleq \left\{x \in \mathbb{R}^n \,|\, x^T P x \leq \rho\right\} \tag{5}$$

can be proved to be a robustly positive invariant region for the state evolution of the closed-loop system, viz. $x(0) \in C(P,\rho)$ implies that $\Phi_K^t x(0) \in C(P,\rho)$ for all $t$. When $N=0$, given the input constraint (2), the problem can be attacked as in Kothare et al. [2] by solving the following LMI optimization problem in the unknowns $\rho, Q, Y$ and $\lambda$:

$$\min_{Q,Y,\rho,\lambda} \rho \tag{6}$$

subject to

$$\left[\begin{array}{cc} 1 & x(t)^T \\ x(t) & Q \end{array}\right] \geq 0\,, \tag{7}$$

$$\left[\begin{array}{ccccc} Q & (*)^T & (*)^T & (*)^T & (*)^T \\ R_u^{1/2}Y & \rho I_{n_u} & 0 & 0 & 0 \\ R_x^{1/2}Q & 0 & \rho I_{n_x} & 0 & 0 \\ C_q Q + D_{qu} Y & 0 & 0 & \lambda I_{n_x} & 0 \\ \Phi Q + GY & 0 & 0 & 0 & Q - \lambda B_p B_p^T \end{array}\right] \geq 0\,, \tag{8}$$

$$\lambda > 0\,, \tag{9}$$

$$\left[\begin{array}{cc} \bar{u}^2 I & Q_u^{\frac{1}{2}} Y \\ Y^T Q_u^{\frac{1}{2}T} & Q \end{array}\right] \geq 0\,, \quad \begin{array}{l} Q_u^{\frac{1}{2}T} Q_u^{\frac{1}{2}} = Q_u \quad P = \rho Q^{-1} \\ K = Y Q^{-1} \end{array} \tag{10}$$

In order to improve the usually modest control performance pertaining to the above receding horizon quadratically stabilizing controllers, in [3] it has been proposed to introduce $N$ additional input free moves over which the optimization takes place. A further idea for improving performance was recently proposed [9] and consists of adopting during predictions the following family of virtual commands

$$\hat{u}(\cdot|t) = \left\{ \begin{array}{ll} K\hat{x}(t+k|t) + c(t+k|t), & k = 0,1,\ldots,N-1, \\ K\hat{x}(t+k|t) & k \geq N, \end{array}\right. \tag{11}$$

where $c(\cdot|t)$ denotes $N$ free perturbations over the action of an admissible controller $K$ and

$$\hat{x}(t+k|t) \triangleq \left(\prod_0^k \Phi_K\right) x(t) + \sum_{i=0}^{k-1}\left(\prod_0^{k-1-i}\Phi_K\right)(Gc(t+i|t) + B_p p(t+i|t)) \tag{12}$$

represents set-valued state predictions, computed under the condition

$$p(t+k|t) \in S_{t+k|t} \triangleq \{p \,:\, \|p\|_2^2 \leq \|C_K \hat{x}(t+k|t) + D_q c(t+k|t)\|_2^2\},$$
$$k = 0,1,\ldots,N-1,$$

with $S_{t+k|t}$ characterizing all admissible perturbations along the system trajectories corresponding to the virtual command sequences (11). The free perturbations $c(\cdot|t)$ are computed by minimizing the following minmax quadratic index

$$\begin{aligned} V(x(t),P,c(\cdot|t)) \;\; \triangleq \;\; & \sum_{k=0}^{N-1}\left(\max_{p(k|t)\in S_{t+k|t}} \|\hat{x}(t+k|t)\|_{R_x}^2 + \|c(t+k|t)\|_{R_u}^2\right) + \\ & \max_{p(t+N|t)\in S_{t+N|t}} \|\hat{x}(t+N|t)\|_P^2, \end{aligned} \tag{13}$$

$R_x \geq 0$, $R_u \geq 0$ are symmetric state and input weighting matrices and $P \geq 0$. Specifically, at each time instant $t$ the problem to be solved on-line consists of computing

$$c^*(\cdot|t) \triangleq \arg\min_{c(\cdot|t)} V(x(t),P,c(\cdot|t)) \tag{14}$$

subject to

$$K\hat{x}(t+k|t) + c(t+k|t) \in \Omega_u, \;\; k = 0,1,...,N-1 \tag{15}$$

$$\hat{x}(t+N|t) \in C(P,\rho) \subset \Omega_u \tag{16}$$

where $C(P,\rho)$ is a robust invariant set under $K \triangleq YQ^{-1}$ with $(P,Q,\rho)$ solution of the LMI conditions (7)-(10). In (14), the constraints (15) are used to enforce input constraints during prediction whereas the constraint (16), hereafter referred to as terminal constrain, is used to ensure closed-loop stability.

## 3 LMI formulation of cost upper-bounds, input and terminal constraints

In this section we determine a suitable upper-bound to the quadratic cost in terms of LMI conditions. A convenient upper-bound to the cost (13) can be achieved by introducing non-negative reals $J_0,\ldots,J_{N-1}$ such that, for arbitrary $P$, $K$ and $c_k \triangleq c(t+k|t)$, $p_k \triangleq p(t+k|t)$, $k=0,..,N-1$, the following inequalities hold true

$$\max_{p_0\in S_0} \hat{x}_1^T R_x \hat{x}_1 + c_0^T R_u c_0 \;\; \leq \;\; J_0 \tag{17}$$

$$\max_{\substack{p_i\in S_i \\ i=0,\ldots,k}} \hat{x}_{k+1}^T R_x \hat{x}_{k+1} + c_k^T R_u c_k \;\; \leq \;\; J_k, \qquad k = 1,\ldots,N-2 \tag{18}$$

$$\max_{\substack{p_i\in S_i \\ i=0,\ldots,N-1}} \hat{x}_N^T P \hat{x}_N + c_{N-1}^T R_u c_{N-1} \;\; \leq \;\; J_{N-1} \tag{19}$$

Then, it results that

$$V(x(t),P,c(\cdot|t)) \leq x_0^T R_x x_0 + \sum_{k=0}^{N-1} J_k \tag{20}$$

In particular, we are interested in determining LMI conditions relating any arbitrary triplet $(x_0,c_k,K)$, $k=0,..,N-1$ to the class of all $J_0,..,J_{N-1}$ that satisfy (17)-(19). This will be accomplished by directly exploiting standard $S$-procedure arguments [8]. Let's consider first (17) for a generic triplet $(x_0,c_0,J_0)$. By recalling that

$$\hat{x}_1 = \Phi_K x_0 + Gc_0 + B_p p_0, \quad \forall p_0 \in S_0 \tag{21}$$

one has that (17) is satisfied provided that

$$(\Phi_K x_0 + Gc_0 + B_p p_0)^T R_x (\Phi_K x_0 + Gc_0 + B_p p_0) + c_0^T R_u c_0 \leq J_0 \tag{22}$$

for all $p_0$ such that

$$p_0^T p_0 \leq (C_K x_0 + D_q c_0)^T (C_K x_0 + D_q c_0)\,. \tag{23}$$

Conditions (22) and (23) can be rearranged respectively as

$$-p_0^T B_p^T R_x B_p p_0 - 2[x_0^T \; c_0^T] D_0^T p_0 + J_0 - [x_0^T \; c_0^T] E_0 \left[\begin{array}{c} x_0 \\ c_0 \end{array}\right] \geq 0 \tag{24}$$

$$-p_0^T p_0 + [x_0^T \; c_0^T] F_0 \left[\begin{array}{c} x_0 \\ c_0 \end{array}\right] \geq 0 \tag{25}$$

where $D_0^T$, $E_0 = E_0^T \geq 0$ and $F_0 = F_0^T \geq 0$ are matrices of appropriate dimensions defined by

$$D_0^T \triangleq \begin{bmatrix} \Phi_K^T \\ G^T \end{bmatrix} R_x B_p, \quad E_0 \triangleq \begin{bmatrix} \Phi_K^T R_x \Phi_K & \Phi_K^T R_x G \\ * & G^T R_x G + R_u \end{bmatrix},$$
$$F_0 \triangleq \begin{bmatrix} C_K^T \\ D_q^T \end{bmatrix} [C_K \;\; D_q] \qquad . \tag{26}$$

Then, the implication:

$$(22) \text{ holds true for all } p_0 \text{ satisfying } (23) \tag{27}$$

can be shown, via the $S$-procedure [8], to be true iff there exists a scalar coefficient $\tau_0 \geq 0$ such that the following matrix

$$\begin{bmatrix} -B_p^T R_x B_p + \tau_0 I & -D_0 \begin{bmatrix} x_0 \\ c_0 \end{bmatrix} \\ * & J_0 - [x_0^T \;\; c_0^T]\,(E_0 + \tau_0 F_0) \begin{bmatrix} x_0 \\ c_0 \end{bmatrix} \end{bmatrix} \geq 0 \tag{28}$$

is semidefinite positive for the triplet $(x_0, c_0, J_0)$. By Schur complements, semidefinitedness of (28) is equivalent to both

$$-B_p^T R_x B_p + \tau_0 I > 0 \tag{29}$$

$$J_0 - [x_0^T c_0^T]\,(E_0 + \tau_0 F_0) \begin{bmatrix} x_0 \\ c_0 \end{bmatrix} -$$
$$[x_0^T c_0^T] D_0^T \left(-B_p^T R_x B_p + \tau_0 I\right)^{-1} D_0 \begin{bmatrix} x_0 \\ c_0 \end{bmatrix} \geq 0 \tag{30}$$

be true. Condition (29) can be satisfied independently by the triplet $(x_0, c_0, J_0)$ by selecting a sufficiently large $\tau_0$. Under (29), (30) characterizes the class of all admissible triplets $(x_0, c_0, J_0)$ for a given $\tau_0$. In order to enlarge this class, a convenient choice is

$$\begin{aligned} \hat{\tau}_0 \triangleq \arg\min_{\tau_0 \geq 0} \quad & \bar{\lambda}\left(E_0 + \tau_0 F_0 + D_0^T \left(-B_p^T R_x B_p + \tau_0 I\right)^{-1} D_0\right) \\ & \text{subject to} \\ & -B_p^T R_x B_p + \tau_0 I > 0 \end{aligned} \tag{31}$$

where $\bar{\lambda}$ denotes the largest eigenvalues. Finally, by performing a Cholesky factorization

$$L_0^T L_0 = E_0 + \hat{\tau} F_0 + D_0^T \left(-B_p^T R_x B_p + \tau_0 I\right)^{-1} D_0 \tag{32}$$

(see [10] if the matrix is only semi-definite positive), one can equivalently rearrange condition (28) as the following LMI condition

$$\Sigma_0 \triangleq \begin{bmatrix} J_0 & -[x_0^T \; c_0^T] L_0^T \\ * & I \end{bmatrix} \geq 0 \tag{33}$$

which is linear in the terms $x_0$, $c_0$ and $J_0$. Then, for a given $x_0$, the LMI condition $\Sigma_0 \geq 0$ can be used to characterize all $J_0$ that satisfy the implication (27) for any $c_0$, Moreover, one can minimize the upper-bound to the cost by selecting the minimum $J_0$ which satisfies $\Sigma_0 \geq 0$. The same procedures can be repeated for conditions (18) and (19). Specifically, consider (18) for the generic $k = 1, ...., N-2$. Define vectors

$$\underline{c}_k \triangleq [c_0^T \; c_1^T \; \cdots \; c_k^T]^T \in \mathbb{R}^{(k+1)n_u}, \quad \underline{p}_k \triangleq [p_0^T \; p_1^T \; \cdots \; p_k^T]^T \in \mathbb{R}^{(k+1)n_p} \tag{34}$$

and matrices

$$\bar{\Phi}_k \triangleq \Phi_K^k \in \mathbb{R}^{n_x \times n_x}, \quad \tilde{\Phi}_k \triangleq [\Phi_K^k \;\; \Phi_K^{k-1} \cdots \Phi_K \; I] \in \mathbb{R}^{n_x \times (k+1)n_x}. \tag{35}$$

Then, the set of all k-steps ahead state predictions can be reformulated as

$$\hat{x}_{k+1} = \bar{\Phi}_k x_0 + \tilde{\Phi}_k G \underline{c}_k + \tilde{\Phi}_k B_p \underline{p}_k, \;\; \forall p_i \in S_i, \;\; i = 0, ..., k \tag{36}$$

and the condition (18) rearranged as

$$\hat{x}_{k+1}^T \hat{R}_x \hat{x}_{k+1} + c_k^T R_u c_k \leq J_k \tag{37}$$

for all $p_i$, $i = 0, ..., k$ such that

$$p_i^T p_i \leq \left(C_K \hat{x}_i + D_q c_i\right)^T \left(C_K \hat{x}_i + D_q c_i\right). \tag{38}$$

Again, the above two conditions (37) and (38) can be rewritten respectively as

$$-\underline{p}_k^T B_p^T \bar{\Phi}_k^T R_x \bar{\Phi}_k B_p \underline{p}_k - 2[x_0^T \; \underline{c}_k^T] D_k^T \underline{p}_k + J_k - [x_0^T \; \underline{c}_k^T] E_k \begin{bmatrix} x_0 \\ \underline{c}_k \end{bmatrix} \geq 0 \tag{39}$$

$$-\underline{p}_k^T \tilde{G}_i \underline{p}_k + 2[x_0^T \; \underline{c}_k^T] \tilde{H}_i^T \underline{p}_k + [x_0^T \; \underline{c}_k^T] \tilde{F}_i \begin{bmatrix} x_0 \\ \underline{c}_k \end{bmatrix} \geq 0, \quad i = 0, ..., k \tag{40}$$

where $D_k^T$, $E_k = E_k^T \geq 0$ are matrices of appropriate dimensions defined by

$$D_k^T \triangleq \begin{bmatrix} \bar{\Phi}_k^T \\ G^T \tilde{\Phi}_k^T \end{bmatrix} R_x \tilde{\Phi}_k^T B_p,$$
$$E_k \triangleq \begin{bmatrix} \bar{\Phi}_k^T R_x \bar{\Phi}_k & \bar{\Phi}_k^T R_x \tilde{\Phi}_k G \\ * & G^T \tilde{\Phi}_k^T R_x \tilde{\Phi}_k G + \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix} \end{bmatrix} \tag{41}$$

where the square matrix $R_u$ is added to the last $n_u$ rows and columns of the sub-matrix $G^T \tilde{\Phi}_k^T R_x \tilde{\Phi}_k G$ of $E_k$, while $\tilde{H}_i$, $\tilde{G}_i = \tilde{G}_i^T$ and $\tilde{F}_i = \tilde{F}_i^T \geq 0$ are given by

$$\tilde{H}_i^T \triangleq \begin{bmatrix} H_i^T & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{G}_i \triangleq \begin{bmatrix} G_i & 0 \\ * & 0 \end{bmatrix}, \; \tilde{F}_i \triangleq \begin{bmatrix} F_i & 0 \\ * & 0 \end{bmatrix} \tag{42}$$

with

$$H_0^T \triangleq 0_{(n_x+n_u) \times n_p}, G_0 \triangleq -I_{n_p} \;\text{ and } F_0 \text{ as in (26)} \tag{43}$$

and for $i = 1, ...k$

$$H_i^T \triangleq \begin{bmatrix} \bar{\Phi}_{i-1}^T C_K^T \\ G^T \tilde{\Phi}_{i-1}^T C_K^T \\ D_q^T \end{bmatrix} [C_K \tilde{\Phi}_{i-1} \;\; 0] \qquad ,$$
$$G_i \triangleq \begin{bmatrix} B_p^T \tilde{\Phi}_{i-1}^T C_K^T C_K \tilde{\Phi}_{i-1} B_p & 0 \\ * & -I \end{bmatrix} \tag{44}$$

$$F_i \triangleq \begin{bmatrix} \bar{\Phi}_{i-1}^T \\ G^T \tilde{\Phi}_{i-1}^T \\ B_p^T \tilde{\Phi}_{i-1}^T \end{bmatrix} C_K^T C_K [\bar{\Phi}_{i-1} \;\; \tilde{\Phi}_{i-1} G \;\; \tilde{\Phi}_{i-1} B_p]. \tag{45}$$

The rationale for introducing the matrices $\tilde{H}_i^T$, $\tilde{G}_i$ and $\tilde{F}_i$ is that of expressing conditions (18) for $i = 0, ..., N-1$ all in terms of the same vectors $\underline{c}_k$ and $\underline{p}_k$, which is instrumental for the application of the S-procedure, while the exact dependence is maintaining. In fact, observe that

$$[x_0^T \; \underline{c}_k^T]\tilde{H}_i^T \underline{p}_k = [x_0^T \; \underline{c}_{i-1}^T \; c_i^T]H_i^T \begin{bmatrix} x_0 \\ \underline{c}_{i-1} \\ c_i \end{bmatrix},$$

$$\underline{p}_k^T \tilde{G}_i^T \underline{p}_k = [\underline{p}_{i-1}^T \; p_i^T]H_i \begin{bmatrix} \underline{p}_{i-1} \\ p_i \end{bmatrix} \tag{46}$$

$$[x_0^T \; \underline{c}_k^T]\tilde{F}_i \begin{bmatrix} x_0 \\ \underline{c}_k \end{bmatrix} = [x_0^T \; \underline{c}_{i-1}^T \; c_i^T]F_i \begin{bmatrix} x_0 \\ \underline{c}_{i-1} \\ c_i \end{bmatrix}. \tag{47}$$

Again, via the S-procedure it can be shown that the implication

$$(37) \text{ holds true for all } p_0,\ldots,p_k \text{ satisfying } (38) \tag{48}$$

is satisfied if there exist $k+1$ reals $\tau_0^k \geq 0,\ldots,\tau_k^k \geq 0$ such that the following matrix

$$\begin{bmatrix} -B_p^T\tilde{\Phi}_k^T R_x \tilde{\Phi}_k B_p - \sum_{i=0}^k \tau_i^k \tilde{G}_i & -\left(D_k + \sum_{i=0}^k \tau_i^k \tilde{H}_i\right) \begin{bmatrix} x_0 \\ \underline{c}_k \end{bmatrix} \\ * & J_k - [x_0^T \; \underline{c}_k^T]\left(E_k + \sum_{i=0}^k \tau_i^k \tilde{F}_i\right) \begin{bmatrix} x_0 \\ \underline{c}_k \end{bmatrix} \end{bmatrix} \tag{49}$$

is semidefinite positive for the triplet $(x_0, \underline{c}_k, J_k)$. By using the same arguments used to derive (33) one arrives to

$$\Sigma_k \triangleq \begin{bmatrix} J_k & -[x_0^T \; \underline{c}_k^T]L_k^T \\ * & I \end{bmatrix} \geq 0 \tag{50}$$

where

$$L_k^T L_k = \left(E_k + \sum_{i=0}^k \hat{\tau}_i^k \tilde{F}_i\right) + \left(D_k + \sum_{i=0}^k \hat{\tau}_i^k \tilde{H}_i\right)^T \left(-B_p^T \tilde{\Phi}_k^T R_x \tilde{\Phi}_k B_p - \sum_{i=0}^k \hat{\tau}_i^k \tilde{G}_i\right)^{-1} \left(D_k + \sum_{i=0}^k \hat{\tau}_i^k \tilde{H}_i\right) \tag{51}$$

and $\hat{\tau}_i^k, \; i = 0,\ldots,k$ are given by

$$\begin{array}{c} [\hat{\tau}_0^k,\ldots,\hat{\tau}_k^k] \triangleq \arg\min_{\tau_i^k \geq 0} \;\; \bar{\lambda}\left(L_k^T L_k\right) \\ \text{subject to} \\ \left(-B_p^T \tilde{\Phi}_k^T R_x \tilde{\Phi}_k B_p - \sum_{i=0}^k \tau_i^k \tilde{G}_i\right) > 0. \end{array} \tag{52}$$

Finally, the following LMI condition

$$\Sigma_{N-1} \triangleq \begin{bmatrix} J_{N-1} & -[x_0^T \; \underline{c}_{N-1}^T]L_{N-1}^T \\ * & I \end{bmatrix} \geq 0 \tag{53}$$

results a sufficient condition for (19) to hold true where the matrix $L_{N-1}$ factorizes

$$L_{N-1}^T L_{N-1} = \left(E_{N-1} + \sum_{i=0}^{N-1} \hat{\tau}_i^{N-1} \tilde{F}_i\right) + \left(D_{N-1} + \sum_{i=0}^k \hat{\tau}_i^{N-1} \tilde{H}_i\right)^T \left(-B_p^T \tilde{\Phi}_{N-1}^T P \tilde{\Phi}_{N-1} B_p - \sum_{i=0}^{N-1} \hat{\tau}_i^{N-1} \tilde{G}_i\right)^{-1} \left(D_{N-1} + \sum_{i=0}^{N-1} \hat{\tau}_i^{N-1} \tilde{H}_i\right) \tag{54}$$

with $D_{N-1}^T$, $E_{N-1} = E_{N-1}^T \geq 0$ matrices defined by

$$D_{N-1}^T \triangleq \begin{bmatrix} \bar{\Phi}_{N-1}^T \\ G^T \tilde{\Phi}_{N-1}^T \end{bmatrix} P \tilde{\Phi}_{N-1}^T B_p,$$

$$E_{N-1} \triangleq \begin{bmatrix} \bar{\Phi}_{N-1}^T P \bar{\Phi}_{N-1} & \bar{\Phi}_{N-1}^T P \tilde{\Phi}_{N-1} G \\ * & G^T \tilde{\Phi}_{N-1}^T P \tilde{\Phi}_{N-1} G + \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix} \end{bmatrix} \tag{55}$$

and

$$\begin{array}{c} [\hat{\tau}_0^{N-1},\ldots,\hat{\tau}_{N-1}^{N-1}] \triangleq \arg\min_{\tau_i^{N-1} \geq 0} \;\; \bar{\lambda}\left(L_{N-1}^T L_{N-1}\right) \\ \text{subject to} \\ \left(-B_p^T \tilde{\Phi}_{N-1}^T P \tilde{\Phi}_{N-1} B_p - \sum_{i=0}^{N-1} \tau_i^{N-1} \tilde{G}_i\right) > 0. \end{array} \tag{56}$$

All the above discussion can be summarized in the following result.

**Lemma 1** - Let the initial state $x_0$, the stabilizing control law $K$ and the input increments $c_i$ be given for $i = 0,..,N-1$. Then, the set of all non-negative variables $J_0,\ldots,J_{N-1}$ which satisfy the following $N-1$ LMI conditions, $\Sigma_i \geq 0, \;\; i = 0,\ldots,N-1$, provide an upper-bound to the cost as indicated in (20). Notice that, for the specific structure of the matrices $G_i, \;\; i = 0,\ldots,N-1$, problems (31), (52) and (56) have always solution by taking their arguments sufficiently large.
**Proof** - By collecting all the above discussion. □

Next step is to find LMI conditions that allows us to enforce the quadratic input constraints (2) along the predictions for $k = 0,\ldots,N-1$. This consists of imposing that

$$(Kx_0 + c_0)^T Q_u (Kx_0 + c_0) \;\; \leq \;\; \bar{u}^2 \tag{57}$$

$$(K\hat{x}_k + c_k)^T Q_u (K\hat{x}_k + c_k) \;\; \leq \;\; \bar{u}^2, \; \forall p_i \in S_i, \; i = 0,\ldots,k \tag{58}$$

with $\hat{x}_k$ given by (36). Condition (57) directly translates into the following LMI feasibility condition

$$\Upsilon_0 \triangleq \begin{bmatrix} \bar{u}^2 & -(Kx_0 + c_0)^T \\ * & Q_u^{-1} \end{bmatrix} \geq 0 \tag{59}$$

where, for each $k$, condition (58) can be satisfied by determining a suitable LMI $\Upsilon_k \geq 0$. One way is to use the arguments used in determining $\Sigma_k$ by eliminating the universally quantified variables $p_k$. A more direct, though possibly conservative strategy is to consider that an outer approximation to the set of predictions $\hat{x}_k$ is provided by the ellipsoidal set

$$Z_k \triangleq \left\{x \in \mathbb{R}^{n_x} \;:\; x^T R_x x + c_{k-1}^T R_u c_{k-1} \leq J_{k-1}\right\} \tag{60}$$

where $J_{k-1}$ is the variable involved in the cost upper-bound (20). Then, for each $k = 1,\ldots N-1$, condition (47) can be rewritten as

$$(Kx + c_k)^T Q_u (Kx + c_k) \leq \bar{u}^2, \quad \forall x \in Z_k. \tag{61}$$

Via the $S$-procedure it can be shown that (61) holds true iff there exists a non-negative real $\theta_k \geq 0$ such that the following matrix

$$\begin{bmatrix} -K^T Q_u K + \theta_k R_x & -K^T Q_u c_k \\ * & \bar{u}^2 - c_k^T Q_u c_k - \theta_k\left(J_{k-1} - c_{k-1}^T R_u c_{k-1}\right) \end{bmatrix} \tag{62}$$

is positive semidefinite which, in turn, is equivalent to verify that both

$$-K^T Q_u K + \theta_k R_x > 0 \tag{63}$$

$$\bar{u}^2 - c_k^T Q_u c_k - \theta_k\left(J_{k-1} - c_{k-1}^T R_u c_{k-1}\right) - c_k^T Q_u K\left(-K^T Q_u K + \theta_k R_x\right)^{-1} K^T Q_u c_k \geq 0 \tag{64}$$

hold true. Then, by selecting

$$\hat{\theta}_k \triangleq \arg\min_{\theta_k \geq 0} \quad \theta \text{ subject to } \left(-K^T Q_u K + \theta_k R_x\right) > 0 \tag{65}$$

factorizing

$$V_k^T V_k = Q_u + Q_u K \left(-K^T Q_u K + \hat{\theta}_k R_x\right)^{-1} K^T Q_u \tag{66}$$

and observing that

$$(64) \ \geq 0 \text{ for } c_k = 0 \Rightarrow (64) \ \geq 0 \ \forall c_k \tag{67}$$

one can rewrite (64) as the following LMI condition

$$\Upsilon_k \triangleq \begin{bmatrix} \vec{u}^2 - \hat{\theta}_k J_{k-1} & -c_k^T V_k^T \\ * & I \end{bmatrix} \geq 0 \tag{68}$$

LMIs (67) and (68), if satisfied, define sufficient conditions on $J_k$ and $c_k$ that give rise to admissible state predictions.

**Lemma 2** - Let the initial state $x_0$ and the stabilizing control law $K$ be given. Then, all vectors $c_k$ that along with $J_k$, $k = 0,...,N-1$ satisfy the LMI conditions $\Upsilon_k \geq 0$, $k = 0,...,N-1$, generate set-valued state predictions that, along with the corresponding $c_k$, fulfil the input constraint (2) for $k = 0,..,N-1$. Notice that, problems (66) have always solution by taking their arguments sufficiently large.

**Proof** - By collecting all the above discussion. □

It remains to impose that: $\hat{x}_N \in C(P,\rho) \subset \Omega_u$ i.e.

$$\hat{x}_N^T P \hat{x}_N \leq \rho \tag{69}$$

where $\hat{x}_N = \Phi_K \hat{x}_{N-1} + G c_1 + B_p p_{N-1}$, $\forall p_{N-1} \in S_{N-1}$. The constraint (69) can be arranged as follows

$$(\Phi_K \hat{x}_{N-1} + G c_{N-1} + B_p p_{N-1})^T P (\Phi_K \hat{x}_{N-1} + G c_{N-1} + B_p p_{N-1}) \leq \rho \tag{70}$$

subject to

$$\begin{aligned} (\Phi_K \hat{x}_{N-1} + G c_{N-1} + B_p p_{N-1})^T P (\Phi_K \hat{x}_{N-1} + G c_{N-1} + B_p p_{N-1}) \\ + c_{N-1}^T R_u c_{N-1} \leq J_{N-1} \end{aligned} \tag{71}$$

$$\forall \hat{x}_{N-1} \ s.t. \ \hat{x}_{N-1}^T P \hat{x}_{N-1} + c_{N-2}^T R_u c_{N-2} \leq J_{N-2} \tag{72}$$

$$\forall p_{N-1} \ s.t. \ p_{N-1}^T p_{N-1} \leq (C_K \hat{x}_{N-1} + D_q c_{N-1})^T (C_K \hat{x}_{N-1} + D_q c_{N-1}). \tag{73}$$

The above conditions (70)-(73) can be rewritten respectively as

$$\begin{aligned} \rho - \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} E_N \begin{bmatrix} \hat{x}_{N-1} \\ p_{N-1} \end{bmatrix} - \\ 2 \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} D_N \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} - \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} F_N \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \geq 0 \end{aligned} \tag{74}$$

$$\begin{aligned} J_{N-1} - \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} E_N \begin{bmatrix} \hat{x}_{N-1} \\ p_{N-1} \end{bmatrix} - \\ 2 \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} D_N \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} - \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} F_{N_1} \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \geq 0 \end{aligned} \tag{75}$$

$$J_{N-2} - \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} E_{N_0} \begin{bmatrix} \hat{x}_{N-1} \\ p_{N-1} \end{bmatrix} - \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} D_{N_0} \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \geq 0. \tag{76}$$

$$\begin{aligned} \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} E_{Np_{N-1}} \begin{bmatrix} \hat{x}_{N-1} \\ p_{N-1} \end{bmatrix} + 2 \begin{bmatrix} \hat{x}_{N-1}^T & p_{N-1}^T \end{bmatrix} D_{Np_{N-1}} \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} + \\ \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} F_{Np_{N-1}} \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \geq 0. \end{aligned} \tag{77}$$

where

- $E_N = \begin{bmatrix} \Phi_K^T P \Phi_K & \Phi_K^T P B_p \\ * & B_p^T P B_p \end{bmatrix}$, $F_N = \begin{bmatrix} G^T P G & 0 \\ 0 & 0 \end{bmatrix}$, $D_N = \begin{bmatrix} \Phi_K^T P \\ B_p^T P \end{bmatrix} [G \ 0]$
- $F_{N_1} = \begin{bmatrix} G^T P G + R_u & 0 \\ 0 & 0 \end{bmatrix}$, $E_{N_0} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$, $D_{N_0} = \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix}$
- $E_{Np_{N-1}} = \begin{bmatrix} C_K^T C_K & 0 \\ 0 & -I \end{bmatrix}$, $F_{Np_{N-1}} = \begin{bmatrix} D_q^T D_q & 0 \\ 0 & 0 \end{bmatrix}$, $D_{Np_{N-1}} = \begin{bmatrix} C_K^T \\ 0 \end{bmatrix} [D_q \ 0]$.

Then the implication

(70) holds true for all $\hat{x}_{N-1}$ satisfying (70)-(72)
and for all $p_{N-1}$ satisfying (73)

can be shown via S-procedure to be true if there exist scalars $\tau_1^N \geq 0$, $\tau_2^N \geq 0$, $\tau_3^N \geq 0$ such that the following matrix

$$\begin{bmatrix} -E_N + \tau_1^N E_N + \tau_2^N E_{N_0} - \tau_3^N E_{Np_{N-1}} & (D_N + \tau_1^N D_N - \tau_3^N D_{Np_{N-1}}) \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \\ * & \begin{matrix} \rho - \tau_1^N J_{N-1} - \tau_2^N J_{N-2} - \\ - \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} F_N + \\ + \tau_1^N F_{N_1} + \tau_2^N D_{N_0} - \tau_3^N F_{Np_{N-1}} \begin{bmatrix} c_{N-1} \\ c_{N-2} \end{bmatrix} \end{matrix} \end{bmatrix} \tag{78}$$

is semidefinite positive for $(c_{N-1}, c_{N-2}, J_{N-1}, J_{N-2})$. By the same arguments used in the section, one obtains

$$\Sigma_N \triangleq \begin{bmatrix} \rho - \tau_1^N J_{N-1} - \tau_2^N J_{N-2} & \begin{bmatrix} c_{N-1}^T & c_{N-2}^T \end{bmatrix} L_N^T \\ * & I \end{bmatrix} \tag{79}$$

where

$$\begin{aligned} L_N^T L_N = F_F + \left(D_N + \tau_1^N D_N - \tau_3^N D_{Np_{N-1}}\right)^T \\ \left(-E_N + \tau_1^N E_N + \tau_2^N E_{N_0} - \tau_3^N E_{Np_{N-1}}\right)^{-1} \left(D_N + \tau_1^N D_N - \tau_3^N D_{Np_{N-1}}\right) \end{aligned} \tag{80}$$

where $\tau_i^N$, $i = 1,2,3$, are given by

$$\begin{gathered} [\tau_1^N, \tau_2^N, \tau_3^N] \triangleq \arg\min_{\tau_i^N \geq 0} \ \bar{\lambda}\left(L_N^T L_N\right) \\ \text{subject to} \\ \left(-E_N + \tau_1^N E_N + \tau_2^N E_{N_0} - \tau_3^N E_{N_{p_{N-1}}}\right) > 0. \end{gathered} \tag{81}$$

### 3.1 Algorithm NB-Frozen

An implementable algorithm which uses the previous result under the assumption that $P$ and $K$ are held constant to their values computed in the initialization phase is as follows:

1. At time $t = 0$ given $x(0)$, find
$$[Y_{\text{opt}}, Q_{\text{opt}}] \triangleq \arg \min_{Q,Y,\rho,\lambda} \rho \tag{82}$$
subject to the constraints (7), (8), (9), (10).
Let $K \leftarrow Y_{\text{opt}} Q_{\text{opt}}^{-1}$, $P \leftarrow \rho_{\text{opt}} Q_{\text{opt}}^{-1}$.
2. Compute the scalars: $\tau_0$ by solving (31); $\tau_i^k$, $k = 1,\ldots,N-2$, $i = 0,\ldots,k$ by solving (52); $\tau_i^{N-1}$, $i = 0,\ldots,N-1$ by solving (56); $\theta_k$, by solving (65); $\tau_i^N$, $i = 1,2,3$ by solving (81);
3. At each time $t \geq 0$, find $\hat{c}_{\text{opt}}(t|t)$, $\hat{c}_{\text{opt}}(t+1|t)$, $\ldots$, $\hat{c}_{\text{opt}}(t+N-1|t)$, the minimizer of
$$\min_{J_i, \hat{c}(t+i|t) \ i=0\ldots N-1} \bar{J} \tag{83}$$
subject to $\Sigma_i \geq 0$, $i = 0,\ldots,N$, $\Upsilon_i \geq 0$, $i = 0,\ldots,N-1$;
4. feed the plant by $\hat{u}_{\text{opt}}(t|t) \rightarrow Kx(t) + \hat{c}_{\text{opt}}(t|t)$;

5. $t \leftarrow t+1$ and go to step 3.

The following result solves feasibility and closed loop stability questions:

**Proposition 1** *Let the NB-Frozen scheme have solution at time $t = 0$. Then, it has solution at each future time instant $t$, satisfies the input constraints and yields an asymptotically (quadratically) stable closed-loop system.*

*Proof:* See [3].

## 4 A numerical experiment

Consider the same two-carts/spring system of [2, 9]. In all simulations we have used $R_u = 1$, $R_x = H' R_y H$, with $R_y = 1$ and input constraints $\bar{u} = 0.1$. Fig. 1 reports the output and input for the proposed NB-Frozen algorithm and the Polytopic-Frozen MPC scheme of [3] for $N = 2$, respectively. As there clearly results, an identical control performance has been obtained by using the two different descriptions for the uncertain system. However, as reported in Table 1 the NB-Frozen algorithm shows a remarkable reduction of the computational complexity, measured in Flops per step, as the control horizon increases. In order to give a statistical measure of how the Algorithm NB-Frozen performs w.r.t. the Polytopic-Frozen Algorithm, a series of 200 randomly chosen plants $(\Phi_m, G_m)$, $m = 1, \ldots, 200$, belonging to the uncertainty structure given by the plant (see [9]), have been generated. For each couple $(\Phi_m, G_m)$, $m = 1, \ldots, 200$, a 100-time step MPC nominal simulation have been run and an estimate of the optimal quadratic cost computed. These results have been compared with the upper bound to the optimal cost after 100-time step for the MPC-Polytopic and the MPC-NB frozen algorithms. The relative error has been finally computed and in the Tables 2, 3 an estimate of the relative error mean and the standard deviation are given.

## 5 Conclusions

We have presented a novel robust predictive control strategy robustly which asymptotically stabilizes an input constrained uncertain linear system with norm-bounded uncertainties. The numerical procedure is based on the minimization, at each time instant, of an upper bound of a minmax quadratic index, under the constraint that all future states are robustly steered within $N$-steps into a feasible positively invariant set. The $S$-procedure plays a crucial role in determining the convex constraints of such an optimization problem. A significant reduction of the computational burden and no control performance loss with respect to the polytopic paradigm has been observed from the numerical experiments.


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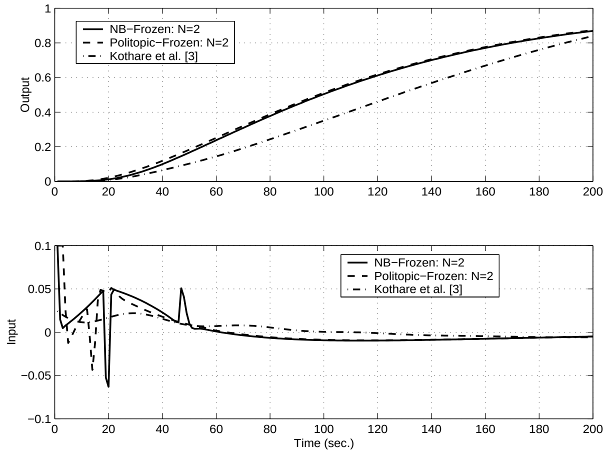


Figure 1: Regulated plant output and input

Table 1: Comparison of numerical complexity per step

| | N=1 | N=2 | N=3 |
|---|---|---|---|
| Polytopic-Frozen | 3686 | 9586 | 27946 |
| Nb-Frozen | 2119 | 4505 | 7835 |

Table 2: Nb-Frozen vs. Exact

| | N=1 | N=2 | N=3 |
|---|---|---|---|
| Mean | 0.3011 | 0.2838 | 0.2636 |
| Standard Deviation | 0.0113 | 0.0145 | 0.0101 |

Table 3: Polytopic-Frozen vs. Exact

| | N=1 | N=2 | N=3 |
|---|---|---|---|
| Mean | 0.2935 | 0.2261 | 0.1581 |
| Standard Deviation | 0.0156 | 0.0198 | 0.0156 |

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