ROBUST STABILIZATION OF JUMPING SYSTEM VIA STATIC OUTPUT FEEDBACK

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Abstract

The paper considers a class of control systems, described by a finite set of linear systems with the transition between them determined by a homogeneous Markov chain. Every state of this chain correspond to some mode of the system. When the mode is fixed, the plant state evolves according to the corresponding individual dynamic. At the moment of discontinuous mode change the plant state vector can be changed by jump. The necessary and sufficient conditions of stabilizability in the mean square via static output feedback control are formulated. These conditions are generalized to the case of the mode change parameters uncertainty. The sufficient conditions which guarantee, that the static output feedback control stabilizes the system for plant parameters uncertainty are obtained. Some heuristic algorithms for computing of the gain matrix of robust stabilizing control are also developed.

1 Introduction

In control practice we can find a lot of dynamical systems with random jumping changes of their structure or parameters, such as aerospace systems, manufacturing systems, economic systems, etc., see, for example [11, 13, 14, 17] and the references therein. Systems with random jumps are hybrid ones with many operating modes. Every mode corresponds to an individual deterministic or stochastic dynamics. The system mode switching is governed by a Markov process with a finite set of states $\mathbb{N} = \{1, 2, \dots, \nu\}$ (Markov chain). When the mode $i \in \mathbb{N}$ is fixed, the plant state evolves according to the corresponding individual dynamic. The state space of these systems is naturally hybrid: to the usual plant state in $\mathbb{R}^n$ we append a discrete variable taking values in the set $\mathbb{N}$.

The stability and control theory for the systems with random jumps began to develop since the pioneering works of Kats and Krasovskii [12], Krasovskii and Lidskii [16] correspondingly. The stochastic moment approach to the stability problem was introduced by Mil'stein [18]. The linear quadratic control problem was solved by Sworder [23] using stochastic maximum principle for state feedback in finite horizon case. Wonham [26] obtained the same results using dynamic programming for both finite and infinite horizon cases. He also obtained a set of sufficient conditions for the existence of a finite solution. Kazakov and Artem'ev [14] have developed a general theory of random structure systems based on Fokker-Planck-Kolmogorov type equation approach. Now, due the large number of applications several results for this class of systems can be found in the current literature, regarding stability, optimal control, stabilization, controllability and observability problems, see for instance [3, 11, 13, 17] and the references therein.

Robust control offers the advantage to design a controller which enables us to cope with the uncertainties which appear in the more realistic models. The series of papers dealing with the robustness of the class of systems with random jumps have been published. The stability, stabilization, H_2, H_∞, mixed H_2/H_∞ problems and their robustness have been investigated. Without any intention of being exhaustive here, we quote the papers [1, 2, 4–7, 9, 10, 19–22] and the references therein.

One of the most important open questions in control theory is the static output feedback problem [24]. In the case of jumping systems we have especially weak development of the analytical and computational methods of solution of this problem [17]. In this connection the purpose of this paper is to obtain stabilizability and robust stabilizability conditions of jumping systems via static output feedback.

In this paper we study the robust static output feedback control problem in two cases. First, we consider the system with mode change parameters uncertainty. In this case we suppose that the matrix of transition intensities is an affine functions of uncertain vector parameter. Second we consider the system with plant parameter uncertainty and sector bounded Lur'e type nonlinearities. In both the cases at the moment of mode change the plant state vector can be changed by jump.

The paper is organized as follows. In Section 2 we give the mathematical description of the considered system. In section 3 necessary and sufficient conditions of stabilizability in the mean square via static output feedback are formulated. In section 4 we obtain necessary and sufficient conditions for static output feedback control to be robust control against mode change parameters uncertainty. In section 5 we obtain the sufficient conditions for the mean square stabilizing control to be robust control against both the plant parameters uncertainty and the uncertain sector bounded nonlinearity. In conclusion some heuristic LMI-based [8] algorithms for computing of the gain matrix of robust stabilizing control are developed and computational aspects of the considered problems are shortly discussed.

2 System description

Consider a control system described by the family of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}[r(t)]\mathbf{x}(t) + \mathbf{B}[r(t)]\mathbf{u}(t), \quad (1)$$
$$\mathbf{y}(t) = \mathbf{C}[r(t)]\mathbf{x}(t),$$

where $\mathbf{x}(t)$ is the n-dimensional plant state vector; $\mathbf{u}(t)$ is the k-dimensional control vector; $\mathbf{y}(t)$ is the s-dimensional output vector; $r(t)$ is homogeneous discrete state Markov process (Markov chain) representing a mode (or regime) of system and taking values in a finite set $\mathbb{N} = \{1, \ldots, \nu\}$ with a matrix of transition probabilities $\mathbf{P}(\Delta) = [p_{ij}(\Delta)]_1^\nu$, from mode i to mode j during the time interval $[t, t+\Delta]$ given by $\mathbf{P}(\Delta) = \exp(\mathbf{\Pi}\Delta)$, $p_{ij}(\tau) = \mathrm{Prob}\{r(t+\Delta) = j \mid r(t) = i\}$ $(i, j \in \mathbb{N})$, $\mathbf{\Pi} = [\pi_{ij}]_1^\nu$, $\pi_{ij} \geq 0$ $(i \neq j)$, $\pi_{ii} = -\sum_{j \neq i}^\nu \pi_{ij}$; $\mathbf{A}(i) = \mathbf{A}_i, \mathbf{B}(i) = \mathbf{B}_i, \mathbf{C}(i) = \mathbf{C}_i$ $(i \in \mathbb{N})$ are known matrices of appropriate dimensions.

Let $\tau > t_0$ be the moment of discontinuous mode change, i.e. the moment of transition from $r(\tau - 0) = i$ to $r(\tau) = j \neq i$. It is supposed that at the moment τ the plant state vector $\mathbf{x}$ can be changed discontinuously too and its value after jump is linearly dependent on the same value before the jump:

$$\mathbf{x}(\tau) = \mathbf{\Phi}_{ij}\mathbf{x}(\tau - 0), \quad (2)$$

where $\mathbf{\Phi}_{ij}, (i, j \in \mathbb{N})$ are $n \times n$ constant matrices, such that $(\mathbf{\Phi}_{ii} = \mathbf{I})$.

Note that as a rule the case of continuous change of the plant state vector is considered ($\mathbf{\Phi}_{ij} = \mathbf{I}$), but in many real systems the situation, when some plant state variables are changed by jump is more typical. This situation is natural for mechanical systems with sudden change of mass or moment of inertia; in this case the linear or angular velocity will be changed by jump, see [13] for more detail.

For every $i \in \mathbb{N}$ the plant state space of the system (1) can be presented in the form of the following partition

$$\mathbb{R}^n = \mathrm{Im}(\mathbf{C}^T(i)) \oplus \mathrm{Ker}(\mathbf{C}(i)), \quad (3)$$

where $\mathrm{Im}(\mathbf{C}^T(i))$ and $\mathrm{Ker}(\mathbf{C}(i))$ are orthogonal subspaces. For any $\mathbf{x} \in \mathbb{R}^n$ we can write

$$\mathbf{x} = \mathbf{x}_\mathrm{I} + \mathbf{x}_\mathrm{K},$$

where $\mathbf{x}_I \in \mathrm{Im}(\mathbf{C}^T(i))$ and $\mathbf{x}_K \in \mathrm{Ker}(\mathbf{C}(i))$. Define the matrices

$$\mathbf{E}_\mathrm{I}(i) = \mathbf{C}_i^+\mathbf{C}_i, \quad (4)$$
$$\mathbf{E}_\mathrm{K}(i) = \mathbf{I} - \mathbf{E}_\mathrm{I}(i), \quad (5)$$

where $\mathbf{C}_i^+$ is the Moore-Penrose inverse of $\mathbf{C}_i$. According to the partition (3) the matrices (4), (5) are projection matrices on $\mathrm{Im}(\mathbf{C}^T(i))$ and on $\mathrm{Ker}(\mathbf{C}(i))$ correspondingly. These matrices are symmetric and unique.

3 Mean square stabilization via static output feedback

Consider a linear control with the static output feedback in the following form

$$\mathbf{u}(t) = -\mathbf{K}(i)\mathbf{y}(t) \ \text{ if } \ r(t) = i. \quad (6)$$

Definition 1 *The system (1) is said to be mean square stabilizable via static output feedback if there exists a control law in the form of (6) such that the system (1) is exponentially stable in the mean square, i.e. for any* $\mathbf{x}_0 = x \in \mathbb{R}^n,\ r_0 = i \in \mathbb{N},\ t \geq 0$ *and for some* $\alpha > 0,\ \beta > 0$ *the following inequality is true [15]:*

$$\mathcal{E}[\| \mathbf{x}(t) \|^2 | \mathbf{x}_0 = x,\ r_0 = i] \leq$$
$$\beta \| x \|^2 \exp(-\alpha(t - t_0)),\ t \geq t_0,$$

where $\mathcal{E}$ *is the expectation operator.*

In this section we generalize the result by Trofino-Neto and Kučera [25] and obtain some necessary and sufficient conditions of stabilizability via static output feedback of the system (1).

Lemma 1 *Let* $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in \mathbb{R}^n$, $\mathbf{z} = \mathbf{x} + \mathbf{y},\ \mathbf{x}^T\mathbf{y} = 0,\ \mathbf{x} \in \mathrm{Im}(\mathbf{B}^T),\ \mathbf{y} \in \mathrm{Ker}(\mathbf{B}),\ \mathbf{x}^T\mathbf{A}\mathbf{x} \leq \mathbf{x}^T\mathbf{B}^T\mathbf{B}\mathbf{x},\ \mathbf{y}^T\mathbf{A}\mathbf{y} < 0,$ *where* $\mathbf{A} = \mathbf{A}^T$ *and* $\mathbf{B}$ *are some matrices of corresponding dimensions. Then*

$$\mathbf{z}^T\mathbf{A}\mathbf{z} \leq \mathbf{z}^T(\mathbf{B}^T\mathbf{B} + \beta\mathbf{I})\mathbf{z},$$

where $\beta = 2 \| \mathbf{A} \|$.

Proof. Taking into account that

$$\mathbf{B}\mathbf{z} = \mathbf{B}\mathbf{x} + \mathbf{B}\mathbf{y},$$

we have

$$\mathbf{z}^T\mathbf{A}\mathbf{z} = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{x}^T\mathbf{A}\mathbf{y} + \mathbf{y}^T\mathbf{A}\mathbf{y} \leq \mathbf{x}^T\mathbf{B}^T\mathbf{B}\mathbf{x} +$$
$$2\mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{z}^T\mathbf{B}^T\mathbf{B}\mathbf{z} + 2\mathbf{x}^T\mathbf{A}\mathbf{y}.$$

According to the Schwartz inequality

$$\mathbf{x}^T\mathbf{A}\mathbf{y} \leq \| \mathbf{x} \| \| \mathbf{A}\mathbf{y} \| \leq$$
$$\| \mathbf{x} \| \| \mathbf{A} \| \| \mathbf{y} \| \leq \| \mathbf{x} \| \| \mathbf{A} \| \| \mathbf{y} \| = \| \mathbf{A} \| \mathbf{z}^T\mathbf{z}.$$

Now from the previous inequality we have the result of the lemma.

Theorem 1 *The system (1) is stabilizable via static output feedback if and only if there exist matrices* $\mathbf{Q}_i$, $\mathbf{R}_i > 0$ *and* $\mathbf{L}_i$ *of compatible dimensions such that the system of algebraic equations and inequalities*

$$\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i - \mathbf{E}_\mathrm{I}(i)[\mathbf{H}_i\mathbf{B}_i +$$
$$\mathbf{L}_i^T]\mathbf{R}_i^{-1}[\mathbf{B}_i^T\mathbf{H}_i + \mathbf{L}_i]\mathbf{E}_\mathrm{I}(i) + \mathbf{Q}_i +$$
$$\sum_{j=1}^{\nu} \mathbf{\Phi}_{ij}^T\mathbf{H}_j\mathbf{\Phi}_{ij}\pi_{ij} = 0, \quad (7)$$

$$\mathbf{H}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i^T\mathbf{H}_i - \mathbf{S}_i^T\mathbf{R}_i^{-1}\mathbf{S}_i + \mathbf{Q}_i > 0,\ i \in \mathbb{N}, \quad (8)$$

where $\mathbf{S}_i = \mathbf{L}_i\mathbf{E}_{\mathrm{I}}(i) - \mathbf{B}_i^T\mathbf{H}_i\mathbf{E}_{\mathrm{K}}(i)$ has positive definite solution $\mathbf{H}_i,\ i \in \mathbb{N}$. The stabilizing control has the form of (6) with the gain matrix given by the formula

$$\mathbf{K}_i = \mathbf{R}_i^{-1}(\mathbf{B}_i^T\mathbf{H}_i + \mathbf{L}_i)\mathbf{C}_i^+,\ i \in \mathbb{N}. \tag{9}$$

Proof. *Necessity.* Let the system (1) is mean square stabilizable via static output feedback. Then without loss the generality we may suppose that there exists a matrix $\mathbf{G}_i = \mathbf{K}_i\mathbf{C}_i\mathbf{E}_{\mathrm{I}}(i) = \mathbf{\Theta}_i\mathbf{E}_{\mathrm{I}}(i)$ and a positive definite matrix $\mathbf{H}_i\ (i \in \mathbb{N})$ such that for all $\mathbf{x} \in \mathbb{R}^n$ the following system of the coupled matrix inequalities holds

$$\mathbf{x}^T[(\mathbf{A}_i - \mathbf{B}_i\mathbf{G}_i)^T\mathbf{H}_i + \mathbf{H}_i(\mathbf{A}_i - \mathbf{B}_i\mathbf{G}_i) + \mathbf{P}_i]\mathbf{x} < 0,\ i \in \mathbb{N}, \tag{10}$$

where $\mathbf{\Theta}_i$ is some matrix of suitable dimension and

$$\mathbf{P}_i = \sum_{j=1}^{\nu}\mathbf{\Phi}_{ij}^T\mathbf{H}(j)\mathbf{\Phi}_{ij}\pi_{ij}.$$

If $\mathbf{x} \in \mathrm{Ker}\mathbf{G}_i$, then from (10) we have

$$\mathbf{x}^T[\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{P}_i]\mathbf{x} < 0,\ i \in \mathbb{N}. \tag{11}$$

Define for every $i \in \mathbb{N}$ the scalars α_i^* as follows

$$\alpha_i^* = \max_{\mathbf{x}\in \mathrm{Im}(\mathbf{G}_i^T)}\frac{\mathbf{x}^T[\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{P}_i]\mathbf{x}}{\mathbf{x}^T\mathbf{G}_i^T\mathbf{G}_i\mathbf{x}}. \tag{12}$$

From (11) and (12) we obtain

$$\mathbf{x}^T[\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{P}_i]\mathbf{x} \le \alpha_i\mathbf{x}^T\mathbf{E}_{\mathrm{I}}(i)\mathbf{\Theta}_i^T\mathbf{\Theta}_i\mathbf{E}_{\mathrm{I}}(i)\mathbf{x} + \beta_i\mathbf{x}^T\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n,\ i \in \mathbb{N},$$

where $\alpha_i > \max(0, \alpha_i^*)$ and β_i is defined as in Lemma 1. Thus for any symmetric matrix $\mathbf{R}_i > \alpha_i\mathbf{I}$, we have

$$\mathbf{x}^T[\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{P}_i]\mathbf{x} < \mathbf{x}^T\mathbf{E}_{\mathrm{I}}(i)\mathbf{\Theta}_i^T\mathbf{R}_i\mathbf{\Theta}_i\mathbf{E}_{\mathrm{I}}(i)\mathbf{x} + \beta_i\mathbf{x}^T\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n,\ i \in \mathbb{N}.$$

This inequality implies the existence of a symmetric matrix $\mathbf{Q}_i\ (i \in \mathbb{N})$, satisfying the system of matrix equations

$$\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i - \mathbf{E}_{\mathrm{I}}(i)\mathbf{\Theta}_i^T\mathbf{R}_i\mathbf{\Theta}_i\mathbf{E}_{\mathrm{I}}(i) + \mathbf{P}_i + \mathbf{Q}_i = 0,\ i \in \mathbb{N}. \tag{13}$$

Let us define $\mathbf{L}_i = \mathbf{R}_i\mathbf{\Theta}_i - \mathbf{B}_i^T\mathbf{H}_i$, then the equation (13) can be rearranged as in (7), moreover $\mathbf{G}_i = \mathbf{R}_i^{-1}(\mathbf{L}_i + \mathbf{B}_i^T\mathbf{H}_i)\mathbf{E}_I(i)$ and $\mathbf{K}_i$ is given by (9). Rewrite (7) in the following equivalent form

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i)^T\mathbf{H}_i + \mathbf{H}_i(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i) + \mathbf{P}_i + \mathbf{H}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i^T\mathbf{H}_i - \mathbf{S}_i^T\mathbf{R}_i^{-1}\mathbf{S}_i + \mathbf{Q}_i = 0;\ i \in \mathbb{N}. \tag{14}$$

Taking into account (10) it is easy to see from (14) that the inequalities (8) hold.

Sufficiency. Let (7), (8) are valid and $\mathbf{K}_i$ is given by (9). Rewriting (7) in the form of (14) and taking into account (8) we obtain that the system of inequalities (10) is true. This means that the control law (6) with the gain matrix $\mathbf{K}_i$ given by (9) is the mean square stabilizing control. □.

4 Robust control against mode change parameters uncertainty

Suppose that the matrix $\mathbf{\Pi} = \mathbf{\Pi}(\delta)$ is an affine function of the vector parameter δ. That is, suppose that there exist real matrices $\mathbf{\Pi}_0, \ldots, \mathbf{\Pi}_N$ all of the same dimension as $\mathbf{\Pi}$ such that

$$\mathbf{\Pi}(\delta(t)) = \mathbf{\Pi}_0 + \delta_1\mathbf{\Pi}_1 + \ldots + \delta_N\mathbf{\Pi}_N$$

for all $\delta \in \Delta$. Let the uncertain parameters $\delta_j,\ j = 1, \ldots, N$ take values in an interval $[\underline{\delta}_j,\ \bar{\delta}_j]$ i.e.

$$\delta_j \in [\underline{\delta}_j,\ \bar{\delta}_j].$$

This means that the uncertainty of each independent parameter is assumed to be bounded between two extremal values. Define the set of corners of the uncertainty region as

$$\Delta_0 = \{\delta = (\delta_1, \ldots, \delta_N) :\ \delta_j \in \{\underline{\delta}_j,\ \bar{\delta}_j\},\ j = 1, \ldots, N\}.$$

Definition 2 *The system (1) with $u(t) \equiv 0$ is said to be stochastically quadratically stable for perturbations Δ if there exist matrices $\mathbf{H}_i = \mathbf{H}_i^T > 0,\ i \in \mathbb{N}$, such that*

$$\mathbf{A}_i^{\mathrm{T}}\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{P}_i(\delta) < 0,\ i \in \mathbb{N},\ \delta \in \Delta, \tag{15}$$

where

$$\mathbf{P}_i(\delta) = \sum_{j=1}^{\nu}\pi_{ij}(\delta)\mathbf{\Phi}_{ij}^T H_j\mathbf{\Phi}_{ij}.$$

Definition 3 *The system (1) is said to be stochastically quadratically stabilizable via static output feedback if there exists a control law in the form of (6) such that the closed loop system (1) is stochastically quadratically stable i.e. there exists a positive definite solution $\mathbf{H}_i = \mathbf{H}_i^T$ of the system of inequalities*

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i)^{\mathrm{T}}\mathbf{H}_i + \mathbf{H}_i(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i) + \mathbf{P}_i(\delta) < 0,\ i \in \mathbb{N},\ \delta \in \Delta.$$

Lemma 2 *The system (1) is stochastically quadratically stabilizable if and only is there exist matrices $\mathbf{H}_i = \mathbf{H}_i^T > 0$ and $\mathbf{K}_i\ (i \in \mathbb{N})$, such that*

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i)^{\mathrm{T}}\mathbf{H}_i + \mathbf{H}_i(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i) + \mathbf{P}_i(\delta) < 0,\ i \in \mathbb{N},\ \delta \in \Delta_0. \tag{16}$$

The proof is based on the well known results from the convex analysis and it is omitted. Based on this lemma and on the the previous theorem we obtain the following result.

Theorem 2 *The system (1) is stochastically quadratically stabilizable via static output feedback if and only if there exist matrices $\mathbf{Q}_i(\delta),\ \delta \in \Delta_0$, $\mathbf{R}_i > 0$ and $\mathbf{L}_i$ of compatible dimensions such that the system of algebraic equations and inequalities*

$$\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i - \mathbf{E}_I(i)[\mathbf{H}_i\mathbf{B}_i +$$

$$\mathbf{L}_i^T]\mathbf{R}_i^{-1}[\mathbf{B}_i^T\mathbf{H}_i + \mathbf{L}_i]\mathbf{E}_I(i) + \mathbf{P}_i(\delta) + \mathbf{Q}_i(\delta) = 0, \tag{17}$$

$$\mathbf{H}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i^T\mathbf{H}_i - \mathbf{S}_i^T\mathbf{R}_i^{-1}\mathbf{S}_i + \mathbf{Q}_i(\delta) > 0,\ i \in \mathbb{N},\ \delta \in \Delta_0, \tag{18}$$

where $\mathbf{S}_i = \mathbf{L}_i\mathbf{E}_{\mathrm{I}}(i) - \mathbf{B}_i^T\mathbf{H}_i\mathbf{E}_{\mathrm{K}}(i)$ *has positive definite solution* $H_i,\ i \in \mathbb{N}$. *The stabilizing control has the form of (6) with the gain matrix* $\mathbf{K}_i$ *given by formula (9).*

5 Robust control against the plant parameters uncertainty

Consider the system (1) with the plant uncertainty

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(r(t)) + \mathbf{F}(r(t))\mathbf{\Omega}(t, r(t))\mathbf{E}(r(t))]\mathbf{x}(t) + \mathbf{B}(r(t))\mathbf{u}(t) + \mathbf{D}(r(t))\varphi(t, \mathbf{z}(t)), \tag{19}$$

$$\mathbf{y}(t) = \mathbf{C}(r(t))\mathbf{x}(t),\ \ \mathbf{z}(t) = \mathbf{\Lambda}(r(t))\mathbf{x}(t), \tag{20}$$

$$\mathbf{x}(\tau) = \mathbf{\Phi}_{ij}\mathbf{x}(\tau - 0), \tag{21}$$

where $\mathbf{z}(t)$ is m- dimensional output vector; $\mathbf{\Omega}(t, r(t))$ is a matrix of uncertain parameters, satisfying for every t and $r(t)$ the following inequality

$$\mathbf{I} - \mathbf{\Omega}^T(t, r(t))\mathbf{\Omega}(t, r(t)) \geq 0; \tag{22}$$

$\varphi(t, \mathbf{z})$ is a nonlinear m-dimensional vector function, whose components have form

$$\varphi_l(t, \mathbf{z}) = \varphi_l(t, z_l),\ \varphi_l(t, 0) = 0\ \ (l = 1, ..., m) \tag{23}$$

and satisfy constraints

$$0 \leq \varphi_l(t, z_l)z_l \leq \kappa_l(i)z_l^2,\ \text{ if }\ r_t = i \quad (l = 1, \ldots, m,\ i \in \mathbb{N}); \tag{24}$$

$\mathbf{\Lambda}_i, \mathbf{D}_i, \mathbf{E}_i, \mathbf{F}_i$ $(i \in \mathbb{N}$ are known matrices of appropriate dimensions. For simplicity but without loss of generality we assume that $\kappa_l(i) = 1$ $(l = 1, \ldots, m,\ i \in \mathbb{N})$. Then we can write

$$\varphi(t, \mathbf{z})\mathbf{\Gamma}[\varphi(t, \mathbf{z}) - \mathbf{z}] \leq 0, \tag{25}$$

where $\mathbf{\Gamma} = \mathrm{diag}[\gamma_l]_1^m$ $(\gamma_l > 0,\ l = 1, \ldots, m)$.

Suppose that both output vector $\mathbf{y}(t)$ and mode change process $r(t)$ are available for controller. Let the control law has the form of static linear output feedback (6) In this section we obtain an additional conditions which guarantee that the control law (6) stabilizes the original system (19) in the sense of exponential stability in the mean square for all plant parameters uncertainty, satisfying inequality (22), and all the nonlinearities, satisfying (23), (24). We say that such a control is robust stabilizing control.

Theorem 3 *Let for some positive scalars* $\gamma, \gamma_l(l = 1, \ldots, m)$ *and matrices* $\mathbf{M}_i \geq 0,\ \mathbf{N}_i > 0$ *there exists a positive definite solution* $\mathbf{X}_i$ $(i \in \mathbb{N})$, *of the system of coupled matrix quadratic inequalities*

$$\mathbf{X}_i\mathbf{A}_c(i) + \mathbf{A}_c^T(i)\mathbf{X}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{N}_i^{-1}\mathbf{B}_i^T\mathbf{X}_i + \gamma\mathbf{E}_i^T\mathbf{E}_i + \mathbf{M}_i + \gamma^{-1}\mathbf{X}_i\mathbf{F}_i\mathbf{F}_i^T\mathbf{X}_i + (\mathbf{X}_i\mathbf{D}_i + \frac{1}{2}\mathbf{\Lambda}_i^T\mathbf{\Gamma})\mathbf{\Gamma}^{-1}(\mathbf{X}_i\mathbf{D}_i + \frac{1}{2}\mathbf{\Lambda}_i^T\mathbf{\Gamma})^T + \sum_{j\neq i}^{\nu}[\mathbf{\Phi}_{ij}^T\mathbf{X}(j)\mathbf{\Phi}_{ij} - \mathbf{X}_i]\pi_{ij} < 0. \tag{26}$$

Then the output feedback control (6) is robust stabilizing control. The function $V(\mathbf{x}, i) = \mathbf{x}^T\mathbf{X}_i\mathbf{x}$ $(i \in \mathbb{N})$ *is stochastic Lyapunov function which guarantees robust stability of the system (19).*

The proof is based on the results of [20] and it is omitted. The inequalities (26) are easily transformed into LMI's by introducing variables $\mathbf{Y}_i = \mathbf{X}_i^{-1}$ and rewriting (26) as

$$\mathbf{Y}_i(\mathbf{A}_c(i) + \frac{1}{2}\mathbf{D}_i\mathbf{\Lambda}_i)^T + (\mathbf{A}_c(i) + \frac{1}{2}\mathbf{D}_i\mathbf{\Lambda}_i)\mathbf{Y}_i - \mathbf{B}_i\mathbf{N}_i^{-1}\mathbf{B}_i^T + \gamma^{-1}\mathbf{F}_i\mathbf{F}_i^T + \mathbf{D}_i\mathbf{\Gamma}^{-1}\mathbf{D}_i + \sum_{j\neq i}^{\nu}\pi_{ij}\mathbf{Y}_i\mathbf{\Phi}_{ij}^T\mathbf{Y}_j^{-1}\mathbf{\Phi}_{ij}\mathbf{Y}_i + \mathbf{Y}_i\pi_{ii} + \mathbf{Y}_i(\mathbf{M}_i + \gamma\mathbf{E}_i^T\mathbf{E}_i + \frac{1}{4}\mathbf{\Lambda}_i\mathbf{\Gamma}\mathbf{\Lambda}_i)\mathbf{Y}_i < 0,$$
$$\mathbf{Y}_i > 0,\ i \in \mathbb{N}.$$

These inequalities can be expressed by the following LMI's

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{R}_i \\ \mathcal{R}_i^T & \mathcal{D}_i \end{bmatrix} \leq 0, \mathbf{Y}_i > 0,\ i \in \mathbb{N}, \tag{27}$$

where

$$\mathcal{A}_i = \mathbf{Y}_i(\mathbf{A}_c(i) + \frac{1}{2}\mathbf{D}_i\mathbf{\Lambda}_i)^T + (\mathbf{A}_c + \frac{1}{2}\mathbf{D}_i\mathbf{\Lambda}_i)\mathbf{Y}_i - \mathbf{B}_i\mathbf{N}_i^{-1}\mathbf{B}_i^T + \gamma^{-1}\mathbf{F}_i\mathbf{F}_i^T + \mathbf{D}_i\mathbf{\Gamma}^{-1}\mathbf{D}_i + \mathbf{Y}_i\pi_{ii},$$

$$\mathcal{R}_i = [\mathbf{Y}_i\sqrt{\mathbf{M}_i + \gamma\mathbf{E}_i^T\mathbf{E}_i + \frac{1}{4}\mathbf{\Lambda}_i\mathbf{\Gamma}\mathbf{\Lambda}_i}\ \ \sqrt{\pi_{i1}}\mathbf{\Phi}_{i1}\mathbf{Y}_i \ldots \sqrt{\pi_{i(i-1)}}\mathbf{\Phi}_{i(i-1)}\mathbf{Y}_i\ \ \sqrt{\pi_{i(i+1)}}\mathbf{\Phi}_{i(i+1)}\mathbf{Y}_i \ldots \sqrt{\pi_{i\nu}}\mathbf{\Phi}_{i\nu}\mathbf{Y}_i],$$

$$\mathcal{D}_i = \mathrm{diag}[-\mathbf{I}\ \ -\mathbf{Y}_1 \ldots -\mathbf{Y}_{i-1}\ \ -\mathbf{Y}_{i+1} \ldots -\mathbf{Y}_\nu].$$

6 Concluding remarks

The obtained results give helpful relationship with the linear quadratic regulator theory for discrete-time jumping systems. Consider the cost functional

$$J = \mathcal{E}\int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}(r(t))\mathbf{x}(t) + 2\mathbf{x}^T(t)\mathbf{S}^T(r(t))\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{R}(r(t))\mathbf{u}(t)]dt.$$

The state feedback gain which minimizes this functional along the trajectories of the system (7) is [13]:

$$\mathbf{G}_{0i} = \mathbf{R}_i^{-1}[\mathbf{B}_i^T\mathbf{H}_i + \mathbf{S}_i],$$

where $\mathbf{H}_i$ $(i \in \mathbb{N})$ is positive definite solution of the following system of coupled matrix quadratic equations

$$\mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i - [\mathbf{H}_i\mathbf{B}_i + \mathbf{S}_i^T]\mathbf{R}_i^{-1}[\mathbf{B}_i^T\mathbf{H}_i\mathbf{A}_i + \mathbf{S}_i]) + \mathbf{Q}_i + \sum_{j=1}^{\nu}\pi_{ij}\mathbf{\Phi}_{ij}^T\mathbf{H}_j\mathbf{\Phi}_{ij} = 0,\ i \in \mathbb{N}. \qquad (28)$$

It is easy to see that the system of equations (7) is the special case of (28), satisfying the linear constraints

$$\mathbf{S}_i = \mathbf{L}_i\mathbf{E}_{\mathrm{I}}(i) - \mathbf{B}_i^T\mathbf{H}_i\mathbf{E}_{\mathrm{K}}(i),\ i \in \mathbb{N}. \qquad (29)$$

This relationship with the linear quadratic regulator and some ideas of [1, 8, 10] allow to propose some heuristic algorithm to calculate the stabilizing output feedback gain in the form of (9). To find this gain it is necessary to solve the system of nonstandard matrix quadratic equations (7) and inequalities (8). Moreover it is necessary to select by some way the matrices $\mathbf{Q}(i)$, $\mathbf{R}(i) > 0$ and $\mathbf{L}(i)$.

The system of inequalities (8) is true if

$$\begin{bmatrix} \mathbf{Q}_i & \mathbf{S}_i^T \\ \mathbf{S}_i & \mathbf{R}_i \end{bmatrix} \geq 0,$$

where $\mathbf{S}_i$ is given by (29)

Then we can propose the following heuristic LME/LMI-based algorithm

Step1.

Find $\mathbf{H}_i,\ \mathbf{L}_i\ i = 1, \dots, \nu,$ to satisfy:
OBJ: trace$[\mathbf{H}_1 + \dots + \mathbf{H}_\nu] \to \max$,

LME1: $\mathbf{H}_i = \mathbf{H}_i^T,\ i = 1, \dots, \nu,$

LME2: $\mathbf{S}_i = \mathbf{L}_i\mathbf{E}_{\mathrm{I}}(i) - \mathbf{B}_i^T\mathbf{H}_i\mathbf{E}_{\mathrm{K}}(i),\ i = 1, \dots, \nu,$

LMI1: $\mathbf{H}_i > 0,\ i = 1, \dots, \nu,$

LMI2: $\begin{bmatrix} \mathbf{Q}_i & \mathbf{S}_i^T \\ \mathbf{S}_i & \mathbf{R}_i \end{bmatrix} \geq 0,$

LMI3: $\begin{bmatrix} \mathcal{M}_i & \mathcal{N}_i^T \\ \mathcal{N}_i & \mathbf{R}_i \end{bmatrix} \geq 0,\ i = 1, \dots, \nu,$

where

$$\mathcal{M}_i = \mathbf{A}_i^T\mathbf{H}_i + \mathbf{H}_i\mathbf{A}_i + \mathbf{Q}_i + \sum_{j=1}^{\nu}\pi_{ij}\mathbf{\Phi}_{ij}^T\mathbf{H}_j\mathbf{\Phi}_{ij},$$
$$\mathcal{N}_i = \mathbf{B}_i^T\mathbf{H}_i\mathbf{A}_i + \mathbf{S}_i,\ i = 1, \dots, \nu.$$

Step 2.

Find the gain matrix $\mathbf{K}_i$ $(i = 1, \dots, \nu)$ by the formula (9).

We have proved based on result of [1, 10] that in particular case $\mathbf{C}(i) = \mathbf{I}$ $(i \in \mathbb{N})$ this algorithm converges to the solution of (7). This property was not true in the numerical examples we have considered using toolbox LMISOLVER from SCILAB software if $\mathbf{C}(i) \neq \mathbf{I}$ $(i \in \mathbb{N})$, but the obtained control law (6) was stabilizing in all the cases. From this point of view it is interesting to study the property of this algorithm.

In the case of mode change uncertainty we can use the same algorithm with replacing $\mathbf{Q}$ by $\mathbf{Q}(\delta)$ $(\delta \in \Delta_0)$.

Finally in the case of the plant parameters uncertainty it is possible to propose the following algorithm.

*Step 1.*Find the gain matrix $\mathbf{K}_i$ $(i = 1, \dots, \nu)$ by the previous algorithm.

Step 2. Solve the LMI problem (27). If this problem is feasible, then $\mathbf{K}_i$ $(i = 1, \dots, \nu)$ is robust stabilizing gain matrix, else correct the matrix $\mathbf{Q}_i$ $(i = 1, \dots, \nu)$ and go to step1.

The convergence of these algorithms remains to be proved. Nevertheless the relationship of the robust output feedback regulator with the linear quadratic regulator established by this paper is helpful and important result.

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