TWO-NORM OPTIMAL CONTROLLERS DELIVER OPTIMAL ROBUST DISTURBANCE ATTENUATION

Anthony M. Holohan*

* School of Electronic Engineering, Dublin City University, Glasnevin, Dublin 9, Ireland.
E-mail: holohana@eeng.dcu.ie Phone: +353-1-700 5107 Fax: +1-353-1 700 5508

Keywords: Robust Disturbance Attenuation, Sensitivity Reduction.

Abstract

This paper shows that, in the SISO case, optimal mixed-sensitivity 2-norm controllers are also solutions to the optimal robust disturbance attenuation problem (ORDAP). That is, they deliver robust sensitivity reduction despite unstructured uncertainty at the optimal level. We explicitely identify one set of ORDAP weights for which such a controller is ORDAP optimal. Functional analysis duality theory is used to establish our conclusions.

1 Introduction

Let us begin by stating briefly the control problems with which this paper deals.

1.1 Problem Statement

Consider the standard setup shown below in Figure 1.

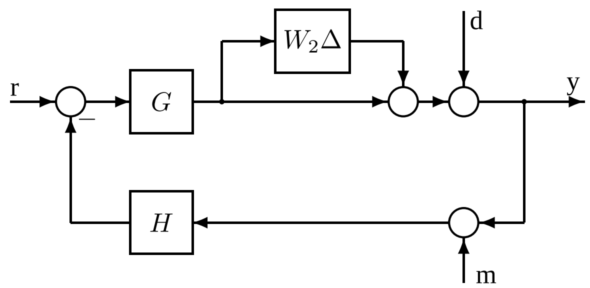

Figure 1

In the figure, signals entering a summer should be taken with a positive sign unless indicated otherwise. All system blocks are single-input single-output (SISO). The dependence on the Laplace transform variable s or on angular frequency ω will generally be suppressed in our notation. The plant model, the nominal plant, is G, and the actual plant is

$$G_\Delta = (1 + W_2\Delta)G$$

Here, $W_2\Delta$ represents the (unstructured) uncertainty in the plant model. So G is G_Δ with $\Delta = 0$. The value of the transfer function $\Delta(j\omega)$ which corresponds to the actual physical system is unknown, but it is internally stable and obeys $|\Delta(j\omega)| \leq 1$ at all frequencies ω. In other words, Δ belongs to the unit ball in $\mathcal{H}_\infty$, i.e. $\Delta \in \mathcal{BH}_\infty$. The transfer function W_2 is a stable and minimum phase weighting, which describes how the "size" of the plant uncertainty varies with ω. The controller is H. The sensitivity and complementary sensitivity functions are, respectively,

$$S_\Delta = \frac{1}{1 + G_\Delta H}, \qquad T_\Delta = \frac{G_\Delta H}{1 + G_\Delta H}$$

These functions depend on Δ (and also on s, of course). The nominal sensitivity function and nominal complementary sensitivity function will be denoted by S and T respectively, so that

$$S = \frac{1}{1 + GH}, \qquad T = \frac{GH}{1 + GH}$$

These transfer functions are, respectively, S_Δ and T_Δ evaluated at $\Delta = 0$.

This paper inter-relates two mixed-sensitivity frequency-domain optimal synthesis problem. We deal with the SISO case only. We consider the standard 2-norm optimization problem

$$\lambda_2 = \inf_{Q \in \mathcal{H}^\infty} \int_{-\infty}^{+\infty} |V_1 S(j\omega)|^2 + |V_2 T(j\omega)|^2 \mathrm{d}\omega \qquad (1)$$

with weights V_1 and V_2, and the so-called optimal robust disturbance attenuation problem

$$\lambda_R = \inf_{Q \in \mathcal{H}^\infty} \| \, |W_1 S(j\omega)| + |W_2 T(j\omega)| \, \|_\infty \qquad (2)$$

with weights W_1 and W_2. The motivation for the latter is the fact that

$$\|W_1 S_\Delta\|_\infty \leq 1 \;\; \forall \; \Delta \in \mathcal{BH}_\infty \qquad (3)$$
$$\Leftrightarrow \| \, |W_1 S(j\omega)| + |W_2 T(j\omega)| \, \|_\infty \leq 1$$

Note that one needs the constraint $\| \, W_2 T \, \|_\infty < 1$ for robust stability. This is perhaps the "simplest" μ-synthesis problem after the solved $\mathcal{H}_\infty$ control problem. There is a literature on this problem, including [1,26,7, 8,11,14,16].

1.2 Main Result

We show that a solution of the former problem (eqn. (1)) is necessarily a solution of the latter problem (eqn. (2)), and we identify one pair of weights for which it is optimal. Thus, starting with a plant G and weights V_1 and V_2, consider obtaining the optimal 2-norm controller (for eqn. (1)). We show that this controller is also ORDAP optimal (for eqn. (2)) for the weights

$$W_1 = \lambda_R \frac{V_1^2 [S_o]_{op}}{Z^2}, \qquad W_2 = \lambda_R \frac{V_2^2 [T_o]_{op}}{Z^2} \qquad (4)$$

where Z is defined by the spectral factorization

$$Z^*Z = V_1^*V_1S_o^*S_o + V_2^*V_2T_o^*T_o \tag{5}$$

and where S_o and T_o are the 2-norm optimal sensitivities. Here, $[\ldots]_{op}$ denotes the outer part of a transfer function. The parameter λ_R is simply a scaling of the weights, and can be dropped if desired. The author's idea of using 2-norm theory to tackle another synthesis problem owes something to [12].

1.3 Assumptions

Next, we state our assumptions.

The assumptions on the plant are as follows.

[A1] The nominal plant G is SISO, real-rational and has no hidden unstable poles (i.e. is stabilizable and detectable).

The weight selection rules are as follows.

[A2] V_1 and V_2 are real-rational and have no open right half plane poles or zeros. They have no finite imaginary axis zeros. They have no finite imainary axis poles (except as in A3). Their relative degrees are such that $\delta_r(V_1S) = 1$ and $\delta_r(V_2T) = 1$ (with any biproper controller) where δ_r denotes relative degree.

Care is needed in dealing with any finite imaginary axis poles and zeros of the plant.

[A3] The finite imaginary axis poles of V_1 are exactly those of G and with the same multiplicity. The finite imaginary axis poles of V_2 are exactly those of G^{-1} and with the same multiplicity.

These poles will be cancelled in the products V_1S and V_2T. From eqn. (4), W_1 and W_2 will obey A3. (Imaginary axis poles and zeros are put into the outer factor here.) Also, the ORDAP weights will obey A2, but with $\delta_r(V_1S) = 0 = \delta_r(V_2T)$. A2 and A3 involve no loss of generality in the following sense. It can be shown that if a controller is 2-norm optimal for any real-rational weights, then it is also optimal for weights obeying A2 and A3 [9,10].

Section 2 gives some background material which will be needed. It describes the Youla parameterization, which will be used to formulate both problems as approximation problems in certain Banach spaces. It also outlines some standard duality theory for approximation problems on vector spaces. This theory is used to identify the maximization problem which is the dual of these minimization problems. Section 3 is devoted to the proof. It works by comparing the two sets of (alignment) conditions for optimality. Section 4 contains a textbook example. Section 5 contains our concluding remarks. This duality theory has been applied to optimal controller synthesis problems before, for instance [3,4,6, 7,8,11,14,15,16].

2 Background

We begin with some necessary background.

2.1 Notation

The notation to be used is as follows. The parameter s, the Laplace transform variable, will generally be dropped in our notation. The subscript "o" denotes the optimal value of that transfer function or vector. If $A(s)$ is a transfer function matrix, then $A^*(s)$ denotes $\overline{A(-\overline{s})}^T$, i.e. the complex conjugate transpose of $A(-\overline{s})$. Hence, for real-rational scalar $A(s)$, $A^*(j\omega)$ is the complex conjugate of $A(j\omega)$. Vector spaces and their subspaces will be denoted by upper case script letters. The prefix $\mathcal{B}$ then means the unit ball of that space. The inner and outer factors of a transfer function A will be denoted by $[A]_{ip}$ and $[A]_{op}$ respectively.

2.2 Youla Parameterization

First, we apply the Youla Parameterization [5]. We state the result for the SISO case as an excuse to specify our notation.

Theorem 1 (The Youla Parameterization) *Consider the problem of the stabilization of the plant G obeying A1 by the feedback controller H, (as in Figure 1 with $\Delta = 0$). Suppose that N, D, U and V are all stable transfer functions such that*

$$G = ND^{-1}, \qquad 1 = NU + DV \tag{6}$$

Then, as Q ranges over $\mathcal{H}^\infty$, all LTI controllers which yield a stable closed loop system are given by

$$H = \frac{-QD + U}{QN + V} \tag{7}$$

Simple algebra using the above then shows that

$$S = D(QN + V), \qquad T = N(-QD + U) \tag{8}$$

Let B_p and B_z denote the Blaschke products corresponding to all the ORHP poles and zeros of the plant, respectively. Define X and Y by $N = B_zX$ and $D = B_pY$. We use the fact that Blaschke products have unit magnitude on the imaginary axis. Hence, the ORDAP problem is to minimize over all stabilizing controllers (i.e. over $Q \in \mathcal{H}_\infty$)

$$\begin{aligned}\lambda_R &= \| \, |W_1S| + |W_2T| \, \|_\infty \\ &= \| \, |B_z^{-1}B_p^{-1}W_1S| + |B_z^{-1}B_p^{-1}W_2T| \, \|_\infty \\ &= \| \, |W_1DV + W_1DNQ| + |W_2NU - W_2DNQ| \, \|_\infty \\ &= \| \, |W_1YVB_z^{-1} + W_1XYQ| + |W_2XUB_p^{-1} - W_2XYQ| \, \|_\infty\end{aligned}$$

2.3 The 2-Norm Problem as a Vector Space Problem

The 2-norm problem of eqn. (1) can be cast as a vector space problem. The space needed is $\mathcal{L}^2 \times \mathcal{L}^2$ equipped with the norm

$$\| \, (a, b) \, \|_{2\times 2} = \left\| \sqrt{|a|^2 + |b|^2} \right\|_2, \quad (a, b) \in \mathcal{L}^2 \times \mathcal{L}^2 \tag{9}$$

This is a Hilbert space. So the problem is equivalent to the following

$$\lambda_2 = \inf_{Q \in \mathcal{H}_\infty} \| (V_1XYQ, -V_2XYQ) + \ldots$$

$$+ (V_1YVB_z^{-1}, V_2XUB_p^{-1}) \|_{2\times 2}$$

Now the set of vectors in $\mathcal{L}^2 \times \mathcal{L}^2$ which are of the form $(V_1XYQ, -V_2XYQ)$ for some $Q \in \mathcal{H}^\infty$ constitutes a vector space, $\mathcal{M}_0$ say, which is a linear subspace of $\mathcal{L}^2 \times \mathcal{L}^2$,

$$\mathcal{M}_0 = \{(a,b) \in \mathcal{L}^2 \times \mathcal{L}^2 | \; a = V_1XYQ, \ldots$$

$$b = -V_2XYQ, \; Q \in \mathcal{H}^\infty\}$$

So the optimal ORDAP synthesis problem is that of determining the (or a) vector in a subspace which best approximates some fixed given vector. There is a well developed theory for handling such point/subspace approximation problems [13].

2.4 ORDAP as a Vector Space Problem

The ORDAP problem can also be cast as a vector space problem. The space needed is $\mathcal{L}^\infty \times \mathcal{L}^\infty$ equipped with the norm

$$\| (a,b) \|_{\infty\times\infty} = \| \, |a| + |b| \, \|_\infty, \;\; (a,b) \in \mathcal{L}^\infty \times \mathcal{L}^\infty \quad (10)$$

So the problem is equivalent to

$$\lambda_R = \inf_{Q \in \mathcal{H}_\infty} \| (W_1XYQ, -W_2XYQ) + \ldots$$

$$+ (W_1YVB_z^{-1}, W_2XUB_p^{-1}) \|_{\infty\times\infty}$$

Now the set of vectors in $\mathcal{L}^\infty \times \mathcal{L}^\infty$ which are of the form $(W_1XYQ, -W_2XYQ)$ for some $Q \in \mathcal{H}^\infty$ constitutes a vector space, $\mathcal{N}_0$ say, which is a linear subspace of $\mathcal{L}^\infty \times \mathcal{L}^\infty$,

$$\mathcal{N}_0 = \{(a,b) \in \mathcal{L}^\infty \times \mathcal{L}^\infty | \; a = W_1XYQ, \ldots$$

$$b = -W_2XYQ, \text{ some } Q \in \mathcal{H}^\infty\}$$

So the problem is again in the form of finding a vector $(W_1XYQ, -W_2XYQ)$ in the subspace $\mathcal{N}_0$ that best approximates a given vector $-(W_1YVB_z^{-1}, W_2UXB_p^{-1})$.

2.5 Duality Theory

Vector space duality theory provides a powerful methodology for tackling many optimization problems. The following two theorems treat point/subspace optimal approximation problems [13].

Theorem 2 *Suppose that $\mathcal{X}$ is a normed vector space with subspace $\mathcal{M}$, that its dual space is $\mathcal{X}^d$, and that $x \in \mathcal{X}$. Then*

(a)

$$\inf_{m \in \mathcal{M}} \| x - m \| = \max_{m^d \in \mathcal{BM}^\perp} | < x, m^d > | \quad (11)$$

and the maximum on the right is attained for some $m^d \in \mathcal{BM}^\perp$, say by m_o^d.

(b) *A sufficient condition for $m_o \in \mathcal{M}$ to attain the minimum on the left is that there exist $m_o \in \mathcal{BM}^\perp$ which is aligned with $x - m_o$.*

Theorem 3 *Suppose that $\mathcal{X}$ is a normed vector space with subspace $\mathcal{M}$, that its dual space is $\mathcal{X}^d$, and that $x^d \in \mathcal{X}^d$. Then*

(a)

$$\min_{m^d \in \mathcal{M}^\perp} \| x^d - m^d \| = \sup_{m \in \mathcal{BM}} | < m, x^d > | \quad (12)$$

and the minimum on the left is attained for some $m^d \in \mathcal{M}^\perp$, say by m_o^d.

(b) *A sufficient condition for $m_o^d \in \mathcal{M}^\perp$ to attain the minimum on the left is that there exist $m_o \in \mathcal{BM}$ which is aligned with $x^d - m_o^d$.*

For the proofs, see [13]. They involve a straightforward application of the Hahn-Banach Theorem.

To apply the above duality theory, we need to identify the various subspaces required.

2.6 2-Norm Duality

Being a Hilbert space, $\mathcal{L}^2 \times \mathcal{L}^2$ is its own dual, and with the same norm.

Theorem 4 *Assume that A1-A3 are obeyed. Consider the subspace of $\mathcal{L}^2 \times \mathcal{L}^2$ given by*

$$\mathcal{M} = \{(a,b) \in \mathcal{L}^2 \times \mathcal{L}^2 | \; a_2 = \frac{V_2^* l_2 + V_1 h_2^*}{\Lambda_V}, \ldots$$

$$b_2 = \frac{V_1^* l_2 - V_2 h_2^*}{\Lambda_V}, \; l_2 \in \mathcal{L}^2, h_2 \in \mathcal{H}^2\}$$

Then $\mathcal{M}^\perp = \{(x,y) \in \mathcal{L}^2 \times \mathcal{L}^2 | x = V_1XYQ, \ldots$

$$y = -V_2XYQ, Q \in \mathcal{H}^\infty\} = \mathcal{M}_0$$

where Λ_V comes from the spectral factorization

$$\Lambda_V^* \Lambda_V = V_1^* V_1 + V_2^* V_2$$

2.7 ORDAP Duality

The norm of interest on $\mathcal{L}^\infty \times \mathcal{L}^\infty$ is that of eqn. (10). Define

$$\| (a,b) \|_{1\times 1} = \int \max\{|a|, |b|\} \mathrm{d}\omega$$

which is a norm on the space $\mathcal{L}^1 \times \mathcal{L}^1$. These two normed spaces are related as follows.

Theorem 5 *The space $\mathcal{L}^\infty \times \mathcal{L}^\infty$ with norm $\| (x,y) \|_{\infty\times\infty}$ is the dual of the space $\mathcal{L}^1 \times \mathcal{L}^1$ with norm $\| (a,b) \|_{1\times 1}$*

Theorem 6 *Assume that A1-A3 are obeyed. Consider the subspace of $\mathcal{L}^1 \times \mathcal{L}^1$ given by*

$$\mathcal{N} = \{(a,b) \in \mathcal{L}^1 \times \mathcal{L}^1 |\ a = (W_2^* l + W_1 h^*)/\Lambda_W, \ldots$$
$$b = (W_1^* l - W_2 h^*)/\Lambda_W,\ l \in \mathcal{L}^1, h \in \mathcal{H}^1\}$$

where Λ_W comes from the spectral factorization

$$\Lambda_W^* \Lambda_W = W_1^* W_1 + W_2^* W_2$$

Then $\mathcal{N}^\perp = \{(x,y) \in \mathcal{L}^\infty \times \mathcal{L}^\infty | x = W_1 XYQ, \ldots$
$$-W_2 XYQ), Q \in \mathcal{H}^\infty\} = \mathcal{N}_0$$

The proofs of the previous three theorems are straightforward, see [13,11].

Note that if the plant has finite imaginary axis poles or zeros, then the weights have finite imaginary axis poles. However, the multiplicity of such poles in $W_1^* W_1$ and $W_2^* W_2$ will be even, so they present no difficulties when doing the spectral factorization. Such poles will then be poles of Λ_W also.

3 Proof

In this section, we establish the main result in a number of steps. The characterization of optimality by the alignment condition is investigated for both synthesis problems. Then, these two sets of alignment conditions are compared.

3.1 2-Norm Alignment Conditions

Under our assumptions, we know how to explicitely compute the minimizing Q_o and the maximizing $(a_{2o}, b_{2o}) \in \mathcal{BM}$. Hence, they exist. Then, let $(x_{2o}, y_{2o}) = (B_z^{-1} B_p^{-1} V_1 S_o, B_z^{-1} B_p^{-1} V_2 T_o)$, and (x_{2o}, y_{2o}) is aligned with (a_{2o}, b_{2o}). The details are as follows.

$$\lambda_2 = | < (x_{2o}, y_{2o}), (a_{2o}, b_{2o}) > |$$
$$\leq \| (x_{2o}, y_{2o}) \|_{2\times 2} \| (a_{2o}, b_{2o}) \|_{2\times 2}$$

The alignment condition (i.e. the condition for equality above) is easy in the 2-norm case. Equality holds

$$\Leftrightarrow\ \lambda_2 (a_{2o}, b_{2o}) = (x_{2o}, y_{2o})$$
$$\Leftrightarrow\ \lambda_2 a_{2o} = V_1 YVB_z^{-1} + V_1 XYQ,\ \text{ and}$$
$$\lambda_2 b_{2o} = -V_2 XYQ + V_2 XUB_p^{-1}$$
$$\Leftrightarrow\ \lambda_2 a_{2o} = B_z^{-1} B_p^{-1} V_1 S_o,\ \text{ and }\ \lambda_2 b_{2o} = B_z^{-1} B_p^{-1} V_2 T_o$$
$$\Leftrightarrow\ \lambda_2 \frac{V_2^* l_{2o} + V_1 h_{2o}^*}{\Lambda_V} = B_z^{-1} B_p^{-1} V_1 S_o \text{ and}$$
$$\lambda_2 \frac{V_1^* l_{2o} - V_2 h_{2o}^*}{\Lambda_V} = B_z^{-1} B_p^{-1} V_2 T_o$$

Solving for h_{2o}^* and gives l_{2o},

$$\Leftrightarrow\ \lambda_2 h_{2o}^* = B_z^{-1} B_p^{-1} \left(\frac{V_1^* V_1 S_o - V_2^* V_2 T_o}{\Lambda_V^*} \right) \tag{13}$$

and

$$\Leftrightarrow\ \lambda_2 l_{2o} = B_z^{-1} B_p^{-1} \left(\frac{V_1 V_2}{\Lambda_V^*} \right) \tag{14}$$

Under our assumptions, there is a unique choice of Q which makes the right hand side of eqn. (13) anti-stable and strictly proper, and this is the optimal Q. Then h_{2o} and l_{2o} are easily constructed to yield alignment. So this condition is necessary and sufficient for 2-norm optimality.

3.2 ORDAP Alignment Conditions

Our treatment here is modeled after [11]. The analysis is little more than the conditions for equality in Holder's inequalities.

Let $\quad x_o = W_1 XYQ_o + W_1 YVB_z^{-1}$
and $\quad y_o = -W_2 XYQ_o + W_2 XUB_p^{-1})$
so $\quad (x_o, y_o) = (WS_o B_z^{-1} B_p^{-1}, W_2 T_o B_z^{-1} B_p^{-1})$
where Q_o attains the minimum in Theorem (3). Existence is assurred. Suppose that $(a_{1o}, b_{1o}) \in \mathcal{L}^1 \times \mathcal{L}^1$ attains the supremum in Theorem (3). Here, existence is not guaranteed in general. Normalize (a_{1o}, b_{1o}) w.l.o.g. so that

$$\| (a_{1o}, b_{1o}) \|_{1\times 1} = 1$$

Then

$$\lambda_R = \| |W_1 S_o| + |W_2 T_o| \|_\infty$$
$$= | < (a_{1o}, b_{1o}), (x_o, y_o) > |$$
$$= \left| \int_{-\infty}^{\infty} (x_o a_{1o}^* + y_o b_{1o}^*) \mathrm{d}\omega \right|$$
$$\leq \int_{-\infty}^{\infty} |x_o a_{1o}^* + y_o b_{1o}^*| \mathrm{d}\omega$$

with equality holding when either (i) $\angle(x_o a_{1o}^* + y_o b_{1o}^*) = 0$, or (ii) $x_o a_{1o}^* + y_o b_{1o}^* = 0$, a.e.

$$\leq \int_{-\infty}^{\infty} (|x_o a_{1o}| + |y_o b_{1o}|) \mathrm{d}\omega$$

with equality holding when either (iii) $\angle x_o a_{1o}^* = \angle y_o b_{1o}^*$, or (iv) $x_o a_{1o}^* = 0$, or (v) $y_o b_{1o}^* = 0$, a.e.

$$\leq \int_{-\infty}^{\infty} \max\{|a_{1o}|, |b_{1o}|\} (|x_o| + |y_o|) \mathrm{d}\omega$$

with equality holding when either (vi) $|a_{1o}| = |b_{1o}|$, or (vii) $|a_{1o}| > |b_{1o}|$ and $y_o = 0$, or (viii) $|b_{1o}| > |a_{1o}|$ and $x_o = 0$, a.e.

$$\leq \| |x_o| + |y_o| \|_\infty \int_{-\infty}^{\infty} \max\{|a_{1o}|, |b_{1o}|\} \mathrm{d}\omega$$

with equality holding when either (ix) $\lambda_R = |x_o| + |y_o|$, or (x) $\max\{|a_{1o}|, |b_{1o}|\} = 0$, a.e.

$$= \| |W_1 S_o| + |W_2 T_o| \|_\infty = \lambda_R$$

since the norm of (a_{1o}, b_{1o}) has been normalized to one.

Since the above expressions are of the form "$\lambda_R \leq \ldots \leq \lambda_R$", we conclude that the alignment condition implies that each of the inequalities must, in fact, be an equality.

It is clear that sufficient conditions for alignment are then (i), (iii), (vi) and (ix). These may be written as

$$\angle W_1 S_o B_z^{-1} B_p^{-1} a_{1o}^* = 0 \tag{15}$$

$$\angle W_2 T_o B_z^{-1} B_p^{-1} b_{1o}^* = 0 \tag{16}$$

$$|a_{1o}| = |b_{1o}| \tag{17}$$

$$|W_1 S_o| + |W_2 T_o| = \lambda_R \tag{18}$$

3.3 Comparing the 2-norm and ORDAP Alignment Conditions

Given the weights V_1 and V_2, suppose that we have found the solution of the 2-norm problem of eqn. (1). We claim that this solution of eqn (1) (S_o, T_o) is then the solution of a certain ORDAP problem, namely that of eqn. (2) with weights (W_1, W_2) as defined in eqn. (4). This is established by using the 2-norm solution to construct vectors a_{1o} and b_{1o} which achieve ORDAP alignment, so that part (b) of Theorem 3 then establishes ORDAP optimality.

Define the Blaschke products B_1 and B_2 by

$$B_1 B_p = [S_o]_{ip}, \qquad B_2 B_z = [T_o]_{ip}$$

So B_1 and B_2 are the Blaschke products for the ORHP poles and zeros of the controller respectively. Let

$$a_{1o} = \frac{V_1^* V_1 S_o B_z^{-1} B_p^{-1}}{W_1^*} \quad b_{1o} = \frac{V_2^* V_2 T_o B_z^{-1} B_p^{-1}}{W_2^*} \tag{19}$$

Formally, two things must be established. First, we show that this vector (a_{1o}, b_{1o}) has the required structure, viz. $(a_{1o}, b_{1o}) \in \mathcal{N}$. Secondly, we show that this (a_{1o}, b_{1o}) achieves alignment. ORDAP optimality then follows from part (b) of Theorem 3.

Solving

$$a_{1o} = \frac{W_2^* l_{1o} + W_1 h_{1o}^*}{\Lambda_W}, \quad \text{and} \quad b_{1o} = \frac{W_1^* l_{1o} - W_2 h_{1o}^*}{\Lambda_W}$$

for h_{1o}^* and l_{1o} gives

$$l_{1o} = \frac{W_2 a_{1o} + W_1 b_{1o}}{\Lambda_W^*}, \quad \text{and} \quad h_{1o}^* = \frac{W_1^* a_{1o} - W_2^* b_{1o}}{\Lambda_W^*}$$

Using eqn. (19) then gives

$$h_{1o}^* = \left(\frac{B_z^{-1} B_p^{-1}}{\Lambda_W^*} \right) (V_1^* V_1 S_o - V_2^* V_2 T_o)$$

and

$$l_{1o} = \left(\frac{B_z^{-1} B_p^{-1} W_1 W_2}{\Lambda_W^*} \right) \left(\frac{V_1^* V_1 S_o}{W_1^* W_1} + \frac{V_2^* V_2 T_o}{W_2^* W_2} \right)$$

It is easily checked that A1-A3 ensure that h_{1o}^* and l_{1o} are strictly proper, and have no finite imaginary axis poles. Eqn. (13) shows that h_{1o}^* is anti-stable, and so is in $\mathcal{H}_1$. Hence $(a_{1o}, b_{1o}) \in \mathcal{N}$ as required.

Eqn. (19) gives

$$W_1 B_z^{-1} B_p^{-1} S_o a_{1o}^* = V_1^* V_1 S_o^* S_o$$

and

$$W_2 B_z^{-1} B_p^{-1} T_o b_{1o}^* = V_2^* V_2 T_o^* T_o$$

showing that eqn. (15) and eqn. (16) are obeyed. Using eqns. (19) and (4) shows that

$$a_{1o}^* a_{1o} = \frac{(V_1^* V_1 V_2^* V_2)^2}{A^* A} = b_{1o}^* b_{1o}$$

establishing eqn. (17). Using eqn. (4) to substitute for W_1 and W_2 in

$$|W_1 S_o(j\omega)| + |W_2 T_o(j\omega)|$$

and then using eqn. (5) establishes eqn. (18). This shows that the conditions for alignment are obeyed.

Part (b) of Theorem 3 now establishes that this Q_o is an optimal solution of the ORDAP problem of eqn. (2).

4 Example

Suppose that the plant is $G = 1/s$. Solving eqn. (6),

$$N = \frac{1}{s+1}, \quad D = \frac{s}{s+1}, \quad U = 1, \quad D = 1$$

Suppose that the 2-norm weights (obeying A2 and A3) are $V_1 = 2/s$ and $V_2 = 1$. Then, a spectral factorization gives $\Lambda_V = (s+2)/s$. Finally, Q_o must be such that the right hand side of eqn. (13) is anti-stable. It is

$$B_z^{-1} B_p^{-1} \left(\frac{V_1^* V_1 S_o - V_2^* V_2 T_o}{\Lambda_V^*} \right)$$

Simple algebra confirms that it may be written as

$$= B_z^{-1} B_p^{-1} \left(Q_o N D \Lambda_V + \frac{V_1^* V_1 D V - V_1^* V_1 N U}{\Lambda_V} \right)$$

The only unknown in the above is now Q_o, so inserting the other terms,

$$= Q_o \frac{(s+2)}{(s+1)^2} - \frac{(s+4)}{(s+1)(s-2)}$$

Compute the partial fraction expansion of the term on the right. This quantity must be anti-stable. It is easily checked that the (unique) optimal Q_o must be

$$Q_o = -\frac{(s+1)^2}{(s+2)} \frac{1}{s+1} = -\frac{(s+1)}{(s+2)}$$

Using eqn. (8) gives

$$S_o = \frac{s}{s+2}, \quad T_o = \frac{2}{s+2}$$

and the optimal controller is $H = 2$. Then eqn. (4) gives (with $\lambda_R = 1$)

$$W_1 = \frac{(s+2)}{2s}, \quad W_2 = \frac{(s+2)}{4}$$

We may then conclude that (from eqn. (3))

$$\left|\frac{(s+2)}{2s} S_\Delta(s)\right| \leq 1 \;\; \forall\, s = j\omega, \quad \forall\, \Delta \in \mathcal{BH}_\infty$$

or

$$|S_\Delta(j\omega)| \leq \left|\frac{2j\omega}{(j\omega+2)}\right| \;\; \forall\, \omega, \quad \forall\, \Delta \in \mathcal{BH}_\infty$$

Of course, the above fact is easily checked separately. The non-trivial point is that no other controller can do better, in the above ORDAP sense.

5 Conclusion and Discussion

The following has been established.

Theorem 7 *Suppose that the plant $G(s)$ obeys A1 and that the 2-norm weights (V_1, V_2) obey A2 and A3. Let (S_o, T_o) denote the optimal solution of the 2-norm problem of eqn. (1). Then, this (S_o, T_o) is also the optimal solution of the ORDAP problem of eqn. (2) with weights (W_1, W_2) given by eqn. (4).*

It should now be possible to design ORDAP optimal controllers for the SISO case. However, one would have to rely on iterative trial and error to find how (V_1, V_2) should be chosen to obtain the desired (W_1, W_2).

Of course, the real motivation for this line of research is to solve the ORDAP problem by reversing the above analysis. Thus, we should like to begin with the ORDAP weights, and then identify the 2-norm weights for which the same controller is optimal. Research along these lines is at an advanced stage.

References

[1] J. S. Bird and B. A. Francis, "On the Robust Disturbance Attenuation Problem", *Proc. IEEE Conf. on Decision and Control*, Athens, Greece, pp. 1804-1809, Dec. 1986.

[2] M. J. Chen and C. A. Desoer, "The Problem of Guaranteeing Robust Disturbance Rejection in Linear Multivariable Feedback Systems", *Int. J. of Control*, **37**, pp. 305-313, 1983.

[3] M. A. Dahleh and J. B. Pearson, "l^1 Optimal Feedback Controllers for MIMO Discrete-Time Systems", *IEEE Trans. on Automatic Control*, **32**, pp. 314-322, April 1987.

[4] M. A. Dahleh and J. B. Pearson, "L^1 Optimal Compensators for Continuous Time Systems", *IEEE Trans. on Automatic Control*, **32**, pp. 889-895, Oct. 1987.

[5] C. A. Desoer, R.-W. Liu, J. Murray and R. Saeks, "Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis", *IEEE Trans. on Automatic Control*, **25**, pp. 399-412, June 1980.

[6] S.M. Djouadi, "Exact Solution to the Non-standard $\mathcal{H}^\infty$ Problem", *Proc. IEEE Conf. on Decision and Control*, **3**, pp. 2843-2848, 1998.

[7] S.M. Djouadi and G. Zames, "On Optimal Robust Disturbance Minimization", *Proc. American Control Conf.*, **3**, pp. 1531-1535, 1998.

[8] J.A. Donnellan and A.M. Holohan, "Tight Bounds for Robust Sensitivity Reduction from Functional Analysis", *Proc. European Control Conf.*, 2001, pp. 1548–1552.

[9] A.M. Holohan and G. Meinsma, "Inverse Problems in 2-norm and Infinity-norm Controller Synthesis", Memo No. 1459, University of Twente, 1998.

[10] A.M. Holohan and G. Meinsma, "On Inverse Problems in 2-norm and ∞-norm Controller Synthesis", *Proc. IEEE Conf. on Decision and Control*, San Diego, 1999, pp. 3627-3632.

[11] A.M. Holohan and M.G. Safonov, "Neoclassical Control Theory: A Functional Analysis Approach to Optimal Frequency Domain Controller Synthesis", in; Control and Dynamic Systems, Ed. C. T. Leondes, **50**, Academic Press, 1992.

[12] H. Kwakernaak, "Minimax Frequency Domain Performance and Robustness Optimization of Linear Feedback Systems", *IEEE Trans. on Automatic Control*, **30**, pp. 994-1004, Oct. 1985.

[13] D. G. Luenberger, "*Optimization by Vector Space Methods*", Wiley, 1969.

[14] J.G. Owen and G. Zames, "Duality Theory of Robust Disturbance Attenuation", *Automatica*, 29, 1993, pp. 695-705.

[15] G. Zames and B. A. Francis, "Feedback, Minimax Sensitivity, and Optimal Robustness", *IEEE Trans. on Automatic Control*, **28**, pp. 585-601, May 1983.

[16] G. Zames and J.G. Owen, "Duality Theory for MIMO Robust Disturbance Rejection", *IEEE Trans. on Automatic Control*, 38, 1993, pp. 743-752.