STABILITY CRITERIA FOR SYSTEMS WITH BOUNDED UNCERTAIN TIME-VARYING DELAY

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Keywords: Time-varying delay, Robust stability, Integral Quadratic Constraint (IQC), Linear Matrix Inequality (LMI)

Abstract

Stability in presence of bounded uncertain time-varying time delays in the feedback loop of a system is studied. The stability problem is treated in the Integral Quadratic Constraint (IQC) framework. The stability criterion is formulated as frequency dependent linear matrix inequalities. The criterion can be equivalently formulated as a Semi-Definite Program (SDP) using Kalman-Yakubovich-Popov lemma. Therefore, checking the criterion can be done efficiently by using various SDP solvers.

1 Introduction

Time delay often occurs in engineering systems. Since the existence of time delay usually causes instability of the system, the study on the time-delay systems has received considerable attentions, and time-delay robustness has been a large research topic. Many stability criteria for time-delay systems can be found in the literature. Stability criteria for time-delay systems tend to fall into one of the two categories: delay-independent and delay-dependent. As the name implies, delay-independent criteria provide conditions which guarantee stability for any length of the time delay. On the other hand, delay-dependent criteria exploit a priori knowledge of upper-bounds on the amount of time-delay. These criteria are generally less conservative than delay-independent criteria since more information about the time-delay is assumed to be known.

Let us consider the following linear time delay system

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) \tag{1}$$

where $\tau(t)$ is a unknown time-varying parameter which satisfies

$$0 \le \tau(t) \le h, \;\; |\dot{\tau}(t)| \le d, \quad \forall t \ge 0, \tag{2}$$

$x \in \mathbf{R}^n$ is the state, A and $A_d \in \mathbf{R}^{n \times n}$ are constant matrices. We assume that the state $x(t) \equiv 0$ for all $t \le 0$. In this paper, we shall develop delay-dependent conditions for robust stability of time-delay system (1). More specifically, given a pair of scalars (h, d), our objective is to derive conditions under which the delay system (1) is robustly stabile for all $\tau(t)$ that satisfies condition (2).

If the delay parameter τ is unknown but constant, then the energy of $x(t-\tau)$ is the same as the energy of $x(t)$. Hence, a simple but conservative delay-independent stability criterion for the system,

$$\sup_{\omega} \|(j\omega I_n - A)^{-1} A d\| < 1,$$

immediately follows the small gain theorem. The exact condition for delay-independent stability was derived in [2] using structured singular value. Many of the recent research works focus on delay-dependent stability. Some of them were derived using frequency-domain analysis (by μ or IQC analysis) [14, 5, 17, 7], while others use time-domain analysis (by various Lyapunov-Krasovskii functions) [16, 11, 13, 4]. See also [10] and [15] for the recent development on stability analysis of time delay systems.

When the delay parameter is time-varying, stability analysis is more involved. One of the difficulties, for instance, is that the delay operator is no longer energy-preserving. In fact, if there is no restriction on the speed of variation; i.e., no bound on $\dot{\tau}(t)$, then the delay operator is not even a bounded operator on the $\mathbf{L}_2$ space no matter how small the length of the delay is. To see this, let $v(t)$ and $\tau(t)$ be

$$v(t) = \begin{cases} 1 & t = 0, \\ 0 & \text{otherwise} \end{cases} \qquad \tau(t) = \begin{cases} t & t \in [0, h], \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

Then $v(t - \tau(t))$ is equal to 1 for $t \in [0, c]$ and 0 otherwise. The energy of $v(t - \tau(t))$ is equal to h while the energy of of $v(t)$ is equal to 0, which implies that the gain of the delay operator is infinite. Hence, intuitively, systems with time-varying delays are more unstable than those with constant time delays, and it is not obvious that stability criteria for system with constant time delays can be easily generalized for time-varying delay systems. Over the past few years, researchers have been working on stability analysis of linear systems with time-varying delays [12, 6, 1, 9, 3, 8]. All of these results are developed in the time-domain, based on Lyapunov's second method using different Lyapunov-Krasovskii functionals.

In this paper, we consider the stability problem where the delays in a closed-loop continuous-time system are bounded

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Notation: Symbol I_n is used to denote n-dimensional identity matrix. Given a matrix M, the transposition and the conjugate transposition are denoted by M' and M^*, respectively. A matrix M is called *positive definite* if M belongs to $H^{n\times n}$ and $x'Mx > 0$ for all $x \in \mathbf{C}^n, x \neq 0$. The notations $M > 0$ is used to denote positive definiteness. The positive semi-definiteness, negative definiteness, and negative semi-definiteness have similar definitions except that the "$>$" is replaced by "$\geq$", "$<$", and "$\leq$", respectively. We use $\mathbf{L}_2$ to denote the space of square summable functions defined on time interval $[0,\infty)$. Given a signal f in $\mathbf{L}_2$ space, we use $\|f\|_{\mathbf{L}_2}$ to denote the $\mathbf{L}_2$ norm of f. Given an bounded operator G on the $\mathbf{L}_2$ space, we use $\|G\|_{\mathbf{L}_2}$ to denote the $\mathbf{L}_2$ induced norm of G.

2 Main Results

Let $\mathcal{D}_\tau$ denote the time-delay operator and let Δ be $(\mathcal{D}_\tau - I)\circ\frac{1}{s}$. That is, $\mathcal{D}_\tau(v) := v(t-\tau(t))$, and

$$\Delta(v) := \int_t^{t-\tau(t)} v(\theta)d\theta \tag{4}$$

In this paper, we derive stability criteria for linear time delay systems based on Integral Quadratic Constraints (IQC) analysis. Given an operator $\mathcal{H}$ and a quadratic form $\sigma(v,w)$ defined on $\mathbf{L}_2$ space, we said that $\mathcal{H}$ satisfies IQC defined by σ if $\sigma(v,\mathcal{H}(v)) \geq 0$ for all $v \in \mathbf{L}_2$.

2.1 IQCs for Operators $\mathcal{D}_\tau$ and Δ

The following lemmas are the main technical results of this paper.

Lemma 1. *Operator $\mathcal{D}_\tau$ is bounded on $\mathbf{L}_2$ space if $d<1$. The $\mathbf{L}_2$ induced norm of $\mathcal{D}_\tau$ is equal to $1/\sqrt{1-d}$, and $\mathcal{D}_\tau$ satisfies integral quadratic constraints defined by*

$$\sigma_1(v,w) = \int_0^\infty \frac{1}{1-d}v(t)'X_1v(t) - w(t)'X_1w(t)dt, \tag{5}$$

where $X_1 = X_1'$ is any positive definite matrix.

Proof. Let $X_1 = X_1' > 0$, and let $v(t)$ be any $\mathbf{L}_2$ signal. We have

$$\begin{aligned}&\int_0^\infty v(t-\tau(t))'X_1v(t-\tau(t))dt\\ &= \int_{-\tau(0)}^\infty v(\theta)'X_1v(\theta)\frac{1}{1-\dot\tau(t(\theta))}d\theta\\ &\leq \int_0^\infty \frac{1}{1-d}v(\theta)'X_1v(\theta)d\theta\end{aligned}$$

The last inequality follows that $|\dot\tau(t)| \leq d$, $\forall t \geq 0$. This verifies that $\mathcal{D}_\tau$ satisfies IQC defined σ_1. Furthermore, by taking X_1 to be identity matrix, we immediately conclude that $\|\mathcal{D}_\tau\|_{\mathbf{L}_2} \leq \frac{1}{\sqrt{1-d}}$. To see $\|\mathcal{D}_\tau\|_{\mathbf{L}_2} = \frac{1}{\sqrt{1-d}}$, let us consider the following signal

$$v(t) = \begin{cases}1 & t\in[0,\frac{h}{d}-h],\\ 0 & \text{otherwise}\end{cases}\qquad \tau(t) = \begin{cases}dt & t\in[0,\frac{h}{d}],\\ h & \text{otherwise}\end{cases},$$

then one can easily verify that $v(t-\tau(t))$ is equal to 1 for $t\in[0,\frac{h}{d}]$, and equal to 0 otherwise. Hence we have

$$\frac{\|v(t-\tau(t))\|_{\mathbf{L}_2}^2}{\|v(t)\|_{\mathbf{L}_2}^2} = \frac{1}{1-d}.$$

This concludes the proof. We note that the $\mathbf{L}_2$-gain of $\mathcal{D}_\tau$ is **independent** of h as long as h is strictly greater than 0. ☐

Lemma 2. *Operator Δ is bounded on $\mathbf{L}_2$ space. The $\mathbf{L}_2$ induced norm of Δ_τ is equal to h, and Δ satisfies integral quadratic constraints defined by*

$$\sigma_2(v,w) = \int_0^\infty h^2v(t)'X_2v(t) - w(t)'X_2w(t)dt, \tag{6}$$

where $X_2 = X_2'$ is any positive definite matrix.

Proof. Let $X_2 = X_2' > 0$ and $w = \Delta(v)$. We have

$$w(t) = -\int_{t-\tau(t)}^t v(\theta)d\theta,$$

and

$$\begin{aligned}w(t)'X_2w(t) &= \left(\int_{t-\tau(t)}^t v(\theta)d\theta\right)' X_2 \left(\int_{t-\tau(t)}^t v(\eta)d\eta\right)\\ &= \int_{t-\tau(t)}^t \int_{t-\tau(t)}^t v(\theta)'X_2v(\eta)d\theta d\eta\end{aligned}$$

Hence, the followings follow the Cauchy-Schwartz inequality

$$\begin{aligned}w'X_2w &\leq \int_{t-h}^t (h\cdot v(\eta)'X_2v(\eta))^{\frac{1}{2}}\left(\int_{t-h}^t v(\theta)'X_2v(\theta)d\theta\right)^{\frac{1}{2}}d\eta\\ &\leq h\left(\int_{t-h}^t v(\theta)'X_2v(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{t-h}^t v(\eta)'X_2v(\eta)d\eta\right)^{\frac{1}{2}}\\ &\leq h\int_{t-h}^t v(\theta)'X_2v(\theta)d\theta\end{aligned}$$

This in turn implies

$$\int_0^\infty w(t)'X_2w(t)dt \le \int_0^\infty h\int_{t-h}^t v(\theta)'X_2v(\theta)d\theta dt$$
$$= h\int_0^\infty \left(\int_{-h}^0 v(t+s)'X_2v(t+s)ds\right)dt$$
$$\le h\int_{-h}^0 \left(\int_0^\infty v(t)'X_2v(t)dt\right)ds = \int_0^\infty h^2v(t)'X_2v(t)dt$$

This verifies the IQC defined by σ_2. By taking $X_2 = I$, it is obvious that the $\mathbf{L}_2$-gain of Δ is bounded by h.

To show that $\|\Delta\| = h$, let $\tau(t) = h$ for all t. Then operator Δ is an linear time-invariant operator such that the transfer function of Δ is equal to

$$\Delta(s) = \frac{e^{-sh}-1}{s}\cdot I$$

The $\mathbf{L}_2$ induced norm of Δ satisfies

$$\|\Delta\|^2_{\mathbf{L}_2} \ge \lim_{\omega\to 0}\left|\frac{e^{-j\omega h}-1}{j\omega}\right|^2 = h^2$$

Hence, $\|\Delta\|_{\mathbf{L}_2} \ge h$ which in turn implies that $\|\Delta\| = h$. Again, we note that the $\mathbf{L}_2$-gain of Δ is **independent** of d; that is, the variation of τ has no effect on the worst-case energy amplification. □

2.2 Stability Criteria based on IQC Analysis

Our results are based on the following transformation, similar to the one introduced in [17].

Lemma 3. *Let $M \in \mathbf{R}^{n\times n}$ be a constant matrix. Then the time-delay system (1) can be equivalently formulated as*

$$\begin{aligned}\dot{x}(t) &= (A+MA_d)x(t) + (I_n - M)A_dw_1(t)\\ &\quad + MA_dAw_2(t) + MA_d^2w_3(t)\end{aligned} \tag{7}$$

where $w_1(t) = x(t-\tau(t))$,

$$w_2(t) = \int_t^{t-\tau(t)} x(\theta)d\theta, \quad w_3(t) = \int_t^{t-\tau(t)} w_1(\theta)d\theta$$

Using (7), one can put the linear time delay system in standard Linear Fractional Transformation (LFT) setup for robustness analysis, as shown in Figure (1). The LTI system G has a state space representation

$$\begin{aligned}\dot{x}(t) &= \bar{A}x(t) + \bar{B}_1\tilde{w}_1(t) + \bar{B}_2\tilde{w}_2(t)\\ \tilde{v}_1(t) &= x(t)\\ \tilde{v}_2(t) &= \begin{bmatrix} x(t)\\ \tilde{w}_1(t)\end{bmatrix}\end{aligned} \tag{8}$$

where $\tilde{w}_1 = \mathcal{D}_\tau(\tilde{v}_1)$, $\tilde{w}_2(t) = \Delta(\tilde{v}_2)$, and matrices $\bar{A} = A + MA_d$, $\bar{B}_1 = (I_n - M)A_d$, $\bar{B}_2 = \begin{bmatrix} MA_dA & MA_d^2\end{bmatrix}$. Since systems (1), (7) and (8) are equivalent, stability of any

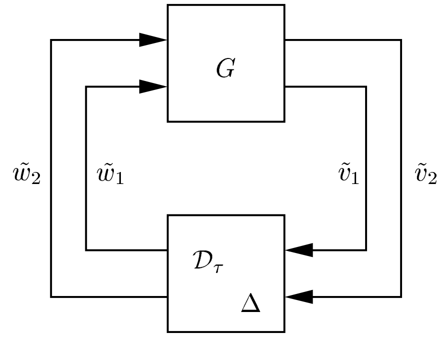

Figure 1: Equivalent System of (1)

one of the systems implies stability of the other two. Applying standard IQC analysis [14] to (8) and using the IQCs derived in Lemmas 1 and Lemma 2 for $\mathcal{D}_\tau$ and Δ, respectively, we have the following stability result for linear time delay system (1)

Theorem 1. *Linear time delay system (1) is stable if there exist symmetric positive-definite matrices X_1, X_2 of suitable dimensions, such that for some $\epsilon > 0$,*

$$G(j\omega)^*\begin{bmatrix}\frac{1}{1-d}X_1 & 0\\ 0 & h^2X_2\end{bmatrix}G(j\omega) - \begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix} \le -\epsilon I, \quad \forall\omega\in[0,\infty]$$

The above stability condition can be formulated as a semi-definite problem (or linear matrix inequality) using the well-known Kalman-Yakubovich-Popov lemma [18].

Theorem 2. *Linear time delay system (1) is stable if there exist matrices $P = P' > 0$ $X_1 = X_1' > 0$, $X_{21} = X_{21}'$, X_{22}, $X_{23} = X_{23}'$, and Q of suitable dimensions, such that*

$$\begin{bmatrix}X_{21} & X_{22}\\ X_{22}' & X_{23}\end{bmatrix} > 0$$

$$\begin{bmatrix}\Pi_{11} & (P-Q)A_d + h^2X_{22} & QA_dA & QA_d^2\\ * & h^2X_{23} - X_1 & 0 & 0\\ * & * & -X_{21} & -X_{22}\\ * & * & * & -X_{23}\end{bmatrix} < 0$$

where $\Pi_{11} = PA + A'P + QA_d + A_d'Q + \frac{1}{1-d}X_1 + h^2X_{21}$.

In cases where there is no restriction on $\dot{\tau}(t)$; i.e., $d = \infty$, or $d \ge 1$, we choose M to be I_n and reformulate (7) as

$$\begin{aligned}\dot{x}(t) &= (A+A_d)x(t) + A_d\tilde{w}(t)\\ \tilde{v}(t) &= \dot{x}(t) = (A+A_d)x(t) + A_d\tilde{w}(t)\end{aligned} \tag{9}$$

where $\tilde{w}(t) = \Delta(\tilde{v})$. Stability criterion for (1) can be obtained using IQC defined by (6) and we have the following theorem

Theorem 3. *Linear time delay system (1) is stable if there exist symmetric positive-definite matrix X, such that*

$$\begin{bmatrix}\Pi_{11} & PA_d + h^2(A+A_d)'XA_d\\ * & -X\end{bmatrix} < 0 \tag{10}$$

where $\Pi_{11} = P(A+A_d) + (A+A_d)'P + h^2(A+A_d)'X(A+A_d)$.

2.3 Further Stability Results

Stability criteria derived in the previous section are used on simple norm-bounded type of integral quadratic constraints for $\mathcal{D}_\tau$ and Δ, which might be very conservative. Less conservative criteria can be derived provided IQCs which better characterize $\mathcal{D}_\tau$ and Δ are available. In this section, stronger IQCs for $\mathcal{D}_\tau$ and Δ are derived

Lemma 4 (Swapping lemma for operator $\mathcal{D}_\tau$). *Let H be a stable linear time invariant system with state space representation $\dot{x}_h = A_h x + u, \quad x_h(0) = 0$, and let T denote the operator of multiplying $\dot{\tau}(t)$; i.e., $T(v(t)) := \dot{\tau}(t)v(t)$. Then*

$$\mathcal{D}_\tau \circ H(s) = H(s) \circ \mathcal{D}_\tau - H(s) \circ T \circ \mathcal{D}_\tau \circ sH(s).$$

Proof. Let v be any $\mathbf{L}_2$ function and define y and z to be

$$\dot{y}(t) = A_h y(t) + v(t), \;\; y(0) = 0$$
$$\dot{z}(t) = A_h z(t) + v(t - \tau(t)), \;\; z(0) = 0$$

Let $r(t) = y(t - \tau(t)) - z(t)$, and we have

$$\begin{aligned}\dot{r}(t) &= \dot{y}(t - \tau(t))(1 - \dot{\tau}(t)) - \dot{z}(t)\\ &= A_h(y(t - \tau(t)) - z(t)) - \dot{\tau}(t) \cdot \dot{y}(t - \tau(t))\\ &= A_h r(t) - \dot{\tau}(t) \cdot \mathcal{D}_\tau(\dot{y}(t)).\end{aligned}$$

which implies $\mathcal{D}_\tau(Hv) = H \circ \mathcal{D}_\tau(v) - H \circ T \circ \mathcal{D}_\tau(\frac{d}{dt}(Hv))$. This concludes the proof. □

Lemma 5 (Swapping lemma for operator Δ). *Let H and T be the operators as defined in Lemma 4. Then*

$$\Delta \circ H(s) = H(s) \circ \Delta - H(s) \circ T \circ \mathcal{D}_\tau \circ H(s)$$

Proof. The proof is similar to the one of Lemma 4. Let v be any $\mathbf{L}_2$ function and define y, z and x to be

$$\dot{y}(t) = A_h y(t) + v(t), \;\; y(0) = 0$$
$$\dot{z}(t) = A_h z(t) + \int_t^{t-\tau(t)} v(\theta)d\theta, \;\; z(0) = 0$$
$$r(t) = \int_t^{t-\tau(t)} y(\theta)d\theta - z(t)$$

One can easily verify that

$$\begin{aligned}\dot{r}(t) &= y(t - \tau(t))(1 - \dot{\tau}(t)) - y(t) - \dot{z}(t)\\ &= \int_t^{t-\tau(t)} \dot{y}(\theta)d\theta - \dot{z}(t) - \dot{\tau}(t) \cdot y(t - \tau(t))\\ &= A_h \int_t^{t-\tau(t)} y(\theta)d\theta - A_h z(t) - \dot{\tau}(t) \cdot y(t - \tau(t))\\ &= A_h r(t) - \dot{\tau}(t) \cdot \mathcal{D}_\tau(y(t)).\end{aligned}$$

which implies $\Delta(Hv) = H \circ \Delta(v) - H \circ T \circ \mathcal{D}_\tau(Hv)$. This concludes the proof. □

Using these swapping lemmas, the following integral quadratic constraints for $\mathcal{D}_\tau$ and Δ can be derived.

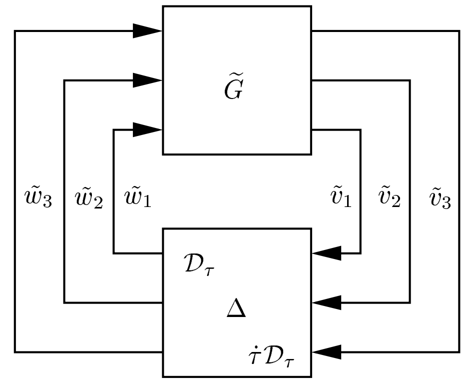

Figure 2: Extended System

Lemma 6. *Let $v(t)$ be any $\mathbf{L}_2$ signal, and $H(s)$ be any strictly proper stable transfer matrix of the form $(sI - A_h)^{-1}$. Let $y = Hv$, $z = Hw$, $r = H\hat{w}$, where $w = \mathcal{D}_\tau(v)$, $\hat{w}(t) = \dot{\tau}(t) \cdot \mathcal{D}_\tau(\dot{y})$. For any given $X_1 = X_1' > 0$, $X_2 = X_2' > 0$, the following integral inequalities hold*

$$\int_0^\infty \frac{1}{1-d} \begin{bmatrix} v \\ y \end{bmatrix}' X_1 \begin{bmatrix} v \\ y \end{bmatrix} - \begin{bmatrix} w \\ (z-r) \end{bmatrix}' X_1 \begin{bmatrix} w \\ (z-r) \end{bmatrix} \, dt \geq 0 \tag{11}$$

$$\int_0^\infty \frac{d^2}{1-d} \dot{y}' X_2 \dot{y} - \hat{w}' X_2 \hat{w} \; dt \geq 0 \tag{12}$$

Proof. By swapping lemma 4, we have $z - r = \mathcal{D}_\tau(y)$, $w = \mathcal{D}_\tau(v)$. Then integral inequality (11) follows immediately from Lemma 1. Let $\tilde{w} = \mathcal{D}_\tau(\dot{y})$. Then

$$\begin{aligned}\int_0^\infty \hat{w}(t)' X_2 \hat{w}(t) dt &= \int_0^\infty \dot{\tau}(t)^2 \tilde{w}(t)' X_2 \tilde{w}(t) dt\\ \leq d^2 \int_0^\infty \tilde{w}(t)' X_2 \tilde{w}(t) dt &\leq d^2 \int_0^\infty \frac{1}{1-d} \dot{y}(t)' X_2 \dot{y}(t) dt\end{aligned}$$

The last inequality follows Lemma 1 and that $\tilde{w} = \mathcal{D}_\tau(\dot{y})$. This concludes the proof. □

Lemma 7. *Let $v(t)$ be any $\mathbf{L}_2$ signal, and $H(s)$ be any strictly proper stable transfer matrix of the form $(sI - A_h)^{-1}$. Let $y = Hv$, $z = Hw$, $r = H\hat{w}$, where $w = \Delta(v)$, $\hat{w}(t) = \dot{\tau}(t) \cdot \mathcal{D}_\tau(y)$. For any given $X_3 = X_3' > 0$, $X_4 = X_4' > 0$, the following integral inequalities hold*

$$\int_0^\infty h^2 \begin{bmatrix} v \\ y \end{bmatrix}' X_3 \begin{bmatrix} v \\ y \end{bmatrix} - \begin{bmatrix} w \\ z-r \end{bmatrix}' X_3 \begin{bmatrix} w \\ z-r \end{bmatrix} \, dt \geq 0 \tag{13}$$

$$\int_0^\infty \frac{d^2}{1-d} y' X_4 y - \hat{w}' X_4 \hat{w} \; dt \geq 0 \tag{14}$$

Proof. By swapping lemma 5 we have $z - r = \Delta(y)$ and $w = \Delta(v)$. Then integral inequality (13) follows immediately from Lemma 2. Integral inequality (14), which is similar to (12), is derived using the same argument for deriving (12). □

$$\Pi(j\omega) = \begin{bmatrix} \frac{1}{1-d}\mathcal{H}_1^* X_1 \mathcal{H}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & h^2 \mathcal{H}_2^* X_2 \mathcal{H}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & \frac{d^2}{1-d} X_3 & 0 & 0 & 0 & 0 \\ * & * & * & -\mathcal{H}_1^* X_1 \mathcal{H}_1 & 0 & -\mathcal{H}_1^* X_1 \mathcal{H}_3 & 0 \\ * & * & * & * & -\mathcal{H}_2^* X_2 \mathcal{H}_2 & 0 & -\mathcal{H}_2^* X_2 \mathcal{H}_4 \\ * & * & * & * & * & -X_{31} - \mathcal{H}_3^* X_1 \mathcal{H}_3 & -X_{32} \\ * & * & * & * & * & * & -X_{33} - \mathcal{H}_4^* X_2 \mathcal{H}_4 \end{bmatrix} \quad (15)$$

Using the above IQCs, one can derive less conservative stability criteria for system (1). Consider system (8) and the following extended system

$$\begin{aligned} \dot{y}_1 &= A_{h1} y_1 + \tilde{v}_1 \\ \dot{z}_1 &= A_{h1} z_1 + \tilde{w}_1 \\ \dot{r}_1 &= A_{h1} r_1 + f_1 \\ \dot{y}_2 &= A_{h2} y_2 + \tilde{v}_2 \\ \dot{z}_2 &= A_{h2} z_2 + \tilde{w}_2 \\ \dot{r}_2 &= A_{h2} r_2 + f_2 \end{aligned} \quad (16)$$

where A_{h1} and A_{h2} are two given Hurwitz matrices. Signals $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{w}_1$, and $\tilde{w}_2$ are as defined in (8). Signals f_1 and f_2 are equal to $\dot{\tau} \cdot \mathcal{D}_\tau(\dot{y})$ and $\dot{\tau} \cdot \mathcal{D}_\tau(y)$, respectively. Let $\tilde{x}' = \begin{bmatrix} x' & y_1' & y_2' & z_1' & z_2' & r_1' & r_2' \end{bmatrix}$. Then systems (8) and (16) can be combined and put into the LFT setup as shown in 2, where $\tilde{v}_3' = \begin{bmatrix} \dot{y}_1' & y_2' \end{bmatrix}$ and $\tilde{w}_3' = \begin{bmatrix} f_1' & f_2' \end{bmatrix}$. State space representation of LTI system $\widetilde{G}$ can be easily derived using state space representation of (8) and (16). Here we omit the details.

Since the original time delay system (1) is embed in the extended system shown in Figure 2, stability of the extended system implies stability of the original system. Applying lemmas 4 to 7, one can verify that the following integral quadratic constraint holds for $v_t' := \begin{bmatrix} \tilde{v}_1' & \tilde{v}_2' & \tilde{v}_3' \end{bmatrix}$, and $w_t' := \begin{bmatrix} \tilde{w}_1' & \tilde{w}_2' & f_1' & f_2' \end{bmatrix}$.

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}_t(j\omega) \\ \hat{w}_t(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}_t(j\omega) \\ \hat{w}_t(j\omega) \end{bmatrix} d\omega \geq 0$$

where $\hat{v}_t(j\omega)$ and $\hat{w}_t(j\omega)$ are Fourier transform of $v_t(t)$ and $w_t(t)$, respectively. $\Pi(j\omega) = \Pi(j\omega)^*$ is of the form (15), where

$$\mathcal{H}_1 = \begin{bmatrix} I \\ H_1(j\omega) \end{bmatrix}, \quad \mathcal{H}_2 = \begin{bmatrix} I \\ H_2(j\omega) \end{bmatrix},$$
$$\mathcal{H}_3 = \begin{bmatrix} 0 \\ H_1(j\omega) \end{bmatrix}, \quad \mathcal{H}_4 = \begin{bmatrix} 0 \\ H_2(j\omega) \end{bmatrix},$$

and $H_1(j\omega) = (j\omega I - A_{h1})^{-1}$, $H_2(j\omega) = (j\omega I - A_{h2})^{-1}$. Matrices X_1, X_2 and

$$X_3 = \begin{bmatrix} X_{31} & X_{32} \\ X_{32}' & X_{33} \end{bmatrix}$$

are symmetric positive-definite matrices of suitable dimensions. We have the following stability theorem for the extended system in Figure 2.

Theorem 4. *The extended system is stable if there exist* $X_1 = X_1' > 0$, $X_2 = X_2' > 0$, *and*

$$X_3 = \begin{bmatrix} X_{31} & X_{32} \\ X_{32}' & X_{33} \end{bmatrix} > 0$$

such that for some $\epsilon > 0$

$$\begin{bmatrix} \widetilde{G}(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widetilde{G}(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \quad \forall \omega \in [0, \infty] \quad (17)$$

where $\Pi(j\omega)$ *is of the form (15).*

Remark 1. Using Kalman-Yakubovich-Popov lemma, one can derive an equivalent condition of (17) in terms of Linear Matrix Inequalities (LMIs). Here we omit the details.

3 Example

In this section, we present a numerical example to test the stability criteria proposed in this paper and compare them with existing criteria from the literature.

Consider the following system (Example 4 of [12]),

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - \tau(t)) \quad (18)$$

As indicated in [12], this system is not asymptotically stable independently of the size of the delay. When delay parameter $\tau(t)$ is arbitrarily time-varying; i.e., $d = \infty$, using the criterion in [12], one obtains asymptotically stability in case $0 \leq \tau(t) \leq 0.8571$. Using criterion (10) in Theorem 3, we obtain the bound $0 \leq \tau(t) \leq 0.9999$.

If we further restrict the variation of $\tau(t)$ to be strictly less than one, then stability regions as shown in Figure 3 can be obtained using Theorem 2. Here we compare our criteria with those provided in [9] and [3]. In Figure 3, the solid line is obtained using the criterion in Theorem 2, while the dotted and dashed lines are obtained using criteria given in [9] and [3], respectively.

As we can see from Figure 3, the stability region obtained using the criterion in [9] is much smaller than the region obtained using the criterion in Theorem 2. On the other hand, the criterion in Theorem 2 is slightly conservative compared to the criterion given in [3]. Currently, we are working on implementing the stability criterion given in Theorem 4, which hopefully is much less conservative and can give a better estimation of the stability region.

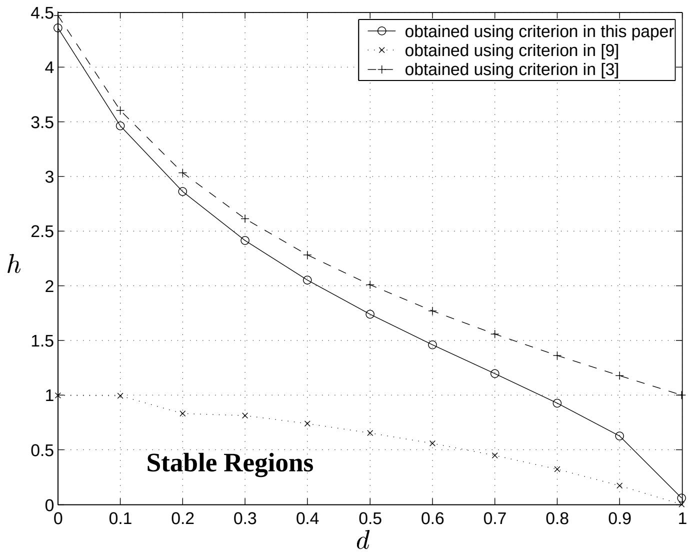

Figure 3: Stability Regions

4 Conclusions

Stability conditions for linear time delay systems were derived. The delay parameter is an unknown time-varying function for which the upper bounds on the magnitude and the variation are given. The influence of time-varying delay is modelled as uncertainties in the system, and integral quadratic constraints were derived to characterize the effect of these uncertain operators. Conditions for stability were then derived based on IQC analysis. The advantage of this approach is that the results can be easily generalized to systems with multiple delays, and extended to deal with systems with parametric uncertainties, unmodelled dynamics, and/or various simple non-linearities.

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