A SIMPLE EXTENSION OF CONTRACTION THEORY TO STUDY INCREMENTAL STABILITY PROPERTIES

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Abstract

Contraction theory is a recent tool enabling to study the stability of nonlinear systems trajectories with respect to one another, and therefore belongs to the class of incremental stability methods. In this paper, we extend the original definition of contraction theory to incorporate in an explicit manner the control input of the considered system. Such an extension, called *universal contraction*, is quite analogous in spirit to the well-known Input-to-State Stability (ISS). It serves as a simple formulation of incremental ISS, external stability, and detectability in a *differential* setting. The hierarchical combination result of contraction theory is restated in this framework, and a differential small-gain theorem is derived from results already available in Lyapunov theory.

1 Introduction

Contraction theory, also called contraction analysis, is a recent tool enabling to study the stability of nonlinear systems trajectories with respect to one another, and therefore belongs to the class of incremental stability methods (see [12, 13] for references on contraction theory, and [1, 3] for other incremental stability approaches).
As in Lyapunov theory, the notations of contraction enable to represent control signals in an *implicit* manner. On the contrary to the original definition of contraction theory, this paper presents a simple extension of contraction theory that enables to *explicitly* incorporate the control input in the process of convergence analysis. One of the advantages of such a consideration is the issue of robustness may be addressed in a very simple way, similar to its Lyapunov counterpart, Sontag's Input-to-State Stability (ISS) [14]. Another similarity with ISS is that the definition of universally contracting systems may lead to a quite general framework for studying different (incrementally) stable behaviors [15].
In the rest of this paper, we first recall the main definition and theorem of contraction in section 2. Then universally contracting systems are briefly introduced in section 3. The section 4 is dedicated to the derivation of the notion of universally contracting systems to consider different aspects of stability as described in [15] in a differential setting. More precisely, after an example, the aspects that are considered are internal stability, external stability, and detectability in relation with observers. Finally, section 5 deals with a restatement of a result on the hierarchical combination of contracting systems under the framework of the newly-introduced extension, and derives a contracting version of the well-known small-gain theorem.

2 Definition and theorem of contraction analysis

The problem considered in contraction theory is to analyze the behavior of a system, possibly subject to control, for which a nonlinear model is known of the following form

$$\dot{x} = f(x,t) \tag{1}$$

where $x \in \mathbb{R}^n$ stands for the state whereas f is a nonlinear function. By this equation, one can notice that the control may easily be expressed *implicitly* for it is merely a function of state and time. Contracting behavior is determined upon the exact differential relation

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x \tag{2}$$

where δx is a virtual displacement, *i.e.* an infinitesimal displacement at fixed time.
From here, and after using a differential coordinate transform $\delta z = \Theta(x,t)\delta x$, define the so-called generalized Jacobian $F = (\dot{\Theta} + \Theta\frac{\partial f}{\partial x})\Theta^{-1}$ which dynamics are

$$\delta\dot{z} = F\delta z \tag{3}$$

For the sake of clarity, thereafter are reproduced the main definition and theorem of contraction taken from [12].

Definition 2.1 *A region of the state space is called a contraction region with respect to a uniformly positive definite metric $M(x,t) = \Theta^T(x,t)\Theta(x,t)$ where Θ stands for a differential coordinate transformation matrix, if equivalently $F = (\dot{\Theta} + \Theta\frac{\partial f}{\partial x})\Theta^{-1}$ or $\frac{\partial f}{\partial x}^T M + \dot{M} + M\frac{\partial f}{\partial x}$ are uniformly negative definite.*

The last expression can be regarded as an extension of the well-known Krasovskii method using a time and state dependent metric. On a historical perspective, note that results very closed from this one —however with a state but not time dependent metric— were established in the early sixties [4], though with a slightly different interpretation.
Definition 2.1 leads to the following convergence result:

Theorem 2.1 *Any trajectory, which starts in a ball of constant radius with respect to the metric $M(x,t)$, centered at a given trajectory and contained at all time in a contraction region, remains in that ball and converges exponentially to this trajectory.*

In the following, only *global* convergence is considered, *i.e.* the contraction region corresponds to the whole state space.

3 Universally contracting systems

Systems to be considered are of the form:

$$\dot{x} = f(x,u,t) \tag{4}$$

where the control signal $u \in \mathcal{U} \subset \mathbb{R}^m$ is this time *explicitly* represented. The system is initialized with x_0. Thanks to the form of equation (4), one can work on the differential expression

$$\delta\dot{x} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u \tag{5}$$

or more generally, by using the local transformation $\delta z = \Theta \delta x$,

$$\delta\dot{z} = F\delta z + \Theta\frac{\partial f}{\partial u}\delta u \tag{6}$$

still with $F = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}$.
We are now ready to state the following definition.

Definition 3.1 *The system $\dot{x} = f(x,u,t)$ is said to be universally contracting in u if it is contracting for all $u \in \mathcal{U}$ and if $\partial f/\partial u$ is uniformly bounded.*

This definition of universality of an input is somewhat different from the usual one that can be found in [2, p. 178] where the issue is to define system observability with respect to specific inputs. However, the relation with the above definition can be regarded as the keeping of a specific property for any change of variable.
For some very special cases, the application of definition 3.1 is pretty simple, as the following trivial example will show.

Example 3.1 *Let the system*

$$\dot{x} = f(x,t) + Bu \tag{7}$$

with B a constant matrix. If $\partial f/\partial x$ is uniformly negative definite, then the system is obviously universally contracting.

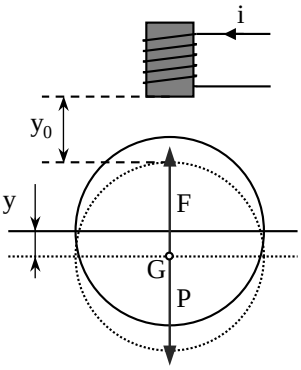


Figure 1: The magnetic levitator

Although very simple, this example is intended to illustrate two main points of universally contracting systems. First, it is precisely because the system is linear in the control input that the value of u has no importance on the contracting behavior. Otherwise the contraction property would generally depend on the values of u. Thus the explicit representation (4) allows one to find conditions on u for which the contraction property is maintained without to deal with a family of systems, as it would have been the case by considering forms like (1) where the control input is only implicitly represented.
As a second point, the presence of u in the model helps to understand in a simple manner how the system behaves for two different control inputs (*i.e.* for example an ideal control and a noise corrupted one), thus addressing the issue of analyzing robustness.
Indeed, it is easy, combining (6) with conditions of Definition 3.1 that universal contraction implies the following inequality [8, 6]

$$||\delta x|| \leq ||\delta x_0||\beta(t) + \gamma||\delta u||_{\mathcal{L}_\infty} \tag{8}$$

where $\beta(t)$ is an exponentially decaying time function, and γ a positive constant that in the following will be termed as *differential gain*. $||\delta u||_{\mathcal{L}_\infty}$ obviously represents the sup norm on the infinitesimal difference between two control signals.

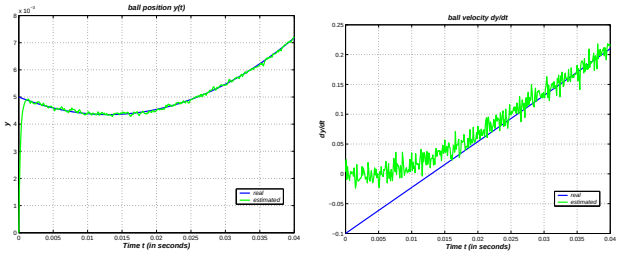
4 A differential framework for incremental stability

4.1 Motivations and example

The previous section allowed us to see that universally contracting systems could be used as a simple means to characterize the impact of input signals on the dynamical behavior of a system. Obviously, such a definition could be more useful in systems more complex that the one of example 3.1.

Indeed, there exist some systems which are not affine in the control. Among these, let us mention the famous magnetic levitator example (see figure 1) which nonlinear model can be described by the following equation:

$$\ddot{y} = g - \frac{Ci^2}{(y_0+y)^2} \tag{9}$$



(a) position (b) velocity

Figure 2: Simulation of the magnetic levitator observer

where y is the vertical position of the ball, i the control current, and g, C, y_0 positive constants. This model can obviously be shaped into the state-space form

$$\begin{cases} \dot{x}_1 = g - \dfrac{Ci^2}{(y_0 + x_2)^2} \\ \dot{x}_2 = x_1 \end{cases} \tag{10}$$

If one's goal is to design an observer for this system, it could be of importance to know whether or not the control inputs to the observer (*i.e.* the control inputs to the ball and beam system as well as its measured outputs) are *universal* inputs for the observer, *i.e.* if the observer is universally contracting in i, but also in y, the ball position which is the only measured output. As an example, consider the following observer:

$$\begin{cases} \dot{\hat{x}}_1 = g - \dfrac{Ci^2}{(y_0 + \hat{x}_2)^2} + k_1(\hat{x}_1 - \dot{y}) \\ \dot{\hat{x}}_2 = \hat{x}_1 + k_2(\hat{x}_2 - y) \end{cases} \tag{11}$$

As the variable $\dot{y}$ is not directly available through measurement, the implementation of the observer will be made using the transform $\bar{x}_1 = \hat{x}_1 + k_1 y$ to finally lead to

$$\begin{cases} \dot{\bar{x}}_1 = g - \dfrac{Ci^2}{(y_0 + \hat{x}_2)^2} + k_1(\bar{x}_1 - k_1 y) \\ \dot{\hat{x}}_2 = \hat{x}_1 + k_2(\hat{x}_2 - y) \end{cases} \tag{12}$$

which can be seen as a nonlinear counterpart of Luenberger reduced-order observers. From here, computing the virtual displacement dynamics of (12), one has

$$\begin{pmatrix} \delta\dot{\bar{x}}_1 \\ \delta\dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} k_1 & \dfrac{2Ci^2}{(y_0 + \hat{x}_2)^3} \\ 1 & k_2 \end{pmatrix} \begin{pmatrix} \delta\bar{x}_1 \\ \delta\hat{x}_2 \end{pmatrix} \tag{13}$$

and the symmetric part of the Jacobian will be given by

$$\frac{\partial f}{\partial x}^T + \frac{\partial f}{\partial x} = \begin{pmatrix} 2k_1 & 1 + \dfrac{2Ci^2}{(y_0 + \hat{x}_2)^3} \\ 1 + \dfrac{2Ci^2}{(y_0 + \hat{x}_2)^3} & 2k_2 \end{pmatrix} \tag{14}$$

so that under the following conditions, the observer is contracting

$$k_1 < 0 \tag{15}$$

$$4k_1 k_2 - \left(1 + \frac{2Ci^2}{(y_0 + \hat{x}_2)^3}\right)^2 > 0 \tag{16}$$

Assuming that $|i| \leq 1.5A$ and that $\hat{x}_2 \geq 0$ for all time, and with parameter values $y_0 = 0.05$, $g = 9.81$ and $C = 0.025$, it is easily checked that the observer is universally contracting in $i \in [-1.5; 1.5]$ and $y \in \mathbb{R}^+$ when the observer gains are tuned as $k_1 = -100$ and $k_2 = -4000$.

Moreover, inequality (8) together with the links of universally contracting systems with Angeli's δISS [1] that were established in [8, 6] ensure the robustness to noise measurement of the observer.

The curves of figure 2 show the behavior of the observer (12) for an additive noise on measurement y. It is also possible, by noticing that

$$\frac{\partial f}{\partial y} = \begin{pmatrix} -k_1^2 \\ -k_2 \end{pmatrix} \tag{17}$$

to estimate quantitatively the impact of the tuning of k_1 and k_2 on the robust properties of the observer with respect to noise measurement.

The previous example thus shown us that the study of universally contracting systems is not limited to the consideration of the control input u, but that they can incorporate the outputs of the observed system. Though it may first seem trivial, let us recall that this last remark is however quite important, especially when defining the notion of detectability for nonlinear systems [16].

Furthermore, assuming now that our goal is not to estimate the *state* of the system but this time only a *function* of this state, we would be more interested in knowing if the error on the estimation remains bounded when the error on the observer inputs is bounded.

Clearly, the objective of the present paper is thus to use both the framework of contraction theory and the notion of universally contracting systems to describe the different aspects of differential stability that just have been briefly depicted.

4.2 A differential triad

In the issue of "generalizing" and opening contraction analysis to a broader context, we will consider the following class of systems

$$\begin{cases} \dot{x} = f(x, u, t) \\ y = h(x, u, t) \end{cases} \tag{18}$$

where y stands as usual (but not always) for external signals that are directly measurable through the use of sensors, or, as an alternative, variables that are to be stabilized, depending on the objectives assigned to the control structure. As often, this system has an initial state vector, noted $x(0) = x_0$, and an input signal u. To (18), let the corresponding "extended"" virtual dynamics be

$$\begin{cases} \delta\dot{x} = \dfrac{\partial f}{\partial x}(x, u, t)\delta x + \dfrac{\partial f}{\partial u}(x, u, t)\delta u \\ \delta y = \dfrac{\partial h}{\partial x}(x, u, t)\delta x + \dfrac{\partial h}{\partial u}(x, u, t)\delta u \end{cases} \tag{19}$$

In [15], stability is described in a broad sense through several aspects grouped in three classes, namely internal stability, external stability, and detectability, which represent three different facets through which stable behavior of a system can be examined. This paper makes use of Input-to-State Stability (ISS) as the core to describe such aspects.
Because of their relatively simple formulation, it seems that universally contracting systems can also exhibit some of the advantages of Input-to-State Stability, thus helping to describe an *incrementally* stable behavior through the differential notation of contraction theory. We will consequently study the implications of this concept in a triad, which main goal is to reunite different aspects of incremental stability under the scheme envisioned by Sontag in a differential setting. As a by-product, some results of already cited Angeli and Fromion could be also related with this description.
Also, note that the declination of the different aspects of stability presented in [15] takes its origin in the field of linear systems, and that consequently, we sincerely think that our differential adaptation makes sense because it also stands as an attempt to make a smooth transition between the linear and the nonlinear worlds.

4.2.1 Internal stability

The first notion that will be considered here is the notion of internal stability, where the interest is mainly to study the evolution of the state, as well as the robustness (to the inputs) of this state, in the case where a stable behavior is observed.
Systems which are universally contracting in u, *i.e.* with respect to the inputs, clearly define this notion. As it has been previously observed, there is also a direct link between universally contracting systems and systems with the δISS property, due to the fact that (8) implies the following relation

$$||\delta x|| \leq \beta_I \left(||\delta x_0||, t\right) + \gamma_I \left(||\delta u||_{\mathcal{L}_\infty}\right) \tag{20}$$

(where β_I is a class-$\mathcal{KL}$ function and γ_I a class-$\mathcal{K}_\infty$ function). This can also be related to the ball to which all the trajectories of a disturbed contracting system converge, which is presented in [12].
On another aspect, note that the ideas in [17] which present a generalization of ISS to time-varying systems, which main purpose is to address tracking issues, seem relatively complex compared to our approach.
Finally, remark that it is in principle possible to conceive universally contracting systems as dissipative transfers from the input to the state since the definition of universal contraction implies the following relation

$$\frac{d}{dt}\left(\delta x^T M \delta x\right) \leq -|\lambda_F|.\delta x^T M \delta x + \frac{\sigma_{max}^2 \sigma_u^2}{|\lambda_F|}||\delta u||^2 \tag{21}$$

and after integration, one gets

$$\begin{aligned}\delta x(t_2)^T M(x,t_2)\delta x(t_2) - \delta x(t_1)^T M(x,t_1)\delta x(t_1) \\ \leq \int_{t_1}^{t_2} w(\delta x(\tau), \delta u(\tau)) d\tau\end{aligned} \tag{22}$$

Such a formulation also enables to link the concept of dissipativity with the feedback combination property of contraction theory. $\delta x(t)^T M(x,t)\delta x(t)$ would thus be regarded as a differential storage function.

4.2.2 External stability

External stability takes into account the output function of a system. In terms of interpretation, this means that *if* it would be possible to define a transfer function in the nonlinear domain (without causal operators), this function would be stable. Moreover, by remembering the local aspect of contraction, it would be possible to get a transfer function for two infinitely close signals. This function would consequently be both state and time dependent [11]. However, as this concept does not really make sense for finite displacements in the state space when speaking of nonlinear systems, we will restrain ourselves to a description of external stability using the following inequality

$$||\delta y|| \leq \beta_E \left(||\delta x_0||, t\right) + \gamma_E \left(||\delta u||_{\mathcal{L}_\infty}\right) \tag{23}$$

It is straightforward to show that if a system is universally contracting in its inputs, combined with the fact that the output function $h(x,u,t)$ is linearly bounded, the system will be differentially externally stable.
Indeed, starting from

$$\delta y = \frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial u}\delta u \tag{24}$$

and assuming bounds on each Jacobian of $h(x,u,t)$ to be positive constants σ_x and σ_h, as

$$\left(\frac{\partial h}{\partial x}\right)^T \left(\frac{\partial h}{\partial x}\right) \leq \sigma_x^2 I \tag{25}$$

and

$$\left(\frac{\partial h}{\partial u}\right)^T \left(\frac{\partial h}{\partial u}\right) \leq \sigma_h^2 I \tag{26}$$

one gets

$$||\delta y|| \leq \sigma_x^2 ||\delta x||^2 + \sigma_h^2 ||\delta u||^2 + 2\sigma_x \sigma_h ||\delta x||.||\delta u|| \tag{27}$$

which finally leads to inequality (23) after completeness of the squares.
The form of external stability described by (23) thus represents an input/ output differential and thus incremental form of stability. Once again, the notation that is used to define it makes such a concept quite general while it remains pretty simple. Also, note that it generalizes the so-called Incremental Quadratic Stability and its extensions, invented by Fromion [3].
But it is clear that if the conditions (25) and (26), together with universal contraction are sufficient conditions to ensure external differential stability (23), they are not necessary. Indeed, the expression (23) only guarantees a partial stability as far as the state is concerned. The system would then be said to be partially contracting, and one could consider the following inequality

$$||\delta x_{reduced}|| \leq \beta_{RE} \left(||\delta x_0||, t\right) + \gamma_{RE} \left(||\delta u||_{\mathcal{L}_\infty}\right) \tag{28}$$

where $x_{reduced}$ stands for the contracting part of the system, which implies that $\dim(x_{reduced}) \leq \dim(x)$ (this idea is also alluded to in [10]).

4.2.3 Detectability and observers

The last element of the triad is quite important for the aspects that were described in section 4.1.
In [16], the authors introduce the notion called IOSS (Input/Output to State Stability) as a nonlinear version of detectability of linear systems[1]. As IOSS is strongly related to the estimation of internal variables of a system, they also introduce a more constraining notion called i-IOSS ("i" for "incremental"), which helps to characterize the convergence of an observer towards the system state, as well as its robustness properties with respect to additive noise on the inputs to the observer, *i.e.* noise on the control input of the system and noise on the measured output.
Hence, Universally contracting observers in the control input and the output injection can be regarded as a differential version of IOSS, as one has the following relation

$$||\delta x|| \leq \beta_D\left(||\delta x_0||, t\right) + \gamma_u\left(||\delta u||_{\mathcal{L}_\infty}\right) + \gamma_y\left(||\delta y||_{\mathcal{L}_\infty}\right) \quad (29)$$

This relation is quite simple because it is independent from the specification of an attractor. As contraction theory, it also stands time-varying systems without any change, and therefore fits quite well the issue of designing nonlinear Luenberger observers.

5 Combination properties of universally contracting systems

We recall hereafter some results of system combinations using the notation of universally contracting systems. The advantage of the notation becomes apparent. The reader familiar with the results on combinations of ISS systems will certainly relate what is presented here with Sontag's framework.

5.1 Cascades

Theorem 5.1 *Let two systems be in cascade form as follows.*

$$\begin{cases} \dot{x}_1 = f_1(x_1, t) \\ \dot{x}_2 = f_2(x_1, x_2, t) \end{cases} \quad (30)$$

If $\dot{x}_1$ is contracting and that $\dot{x}_2$ is universally contracting in x_1, then the global system (30) is contracting.

The proof of such a theorem is rather simple to obtain through the use of estimate functions that are widely used in the context of ISS (see for example [14]). Indeed, starting from (30) together with the hypothesis of contraction of $\dot{x}_1$ and universal contraction in x_1 of $\dot{x}_2$, it comes

$$||\delta x_1(t)|| \leq ||\delta x_1(0)||\beta_1(t) \quad (31)$$

and

$$||\delta x_2(t)|| \leq ||\delta x_2(0)||\beta_2(t) + \gamma \sup_{0\leq\tau\leq t} ||\delta x_1(\tau)|| \quad (32)$$

where $\beta_1(t)$ and $\beta_2(t)$ are exponential functions of the time variable.
From the first of two inequalities, one has

$$\sup_{t/2\leq\tau\leq t} ||\delta x_1(\tau)|| \leq ||\delta x_1(t/2)||\beta_1(t/2) \quad (33)$$

and

$$||\delta x_1\left(t/2\right)|| \leq ||\delta x_1(0)||\beta_1\left(t/2\right) \quad (34)$$

These two expressions ((33) and (34)) lead us to

$$\sup_{t/2\leq\tau\leq t} ||\delta x_1(\tau)|| \leq ||\delta x_1(0)||\beta_1^2\left(t/2\right) \quad (35)$$

By rewriting (32) as

$$||\delta x_2(t)|| \leq ||\delta x_2\left(t/2\right)||\beta_2\left(t/2\right) + \gamma \sup_{t/2\leq\tau\leq t} ||\delta x_1(\tau)|| \quad (36)$$

and by using (35), one gets

$$||\delta x_2(t)|| \leq ||\delta x_2\left(t/2\right)||\beta_2\left(t/2\right) + \gamma||\delta x_1(0)||\beta_1^2\left(t/2\right) \quad (37)$$

Knowing that

$$||\delta x_2\left(t/2\right)|| \leq ||\delta x_2(0)||\beta_2\left(t/2\right) + \gamma||\delta x_1(0)||\beta_1(0) \quad (38)$$

from (39) one can deduce

$$\begin{aligned} ||\delta x_2(t)|| \leq\ & ||\delta x_2(0)||\beta_2^2\left(t/2\right) \\ & + \gamma||\delta x_1(0)||\beta_1(0)\beta_2\left(t/2\right) \\ & + \gamma||\delta x_1(0)||\beta_1^2\left(t/2\right) \end{aligned} \quad (39)$$

Taking into account the fact that $\beta_i(t)$ are exponential functions, this last expression, combined to (31) thanks to the triangle inequality $||\delta x(t)|| \leq ||\delta x_1(t)|| + ||\delta x_2(t)||$, leads to the general bound

$$||\delta x(t)|| \leq ||\delta x(0)||\beta(t) \quad (40)$$

which guarantees that (30) is contracting thanks to the converse theorem in [12, section 3.5].
Note that the proof of this theorem is another way to demonstrate the result of Lohmiller and Slotine on the hierarchical combination of contracting systems. However the use of $\beta_i(t)$ functions enables to give an estimate of the increase in energy on $\dot{x}_2$ brought by subsystem $\dot{x}_1$.

Furthermore, it is quite simple, using this method, to generalize this result and to consider, for example, two (or more) subsystems in cascade form as represented in figure 3 where H_i is written as

$$\begin{cases} \dot{x}_i = f_i(x_i, u_i, t) \\ y_i = h_i(x_i, u_i, t) \end{cases} \quad (41)$$

[1] Recall that detectability can be defined as the stability of the unobservable part of a system.

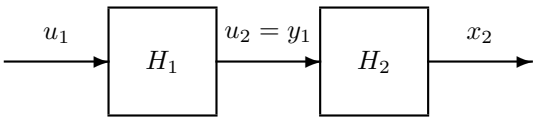


Figure 3: Cascade of two nonlinear systems

5.2 Differential versions of small gain theorem

The so-called small-gain theorem has been presented under many different versions (see for example [18] and [9, p. 430]). The issue of considering initial conditions was included in the work of [5], where the main tool is ISS as well as its practical extension, ISpS. The following theorem, adapted to the notion of universally contracting systems, is stated as follows.

Theorem 5.2 *Let two systems put in a loop as follows.*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, t) \\ \dot{x}_2 = f_2(x_1, x_2, t) \end{cases} \tag{42}$$

If $\dot{x}_1$ is universally contracting in x_2, and $\dot{x}_2$ is universally contracting in x_1, and that their respective differential gains γ_1 and γ_2 are such that

$$\gamma_1 \gamma_2 < 1 \tag{43}$$

then the global system (42) is contracting.

To start the proof of this theorem, we will check that $||\delta x(t)||$ is upper bounded. The hypothesis of universal contraction imply

$$||\delta x_1(t)|| \leq ||\delta x_1(0)||\beta_1(t) + \gamma_1 \sup_{0\leq\tau\leq t} ||\delta x_2(\tau)|| \tag{44}$$

and

$$||\delta x_2(t)|| \leq ||\delta x_2(0)||\beta_2(t) + \gamma_2 \sup_{0\leq\tau\leq t} ||\delta x_1(\tau)|| \tag{45}$$

for all time.
From (44), it comes

$$\sup_{0\leq\tau} ||\delta x_1(\tau)|| \leq ||\delta x_1(0)||\beta_1(0) + \gamma_1 \sup_{0\leq\tau} ||\delta x_2(\tau)|| \tag{46}$$

expression that can be used in (45) to get

$$\begin{aligned} \sup_{0\leq\tau} ||\delta x_2(\tau)|| \leq\ & ||\delta x_2(0)||\beta_2(0) \\ & + \gamma_2||\delta x_1(0)||\beta_1(0) \\ & + \gamma_1\gamma_2 \sup_{0\leq\tau} ||\delta x_2(\tau)|| \end{aligned} \tag{47}$$

and one gets

$$\begin{aligned} &\sup_{0\leq\tau}||\delta x_2(\tau)|| \\ &\quad \leq \frac{1}{1-\gamma_1\gamma_2}\left(||\delta x_2(0)||\beta_2(0) + \gamma_2||\delta x_1(0)||\beta_1(0)\right) \end{aligned} \tag{48}$$

if the condition $\gamma_1\gamma_2 < 1$ is verified.
Taking into account the fact that $||\delta x_2(t)|| \leq \sup_{0\leq\tau}||\delta x_2(\tau)||$ for all time, and by using triangular inequality on the initial displacements, we finally get

$$||\delta x_2(t)|| \leq K_2||\delta x(0)|| \tag{49}$$

The case of $||\delta x_1(t)||$ is symmetric, and it can be written

$$||\delta x_1(t)|| \leq K_1||\delta x(0)|| \tag{50}$$

which with (49) allows to conclude that

$$||\delta x(t)|| \leq K||\delta x(0)|| \tag{51}$$

Then, one has to demonstrate that $\delta x(t)$ goes to 0 in an exponential manner.
This demonstration starts with a temporal shift of the two estimate functions (44) and (45) that will be rewritten as

$$||\delta x_1(T)|| \leq ||\delta x_1(t/4)||\beta_1(T - t/4) + \gamma_1 \sup_{t/4\leq\tau\leq T} ||\delta x_2(\tau)|| \tag{52}$$

and

$$||\delta x_2(t)|| \leq ||\delta x_2(t/2)||\beta_2(t/2) + \gamma_2 \sup_{t/2\leq\tau\leq t} ||\delta x_1(\tau)|| \tag{53}$$

If one decides that $T \in [t/2, t]$, (52) becomes

$$||\delta x_1(T)|| \leq ||\delta x_1(t/4)||\beta_1(t/4) + \gamma_1 \sup_{t/4\leq\tau\leq t} ||\delta x_2(\tau)|| \tag{54}$$

which implies

$$\begin{aligned} \sup_{t/2\leq\tau\leq t} ||\delta x_1(\tau)|| \leq\ & ||\delta x_1(t/4)||\beta_1(t/4) \\ & + \gamma_1 \sup_{t/4\leq\tau\leq t} ||\delta x_2(\tau)|| \end{aligned} \tag{55}$$

expression that can be put in (53) to obtain

$$\begin{aligned} ||\delta x_2(t)|| \leq\ & ||\delta x_2(t/2)||\beta_2(t/2) \\ & + \gamma_2||\delta x_1(t/4)||\beta_1(t/4) \\ & + \gamma_1\gamma_2 \sup_{t/4\leq\tau\leq t} ||\delta x_2(\tau)|| \end{aligned} \tag{56}$$

Then, using the general bound (51), The triangular inequality, and some elementary notions on exponential functions, it comes

$$||\delta x_2(t)|| \leq ||\delta x(0)||\beta_2'(t) + \gamma_1\gamma_2 \sup_{\frac{t}{4}\leq\tau} ||\delta x_2(\tau)|| \tag{57}$$

When $t = 0$, from (57), it is straightforward to get

$$||\delta x_2(0)|| \leq \frac{1}{1-\gamma_1\gamma_2}||\delta x(0)||\beta_2'(0) \tag{58}$$

When $t > 0$, taking $T > 0$ such that $T \leq t/4$ leads to

$$||\delta x_2(t)|| \leq ||\delta x(0)||\beta_2'(T) + \gamma_1\gamma_2 \sup_{T\leq\tau} ||\delta x_2(\tau)|| \tag{59}$$

which is true for all $t \in [T; +\infty[$.
From there, it is easy to get to

$$||\delta x_2(t)|| \leq \frac{1}{1-\gamma_1\gamma_2}||\delta x(0)||\beta_2'(t) \tag{60}$$

The case of $||\delta x_1(t)||$ begin once more symmetric, one finds

$$||\delta x_1(t)|| \leq \frac{1}{1-\gamma_1\gamma_2}||\delta x(0)||\beta_1'(t) \tag{61}$$

which lead us to conclude that

$$||\delta x(t)|| \leq \beta(t)||\delta x(0)|| \tag{62}$$

and that the global system (42) is contracting.
From the point of view of the original definition of contraction analysis, this last theorem can be considered as a result which is complementary to the feedback combination property in [12] (see also [7]) for an application of this combination property).

6 Concluding remarks

In this paper, a simple extension of contraction theory –named universal contraction– was introduced to incorporate in an explicit manner the effect of external input signals on the contracting behavior of systems. We then derived several different aspects of stability as internal and external stability, detectability, in a framework fully compatible with contraction theory. Some combination properties for universal contracting systems were also derived.
This extension would hopefully help to define nice nonlinear extensions to the well-known rank conditions associated with controllability, observability and detectability in linear systems. This, along with the application to several physically-motivated examples, is a subject of current research.

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