# SECOND ORDER VARIABLE STRUCTURE SYSTEMS: BEHAVIOUR UNDER AN UNKNOWN INPUT DELAY

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## Abstract

In this paper it is proposed a solution to the regulation problem of a second order system with delayed input. The delay which affects the input is assumed to be bounded but uncertain. The behaviour of the system, under the action of the proposed control strategy, results to be a persisting oscillating one due to the delay of the input.

## 1 Introduction

The analysis and control of systems with time delays (TDS) are currently receiving much attention from both the engineering and mathematical communities. The increasing interest in this research area is motivated by several factors. As technological progress brings the need to steadily enhance performances, there grows the requirement to increase the precision of mathematical models of the system dynamics. The introduction in the system modeling of previously neglected delays is just an example of this trend. Another major motivation is related to the rapidly spreading of communication networks and information technologies, in which delays play an important role.
On the control side, delays operate in two main forms. Indeed, apart from the need to control systems in which the evolution is governed by equations of retarded type (delay in the state variable), in practice it usually happens that time delays are also introduced through the same control channel (delay in the input variable). This second kind of delay is caused either by the actuators (e.g. in [1]) or the measurement devices (e.g. in [3]) or both (see also [10]). In this paper we focus our attention on this type of time lag, in the framework of the control of second order systems.
Classical control methods performances can be substantially deteriorated by the delays action, thus specific controllers have to be designed to overcome these problems (for a survey see e.g. [13]).
Important requirements such as robustness with respect to external disturbances have to be taken into account. Sliding mode control (SMC) extension to TDS has been studied in view of exploiting its robustness properties (surveys about this topic can be found in [13, 14, 9]). Many of these results only consider delays in the state: here the SMC philosophy does not change, one just has to carefully choose the "right" sliding surface.
The nature of the problem is instead completely altered if we consider the combination of an input delay in a relay-type controller. In this case the delay induces oscillations around the sliding surface and also causes complex bifurcation phenomena [8, 6, 4, 5, 7]. The same authors in [4] also considered second order relay control with time delay. The resulting motions acquire an oscillating behaviour and it is shown that every trajectory has a finite limit frequency. Also, the property of zero limit frequency is a stable one and the stability properties of steady modes and oscillatory solutions are investigated..
In this paper we analyse the effect of an input time-delay on a particular second order sliding mode control approach [2]. This algorithm, which is related to time optimal bang-bang controls, assures a faster transient and lower control bounds when compared to other strategies of the same kind. This of course can be very useful when delays affect the system, since in this case the closed loop behaviour is for example strictly connected to the amplitude of the feedback gain (see e.g. [9] in the SMC case). The aim is to understand the impact of input time delays on the control scheme, in the view of using this knowledge to adapt the algorithm to the control of TDS. In this paper, the analysis is carried out in the case of an unperturbed double integrator. This choice allows one to directly find the relevant terms governing the evolution of the closed loop and thus to study the asymptotic properties of the system trajectories. It is shown that whatever the choice of the control parameters, for any fixed constant input delay, the limit state evolution is periodic. The dependence of the amplitude of the limit cycle in the phase plane on the delay, the control modulus and the control parameters is also presented. In Section 2 we briefly describe the structure of the existent control algorithm, while in Section 3 we state our problem and study the effect of the input delay on the system evolution. Section 4 is devoted to the applica-

tion of the obtained results to the control of a double integrator subject to an unknown bounded input delay.

## 2 Second order control systems

In this section we consider, from a general point of view, the behaviour of second order systems and introduce the control algorithm in [2].

To highlight the ideas behind this control method, we start with an observation about first and second order differential inequalities .

When a system satisfies a differential inequality of the first order (e.g. $\dot{y}_1 y_1 \leq -h^2|y_1|$), either when a Lyapunov like inequality holds (e.g. $\dot{V} \leq -K\sqrt{V}$), the qualitative behaviour of the worst case solution, that is the solution corresponding to the equality sign, is inherited by all the solutions satisfying the inequality. Indeed, when a second order differential inequality is involved, different evolutions are possible.

It is easy to show that the trajectories of a system which satisfies a differential inequality of the type $\ddot{y}_1 y_1 \leq -h^2|y_1|$ are characterized by a focus in the origin of the phase plane. A sequence of singular points $\{y_1(t_{M_i}), t_{M_i} : \dot{y}_1(t_{M_i}) = 0\}$ is generated and the related behaviour can range from either explosive or persistently oscillating to the desired stable one. The control objective of steering both $y_1$ and $\dot{y}_1$ to zero can be stated in terms of the convergence property of the sequence of $\{y_1(t_{M_i})\}$ and the associated sequence $\left\{\Delta t_{M_i} = t_{M_i} - t_{M_{i-1}}\right\}$. Indeed if the two sequences are strictly contractive, that is $\frac{|y_1(t_{M_i})|}{|y_1(t_{M_{i-1}})|} \leq \rho < 1$ and $\frac{\Delta t_{M_i}}{\Delta t_{M_{i-1}}} \leq q < 1$ $\lim_{i\to\infty} y_1(t_{M_i}) = 0$ and $\sum_{i=1}^{\infty}(t_{M_i} - t_{M_{i-1}}) = T < \infty$, at $t = T$ both $y_1$ and $\dot{y}_1$ are steered to zero.

The following control algorithm [2] can be successfully applied to the perturbed double integrator $\dot{z}_1 = z_2$, $\dot{z}_2 = h(z) + d(z)w$ affected by uncertain terms for which constant bounds are known $|h(z)| < H$, $0 < d_1 < d(z) < d_2$.

**Algorithm 1:**

*When $t = 0$, set $z_{1_M} = z_1(0)$, $i = 0$, $t_{M_i} = 0$.*

*During the control interval, that is $\forall t \in [0, \infty)$, the following steps are performed:*

*If $z_2(t) = 0$ then set $z_{1_M} = z_1(t)$, $i = i+1$, and $t_{M_i} = t$.*

*It is applied the control*

$$w(t) = -W(t)\mathrm{sign}\left[z_1(t) - \frac{1}{2}z_{1_M}\right] \qquad (1)$$

$$W(t) = \begin{cases} W_M & z_{1_M}\left[z_1(t) - \frac{1}{2}z_{1_M}\right] > 0 \\ \alpha W_M & z_{1_M}\left[z_1(t) - \frac{1}{2}z_{1_M}\right] < 0 \end{cases}$$

$$W_M > \max\left(\frac{H}{d_1}, \frac{4H}{3\alpha d_1 - d_2}\right) \qquad \alpha > 1, \quad \alpha \neq \frac{d_2}{3d_1}$$

It can be proved [2] that, despite of the uncertainties, if the control amplitude is sufficiently high, the application of the control strategy (1) generates a sequence of successive singular points $\{z_1(t_{M_i}), t_{M_i} : z_2(t_{M_i}) = 0\}$, and this sequence is strictly contractive, that is $\frac{\left|z_1(t_{M_{i+1}})\right|}{\left|z_1(t_{M_i})\right|} \leq q < 1$. Moreover, the reaching time is a series of positive elements upper-bounded by a geometric series with ratio strictly less than one. Therefore, $\sum_{i=1}^{\infty}(t_{M_i} - t_{M_{i-1}}) = T < \infty$.

## 3 Control of a double integrator with delayed input

The control system we are going to study is the following

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u(t-\tau), \quad u(\theta) = u_0(\theta) \quad \theta \in [-\tau, 0] \end{cases} \qquad (2)$$

which is a simple double integrator with scalar control $u$, where we assume that the input is subject to a fixed delay $\tau \in (0, \tau_M]$ with known upper bound $\tau_M$. This delay can be interpreted as either a delay in the actuators or in the sensors in the case of feedback control. Our goal is to study the behaviour of the second order sliding algorithm presented in Section 2 when applied to the retarded system (2).

### 3.1 Control algorithm

We slightly modify the notation of the previous section in the definition of the control law. For simplicity we choose to fix the control modulus to a positive constant $U$, while we introduce a new parameter $\gamma \in (0, 1)$ to analyse the effect of anticipating or retarding the control switching (the situation described in Section 2 corresponds to the choice $\gamma = 1/2$). Thus we define the following control law for the system (2).

**Algorithm 2:** *When $t = 0$, set $x_{1_M} = x_1(0)$.*

*For $t \in [0, \infty)$, we do the following:*

*if $x_2(t) = 0$ we update the value $x_{1_M}$ by setting $x_{1_M} = x_1(t)$;*

*next we apply the control*

$$u(\theta) = -U\,\mathrm{sign}\,(x_1(\theta) - \gamma x_{1_M}). \qquad (3)$$

### 3.2 Closed loop analysis

In this section we analyse the effects of our control law on the system trajectories. In particular, since the system is of retarded type, we show that in the limit the system trajectory is periodic. Thus in the phase space its graph draws a loop, which in this case is an ellipse centered at the origin. We prove the convergence to this limit cycle and evaluate its amplitude, showing how it depends on the control parameters and the delay $\tau$.

The evolution of the closed loop system is closely related to that of the trajectory's intercepts with the $x_1$ axis, on which the structure of the control is based. Therefore let us suppose that at the initial time the position is $x_0$ and the velocity is zero; to fix ideas assume $x_0 > 0$. We apply our control algorithm and find the next intersection between the trajectory and the $x_1$ axis.

The control is set to $u_0(\theta) = -U$ if $\theta \in [-\tau, 0)$ and

$$u(\theta) = -U\,\mathrm{sign}\,(x_1(\theta) - \gamma x_0)\,,\ \theta \geq 0.$$

Would there be no delay, the control sign would change once the trajectory reaches a point $(\gamma x_0, x_2^*)$, for some $x_2^* < 0$. Since the control law acts with delay, the switching will take place $\tau$ instants later. Using the equations of motion it is easy to get the value of the position $z_m$ at the switching time

$$z_m = \gamma x_0 - \frac{U}{2}\tau^2 - \tau\sqrt{2Ux_0(1-\gamma)}. \qquad (4)$$

Since the control modulus is constant, the trajectories describe branches of symmetric parabola, therefore the next intersection with the $x_1$-axis will take place at the position

$$\hat{x}_0 = x_0 - 2(x_0 - z_m) = (2\gamma - 1)x_0 - U\tau^2 - 2\tau\sqrt{2Ux_0(1-\gamma)}.$$

Now, if $\mathrm{sign}\,(x_0\hat{x}_0) > 0$, the system trajectory will produce what we shall call a *curl*. In fact once the position $\hat{x}_0$ is reached, the control structure changes, and so does the control sign. However, due to the input delay, this change will really affect the system after it has acquired a little velocity. The change of sign in the acceleration will then annihilate it and produce another intercept, closer to $x_0$ by a quantity $U\tau^2$ (see Figure 1). As the behaviour of the closed loop is based on the convergence of the sequence of intercepts, in the presence of a curl the significant intercept is the second, so we define

$$x_1 = \hat{x}_0 + U\tau^2 \max\{0, \mathrm{sign}\,(x_0\hat{x}_0)\}.$$

Recursing the process and making the necessary changes for negative values of the position, we get the following sequence: given $x_0 \in \mathbb{R}$, let

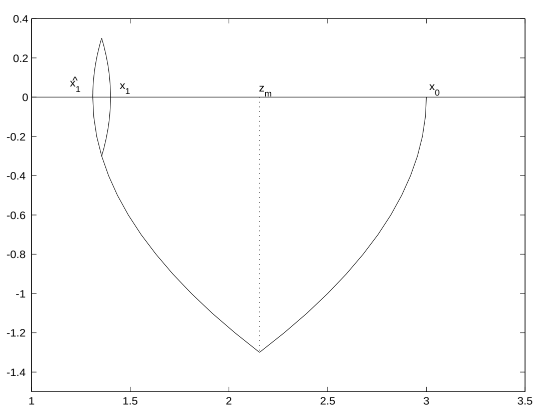


Figure 1: Due to the input delay, the system trajectory produces a curl.

$$\delta = U\tau^2, \quad \alpha = 2\gamma - 1, \quad \beta = 2\sqrt{\delta(1-\alpha)}$$

and for $k = 0, 1, 2, \ldots$

$$\begin{array}{rcl} \hat{x}_k & = & \alpha x_k - \mathrm{sign}\,(x_k)\,(\delta + \beta\sqrt{|x_k|}) \\ x_{k+1} & = & \hat{x}_k + \delta\,\mathrm{sign}\,(x_k)\,\max\{0, \mathrm{sign}\,(x_k\hat{x}_k)\} \end{array} \qquad (5)$$

The following Lemma proves that there can only be a finite number of curls.

**Lemma 3.1** *For all choices of the control parameters and any delay $\tau$ there exists $k_0 \geq 0$ such that*

$$x_k\hat{x}_k < 0 \quad \textit{for all} \quad k \geq k_0.$$

*Moreover, for any $m \geq 0$ we have*

$$\begin{aligned} l|x_{k_0+m+1}| = f(|x_{k_0+m}|) = f(f(|x_{k_0+m-1}|)) = \ldots \\ = f^{m+1}(|x_{k_0}|), \end{aligned} \qquad (6)$$

*with $f(x) = -\alpha x + \beta\sqrt{x} + \delta$.*

**Proof.** See [11].

The analysis of the convergence of the sequence of intercepts is therefore based on the evolution of the discrete dynamical system given by the iterates of the function $f$ in (6). Generally speaking, sequences of this kind can show a chaotic behaviour (see for example [12]). However it can be shown that, in our framework, this is never the case. In fact one can prove that the sequence $\{|x_k|\}$ is either definitely monotone or contractive and thus convergent.

**Proposition 3.1** *For all choices of the control parameters, any delay $\tau$ and any initial value $x_0$ the sequence*

$$|x_{k+1}| = f(|x_k|) = f^{k+1}(|x_0|), \qquad (7)$$

*with $f(x) = -\alpha x + \beta\sqrt{x} + \delta$, converges to the unique fixed point of $f$*

$$\bar{x} = \delta\,\frac{3 - \alpha + 2\sqrt{2(1-\alpha)}}{(\alpha+1)^2}.$$

**Proof.** See [11].

From Lemma 3.1, the values of the intercepts of the system trajectory of the closed loop (2)-(3), possibly after a transient, evolve according to the following algorithm:

$$\text{given } x_k, \quad x_{k+1} = -\mathrm{sign}\,(x_k)\,f(|x_k|), \quad f(x) = -\alpha x + \beta\sqrt{x} + \delta.$$

Proposition 3.1 then shows that in the limit for $k$ tending to infinity the sequence bounces from the point $\bar{x}$ to its symmetric $-\bar{x}$. Therefore the trajectory becomes periodic and its limit cycle can be easily deduced from the motion equations.

**Corollary 3.1** *The solution of the closed loop (2)-(3) shows a limit cycle in the phase plane. This is an ellipse centered at the origin; if we set*

$$g(\alpha) = \frac{2}{(\alpha+1)^2}(3 - \alpha + 2\sqrt{2(1-\alpha)}), \quad \alpha = 2\gamma - 1$$

*the lengths of its axes on $x_1$ and $x_2$ are respectively*

$$d_1 = U\tau^2\,g(\alpha), \qquad d_2 = U\tau\,2\sqrt{2g(\alpha)}.$$

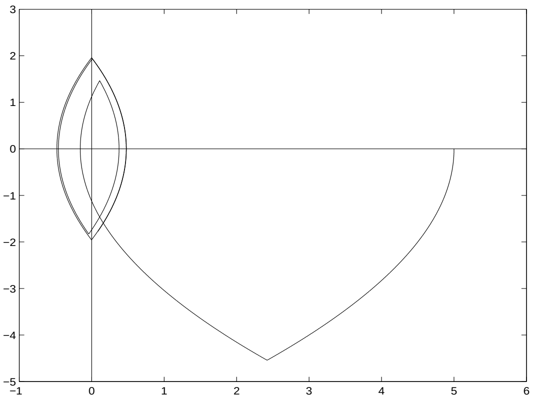


Figure 2: Closed loop phase plot

In Figure 2 we show as an example the simulation results obtained for the following choice of the parameters: the delay is set to $\tau = .2$, the initial condition is given by $x_1(0) = x_0 = 5$, $x_2(0) = 0$, $u_0(\theta) = 0$ for $\theta \in [-\tau, 0)$ and the control parameters are $\gamma = .65$ and $U = 4$ (thus $x \approx 0.5$).
Lastly, we also make this remark about the convergence to the limit trajectory. As the sequence $\{x_k\}$ is absolutely convergent, however small we choose $\varepsilon > 0$, after a finite number of steps the sequence will belong to $(-\bar{x}-\varepsilon, -\bar{x}+\varepsilon) \cup (\bar{x}-\varepsilon, \bar{x}+\varepsilon)$. Consequently for any $\varepsilon > 0$ there exist a finite time $T_\varepsilon$ and a boundary layer set $\Gamma_\varepsilon$ around the limit trajectory such that $(x_1(t), x_2(t)) \in \Gamma_\varepsilon$ for $t \geq T_\varepsilon$ (Figure 3).

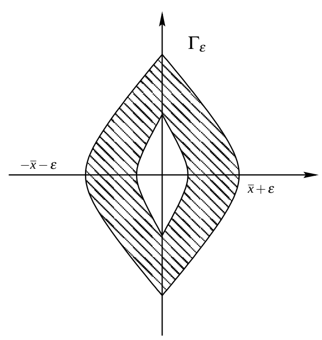


Figure 3: The boundary layer $\Gamma_\varepsilon$

## 4 Control under an unknown bounded input delay: an example

Let us suppose we are given a control system in the form (2), affected by a constant unknown input delay $\tau > 0$. Suppose moreover that a bound $\tau_M$ for this delay is known, i.e. $\tau \in (0, \tau_M]$. Then, using the analysis carried out in the previous section, it is possible to tune the control parameters $\gamma$ and $U$ in such a way that the sequence $\{x_k\}$ has a suitable behaviour under any $\tau \in (0, \tau_M]$. More precisely, using equation (4), one can define a time-varying control modulus $U = U(t)$ and an anticipation $\gamma$ so to obtain a suitable sequence of switching points $z_m$.
In Figures 4 and 5, we show some simulation results obtained using the following technique: let $\tau_M$ be the delay bound. Set the constant $U_M$ to an a priori fixed upper bound for the control modulus and choose two positive constants $r, \rho$ so that $r, r+2\rho < 1$. If $x_k$ is the value of the more recent intercept, we set

$$U = \min\left\{ \frac{|x_k|}{\tau_M^2} r, U_M \right\}, \quad \gamma = \frac{1}{2}(1+r) + \rho.$$

The control strategy is carried out taking into account the necessity to avoid drastic reduction of the control modulus by verifying, at any control time interval, the reduction rate.
In Figure 4 we show the evolution of the position and the velocity and the phase plane for $\tau_M = 0.5$, $\tau = 0.3$, $U_M = 4$, $r = 0.3$ and $\rho = 0.1$, while in Figure 5 we have $\tau_M = 1.1$ and $\tau = 1$. In both cases these choices impose the asymptotically decreasing nature of the sequence $\{|x_k|\}$.

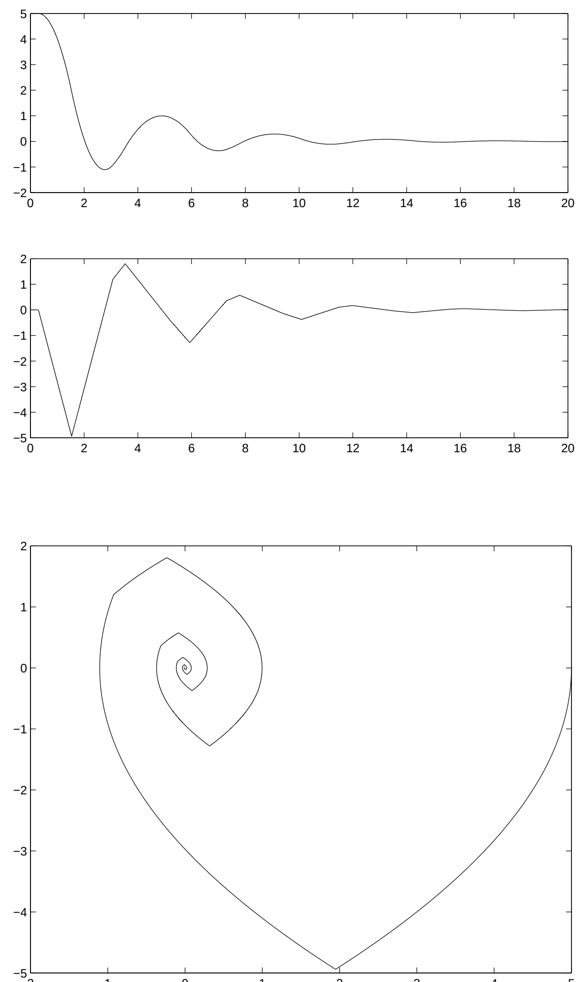


Figure 4: Position, velocity and phase plane with $\tau_M = 0.5$ and $\tau = 0.3$.

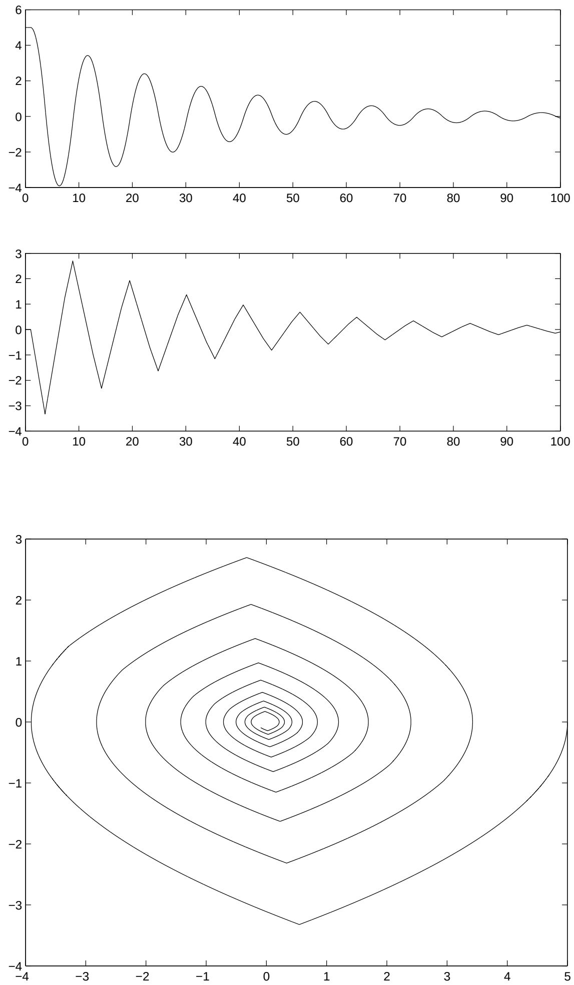


Figure 5: Position, velocity and phase plane with $\tau_M = 1.1$ and $\tau = 1$.

## 5 Conclusions

In this paper we studied the effect of an input delay $\tau$ on a second order sliding mode control. When the system is unperturbed we showed that the system trajectory converges to a limit cycle. The amplitude of the oscillations of the measured variable is shown to be $O(U\tau^2)$ if the modulus $U$ of the control is constant. Based on this analysis, a control algorithm is proposed for the regulation of a double integrator with unknown bounded input delay.

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