LINEAR FRACTIONAL ORDER CONTROL OF A DC-DC BUCK CONVERTER

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Abstract

The aim of this paper is the application of Fractional Order Control (FOC) to the control of a Buck converter. This work deals with the design of a linear controller for a DC-DC Buck power converter. In order to achieve this goal, several design methods are proposed and experimentally validated. Bode's ideal function will be used as reference system in all of them. The employ of this design technique requires the obtention of a continuous linearized model of the dc-dc converter (LC filter + PWM actuator). The resultant pseudo-continuous system is a non-minimum phase system with a Right-Hand-Side (RHS) zero. In this work, two methods are proposed for the design of fractional order controllers in systems that exhibit this behavior. As well, a technique based on the discrete linearized model is applied. In all the cases, discrete approximations of the fractional controllers are obtained.

1 Introduction

Switched mode *DC-DC* power converters are used in a wide variety of applications, including power supplies for personal computers, dc motor drives, active filters, etc. *Pulse-width modulation* (*PWM*), in which the duty ratio changes, sets the basis for the regulation of switched mode converters. The operation of these devices is often based on the control of the output voltage of a passive filter. A basic *DC-DC* converter circuit known as the *Buck converter* is illustrated in Fig.1. The Buck converter consists of a switch network that reduces the *dc* component of voltage and a low-pass filter that removes the high-frequency switching harmonics. Several control strategies have been used for the control of *DC-DC* converters, such as PI, Dead Beat, Sliding Mode Control, etc. [5].

On the other hand, *FOC* have been introduced in the last decades for managing a variety of control problems [9],[10],[11]. One of the main advantages of fractional controllers is the possibility of obtaining an open loop transfer function in the form of a fractional order integrator. Such a system, usually called Bode's ideal transfer function, gives a controlled system robust to changes in process gain. In this way, the purpose of this paper is to propose a method for the control of power electronic converters by using a linear controller based on *FOC* and Bode's ideal function [3, 7] and then

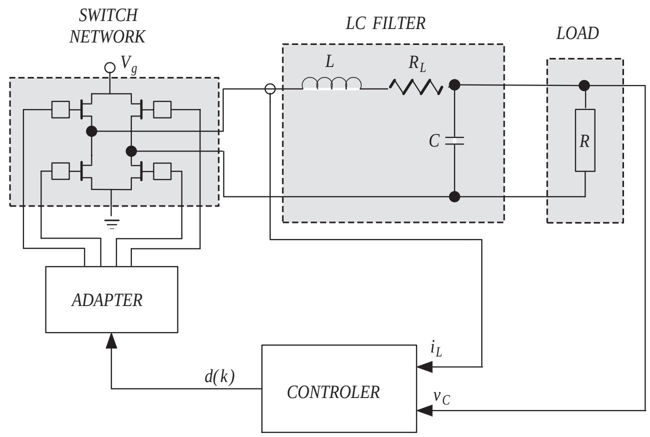

Figure 1: Buck converter

to obtain a discrete equivalent that allows its practical implementation. (First results have been presented in [4]). According to this, the closed loop transfer function of the converter must be the following:

$$F(s) = \frac{k_c}{s^\lambda + k_c} \tag{1}$$

or, in other words, its open loop transfer function will be given by:

$$G(s)D(s) = \frac{k_c}{s^\lambda} \tag{2}$$

where, $G(s)$ is the continuous transfer function of the plant and $D(s)$ is the controller.

The methods of design proposed in this paper are based on frequency domain techniques. Therefore, a linearized model of the plant is necessary in order to apply the design process. First, a pseudo-continuous model is obtained from the discrete linearized one. By using a w-transform the obtained pseudo-continuous system is a non-minimum system, whose phase lag must be cancelled. To take into account this phase lag two methods are proposed. One of them is based on considering the non-minimum phase term like a delay and to use a controller that combines a fractional integrator and a Smith predictor structure in order to achieve the working specifications. The other one deals with the design of the controller taking into account the phase lag at the frequency of interest for determining the order of the Bode's ideal function. Finally, in order to avoid the handling of non-minimum phase systems due to the use of the w transformation, the design problem is broached considering directly the discrete linearized model of the dc-dc converter and applying the discrete approximation of the resultant fractional controller. In order to show the feasibility of the proposed methods, a prototype has been built and experimental results are reported and discussed. The rest of the paper is organized as follows. Section II describes the processes of modeling and linearization of the plant and a real system model is obtained. Section III deals with the controller design techniques proposed in this paper. Section IV shows simulation and experimental results and Section V states some conclusions and guidelines for further work.

2 Plant model

2.1 Discrete plant model

To design the control system of a converter, it is necessary to model the converter dynamic behavior. Unfortunately, modeling of converter dynamic behavior is hampered by the nonlinear time-varying nature of the switching and pulse-width modulation process. In order to obtain a valid model of the dc-dc converter, this system is considered as composed of two subsystems: *LC* filter and a *PWM* actuator. A time-invariant linear *LC* filter may be represented by a state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv_s(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

where $v_s(t)$ is the input of the filter and it is provided by a *PWM* actuator where the input control is the duty cycle, d. In the case of a *ON-OFF-ON PWM* actuator of symmetric pulse with respect $T/2$ its model is

$$v_s(t) = \begin{cases} 0 & for \quad T(k) \leq t < T(k) + t_1(k) \\ V_g & for \quad T(k) + t_1(k) \leq t < T(k) + t_2(k) \\ 0 & for \quad T(k) + t_2(k) \leq t < T(k+1) \end{cases} \tag{4}$$

where $t_1(k) = \frac{T}{2}(1 - d(k))$ and $t_2(k) = \frac{T}{2}(1 + d(k))$.

An equivalent discrete model of the joint system *PWM* actuator-*LC* filter can be obtained from the solution of the equation (3)

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bv_g(\tau)d\tau \tag{5}$$

Particularizing for $t_0 = kT$, $t = T(k+1)$, and a $v_s(t)$ given by (4), the discretized model is given by the expression

$$\begin{aligned} x(k+1) &= e^{AT}x(k) + e^{AT}(e^{-At_1(k)} - e^{-At_2(k)})A^{-1}BV_g \\ y(k) &= Cx(k) \end{aligned} \tag{6}$$

The matrix A is diagonalizable, and its Jordan canonical form is

$$A = V\Lambda V^{-1}, \quad \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n) \tag{7}$$

Then, expression (6) becomes

$$x(k+1) = e^{AT}x(k) + V\Psi(k)V^{-1}BV_g \tag{8}$$

where $\Psi(k)$ is a diagonal control matrix of the form

$$\Psi(k) = e^{\Lambda \frac{T}{2}}(e^{\Lambda \frac{T}{2}d(k)} - e^{-\Lambda \frac{T}{2}d(k)}) \tag{9}$$

which define the relation between Ψ and $d(k)$ in the case of a symmetric *ON-OFF-ON* actuator. Expression (8) defines a nonlinear relation between the control variable $d(k)$ and the state $x(k+1)$.

The control design methods proposed in this paper are based on frequency domain methods. Consequently, the first step in the control design process is to obtain a linear model of the dc-dc converter. Several methods have been proposed to carry out this linearization: model averaging [6], first-order truncation of a Taylor series expansion [1] and methods based on the functional minimization [8]. The last one is used in this paper, which allows the use of lower sampling frequencies.

2.2 Linearized plant model

This method is developed in [8] and is based on the approximation of the diagonal matrix $\Psi(k)$ by

$$\Psi(k) \approx \hat{\Gamma}v(k) \tag{10}$$

where $\hat{\Gamma}$ is a constant diagonal complex matrix of the same dimension of Ψ, and $v(k)$ is a real function of the physical control variable $d(k)$.

The approximation (10) is defined in terms of an optimization problem. Assuming that $\Psi(k) = diag\,(\psi_1(d), ..., \psi_n(d))$ and $\hat{\Gamma} = diag\,(\gamma_1, ..., \gamma_n)$ are complex matrices, will be found the matrix $\hat{\Gamma}$ and the function $v(d)$ which minimize the cost function

$$J = \int_0^1 \sum_{i=1}^{n} (\psi_i(d) - \gamma_i v(d))\,(\psi_i(d) - \gamma_i v(d))^*\,d(d) \tag{11}$$

where $*$ denote a complex conjugate transpose. This cost function penalizes the squared error of the approximation in all the terms of the diagonal of $\Psi(k)$ and through all the values of d.

Applying the variational calculus to determine the optimal $\bar{v}(d)$ for a fixed matrix $\hat{\Gamma}$, the Euler condition gives the optimum

$$\bar{v}(d) = \frac{\sum_{i=1}^{n} \psi_i(d)\gamma_i^*}{\sum_{i=1}^{n} \gamma_i\gamma_i^*} \tag{12}$$

Thus, the linearized model becomes

$$x\left(k+1\right)=e^{AT}x(k)+Vdiag\left(\bar{\Gamma}\right)V^{-1}BV_g\bar{v}(k) \tag{13}$$

where $\bar{v}(k)$ is a fictitious control signal related to the real control signal $d(k)$. Its inverse $\bar{v}^{-1}$ can be tabulated and used in the control algorithm. In the case of an $ON-OFF-ON$ actuator the equation to use is

$$d(k)=\bar{v}^{-1}\left(\bar{v}\left(k\right)\right) \tag{14}$$

2.3 Real system model for design

With parameters values (see Fig. 1) $L=3.24\,mH$, $R_L=0\,\Omega$, $C=48\,\mu F$, $R=117\,\Omega$, and $V_g=20\,V$, for a switching frequency of $2\,kHz$ ($T=0.5\,ms$), the obtained linear discrete model is

$$x\left(k+1\right)=\begin{bmatrix}0.3070 & -0.1100\\ 7.4237 & 0.2585\end{bmatrix}x(k)+ 10^3\begin{bmatrix}0.1798\\ 1.0571\end{bmatrix}V_g\bar{v}(k) \tag{15}$$

where $\bar{v}(k)$ is the fictitious control signal and the duty ratio $d(k)$ is obtained from the expression (14).

Then, the discrete transfer function of the converter is

$$G(z)=\frac{V(z)}{\bar{V}(z)}=\frac{10^3\left[1.0571z^{-1}+1.0100z^{-2}\right]}{1-0.5655z^{-1}+0.8958z^{-2}} \tag{16}$$

where $V(z)$ is the output voltage of the converter and $\bar{V}(z)$ is the fictitious control signal.

As the model system is discrete and the design method used is based on a continuous frequency-domain, the discrete model must be converted into a pseudo-continuous system using the bilinear w-transformation. So, the obtained pseudo-continuous systems is:

$$G(w)=-k\frac{\left(w-p\right)\left(b_1w+b_2\right)}{a_1w^2+a_2w+a_3} \tag{17}$$

being $k=10^3$, $p=4\ 10^3$, $b_1=471$, $b_2=8.2684\ 10^7$, $a_1=2.4613\ 10^4$, $a_2=8.336\ 10^6$, $a_3=2.12848\ 10^{11}$.

As it is observed, the previous system (17) corresponds to a non-minimum phase system, which has a right-hand-side (RHS) zero.

2.3.1 Separation of minimum phase and non-minimum phase systems

The presence of RHS zeros in the continuous model involves certain constraints in the control system behavior of the converter. The transfer function $G(w)$ can be write as the composition of a minimum-phase function $G_{fm}(w)$ with an all-pass filter $A(w)$ [2], so $G(w)=G_{fm}(w)A(w)$, where

$$G_{fm}(w)=k\ \frac{\left(w+p\right)\left(b_1w+b_2\right)}{a_1w^2+a_2w+a_3} \tag{18}$$

and

$$A(w)=\frac{-w+p}{w+p} \tag{19}$$

3 Controller design

3.1 Controller design based on the Smith predictor structure

The aim of this section is to design a fractional controller for the buck converter in the presence of a RHS zero. The controller design will be carried out in two steps. First, minimum-phase subsystem $G_{fm}(w)$ will be compensated. Next, non-minimum-phase subsystem $A(w)$ will be considered.

3.1.1 Minimum-phase subsystem compensation

For this purpose Bode's ideal function will be taken as reference system. Considering the transfer function of the minimum-phase subsystem (18) and a desired open loop transfer function for the compensated minimum-phase subsystem of the form (2), the parameters k_c and λ will be selected to obtain a specified phase margin (PM) and crossover frequency (ω_c). So, the transfer function of the compensator $D_o(w)$ can be obtained as:

$$D_o(w)=\frac{k_c}{w^{\lambda}G_{fm}(s)}=\frac{k_c\left(a_1w^2+a_2w+a_3\right)}{w^{\lambda}k\left(w+p\right)\left(b_1w+b_2\right)} \tag{20}$$

being the design equations:

$$k_c=\omega_c^{\lambda},\qquad \lambda=\tfrac{2}{\pi}(\pi-PM) \tag{21}$$

Taking as working specifications for the design: $PM=64^o$; $\omega_c=1.36\ 10^3\ rad/seg$, the parameters of the compensator are $k_c=1.1\ 10^4$, $\lambda=1.29$. Taking into account the original system, it is, a non-minimum phase system, the compensated system after this first step is:

$$G(w)D_o(w)=\frac{k_c}{w^{\lambda}}\frac{\left(-w+p\right)}{\left(w+p\right)} \tag{22}$$

The previous expression consists of the Bode's ideal function plus an all-pass filter. The all-pass filter introduced a phase lag that must be remove.

3.1.2 Non-minimum-phase subsystem compensation using the Smith predictor structure

The previous analysis reveals that the RHS effects must be considered in the controller design process. In this section, the

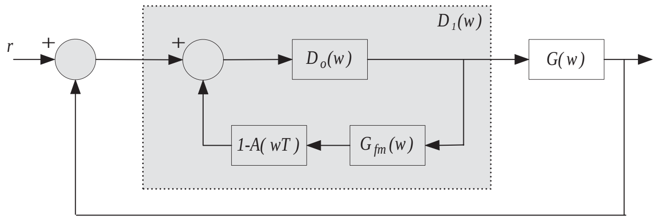

Figure 2: Smith predictor structure

non-minimum phase term will be compensated by using the Smith predictor structure (see Fig. 2)

By using this control structure, the total controller $D_1(w)$, marked with a dotted line in the figure 2, will be

$$D_1(w) = \frac{D_o(w)}{1 + D_o(w)G_{fm}(w) - D_o(w)G(w)} \tag{23}$$

Consequently, the global transfer function of the closed loop compensated system will be

$$F(w) = \frac{k_c}{w^\lambda + k_c} A(w) \tag{24}$$

Equation (24) differs from (1) and so an error between the compensated system and the Bode's ideal function can be observed at the frequency of interest. This error must be remove for preserving the design specifications, and a different phase margin and crossover frequency must be calculated to take into account the errors in module and phase. According to this, the new values of the corrected specifications are: $PM = 84.22^o$; $\omega_c = 2.53\ 10^3\ rad/seg$, and so, the new parameters of the compensator are $k_c = 4.19\ 10^3$, $\lambda = 1.06$.

3.2 Controller design by the phase lag compensation.

The second proposed design technique takes into account the additional phase-lag introduced by the all-pass filter. This additional phase delay can be evaluated at the frequency of interest (ω_c) being:

$$\angle\left(A(jv)\right)|_{v=\omega_c} = -37.55^o$$

The open loop transfer function of the compensated system in this case will be:

$$D(w)G(w) = \frac{1}{G_{fm}(w)} \frac{k_c}{w^\lambda} G(w) \tag{25}$$

The controller $D(w)$ will be obtained from the minimum phase subsystem, considering the phase lag of $A(w)$, resulting a new phase margin for design:

$$PM' = PM - \angle\left(A(jv)\right)|_{v=\omega_c} = 64 + 37.55 = 101.55^o$$

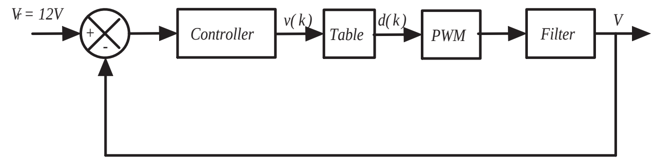

Figure 3: Block diagram for simulation

From the expressions (21) the controller parameters are obtained as: $k_c = 532$, $\lambda = 0.87$.

3.3 Controller design based on the discrete linearized model

In this section a new design alternative for the controller is proposed from the discrete linearized model of the converter. For this purpose a discrete version of the Bode's ideal function will be taken as reference system applying the Tustin's bilinear transformation. So, the open loop transfer function of the desired compensated system will become:

$$G(z)D(z) = \frac{k_c}{\left(\frac{2}{T}\frac{z-1}{z+1}\right)^\lambda} \tag{26}$$

where k_c and λ will be selected to obtain the specified phase margin (PM) and crossover frequency (w_c). The transfer function of the compensator $D(z)$ can be obtained as:

$$D(z) = \frac{K_c}{\left(\frac{2}{T}\frac{z-1}{z+1}\right)^\lambda} \frac{1}{G(z)} \tag{27}$$

$G(z)$ corresponds to (16) and the design equations are (21). Taking as working specifications for the design: $PM = 64^o$, $\omega_c = 1.36\ 10^3\ rad/seg$, the parameters of the compensator are $k_c = 1.1\ 10^4$, $\lambda = 1.29$. The discrete fractional integrator is approximated by continuous fraction expansion (CFE) as proposed in [12].

4 Simulation and experimental results

In order to show the performance of the three proposed methods, simulation and experimental results for all the controllers, with the specifications listed before, are shown here. For obtaining the simulation results of the compensated system, the first step is to find the discrete equivalents of the previous controllers. This is achieved using inverse w-transformation, and, in the case of the fractional pseudo-integrator, it is approximated by CFE method aforementioned.

The simulated system correspond to block diagram in Fig. 3 where: the block *"Table"* perform the conversion between fictitious control signal and duty ratio; the block *PWM* provides to the filter a voltage V_g during the interval $d(k)$; the block *"Filter"* is the LC filter plus the load resistance.

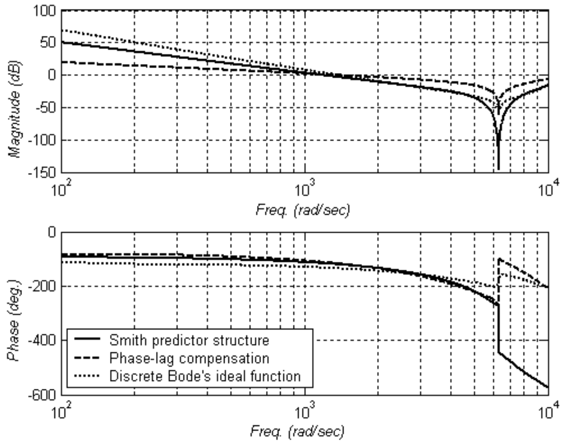

Figure 4: Simulated frequency responses for the designed controllers

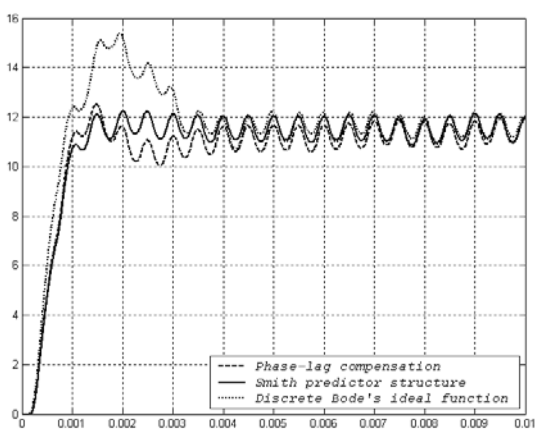

Figure 5: Simulated step responses for the designed controllers

Figure 4 displays the Bode diagrams of the compensated system with the described controllers. These results show that the design specification, phase margin (PM) and crossover frequency (ω_c), are satisfied.

Figure 5 shows the simulated step responses obtained with the described controllers. An overshoot can be observed in the time response of the controlled system using a controller based on the discrete version of the Bode's ideal function. This overshoot is due to $\lambda > 1$, and can be removed by changing the design specifications.

A real prototype of the Buck converter has been built to verify the feasibility of the proposed methods. The controller algorithms has been implemented in a *Pentium 166 MHz* machine. Figures 6, 7 y 8 show the experimental responses obtained with the different controllers. As can be observed, there is a good agreement between simulated and experimental results. All the controllers exhibit a good behavior in transitory and stationary

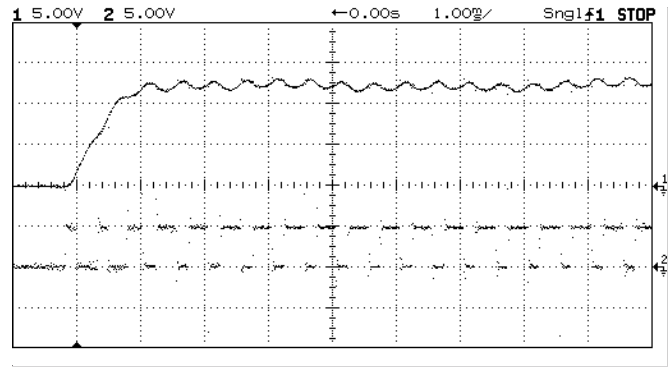

Figure 6: Experimental step response for Smith predictor structure case

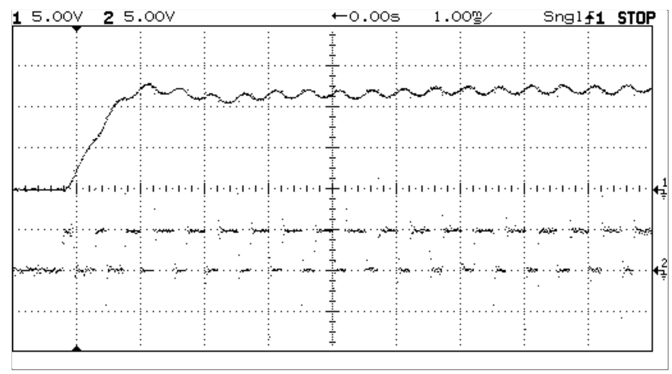

Figure 7: Experimental step response for phase-lag compensation case

regimes.

5 Conclusions

Several alternative new methods to control power electronic converters based on the use of fractional order controllers are proposed in this paper. The controller design methods are given, and simulated and experimental step responses are presented in order to show the performances of the controlled system and the flexibility and feasibility of the methods. This paper has shown that other alternatives can be applied to the control of systems that exhibit non-linearities or RHS zeros, like power electronics converters, and the practical implementation of the obtained controllers is feasible and they give good results. New works are carrying out with the goal of extending the use of fractional order operators to sliding mode control, as the definition of switching surfaces.

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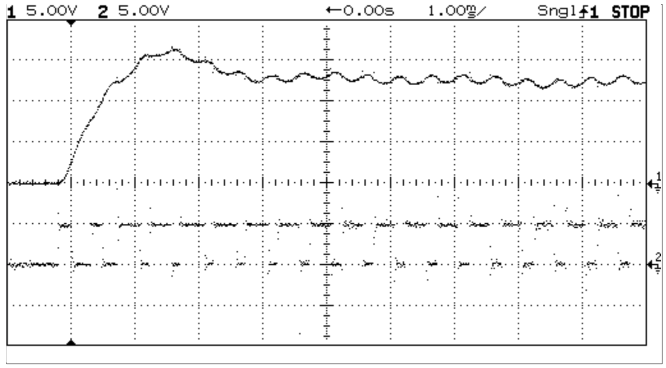

Figure 8: Experimental step response for design based on the discrete model case

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